Journal of Statistical Physics, Vol. 51, Nos. 5/6, 1988

Mean Field Theory of Self-Organized Critical Phenomena

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Received January 30, 1988

A mean field theory is presented for the recently discovered self-organized critical phenomena. The critical exponents are calculated and found to be the same as the mean field values for percolation. The power spectrum has "$1/f$" behavior with exponent $\varphi = 1$.

KEY WORDS: Self-organized criticality; critical exponents; mean field theory; $1/f$ noise.

Recently a new critical phenomenon termed "self-organized criticality" was discovered.(1) It was argued that some extended dissipative systems naturally evolve into a stationary state that is critical, in the sense that both spatial and temporal correlations obey power law behavior. Indeed, the existence of the self-organized critical state was demonstrated on several specific dynamical models. Critical exponents were defined and calculated numerically, and scaling relations derived.(2)

In this paper, we present a mean field theory for the self-organized critical phenomena. The main purpose of the theory is to yield insight into the fundamental mechanisms, and to provide a phenomenological framework and vocabulary. The mean field exponents that we provide are probably not very accurate for low-dimensional systems, as for most other critical phenomena. Nevertheless, they may serve as a first approximation in more general expansions. Moreover, there may exist an upper critical dimension above which these exponents are exact. We found that these mean field exponents are identical with those for the percolation model. This does not mean that they belong to the same universality class. In fact,

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0022-4715/88/0600-0797$06.00/0

the numerically calculated exponents in two and three dimensions are very different from 2D and 3D percolation exponents.[2] Of particular interest is the exponent φ for the power spectrum. We find a pure "$1/f$" spectrum with exponent 1. In real systems, the exponent is usually not precisely 1. The deviations of the exponent from unity can thus be understood as deviations from mean field theory.

We consider the following simple "cellular automaton."[1] On each site $\mathbf{r}$ on a d-dimensional hypercubic lattice of linear size L we define an integer variable $z(\mathbf{r})$, which can be thought of as a local slope or pressure driving a transport process. The dynamics is very simple: if $z(\mathbf{r})$ exceeds a critical value z_c, then at the next time step it "diffuses"

$$\begin{aligned} z(\mathbf{r}) &\to z(\mathbf{r}) - 2d \\ z(\mathbf{r}+\mathbf{n}) &\to z(\mathbf{r}+\mathbf{n}) + 1, \qquad \mathbf{n} = \pm\mathbf{e}_1, \pm\mathbf{e}_2, ..., \pm\mathbf{e}_d \end{aligned} \tag{1}$$

where $\{\mathbf{e}_i\}$ are the unit vectors. Without loss of generality, we can choose $z_c = 2d - 1$. The event above represents a unit "particle flow." The total flow at time t, $J(t)$, is simply the number of lattice sites on which $z > z_c$. The current density $j = J/L^d$ may be viewed as the order parameter for this critical phenomenon: if the average slope $\theta = \langle z \rangle$, where $\langle \cdot \rangle$ denotes the lattice average, is larger than a critical value θ_c, then $j \neq 0$. Otherwise, j will be zero unless an external "field" h is applied. The field represents the probability that

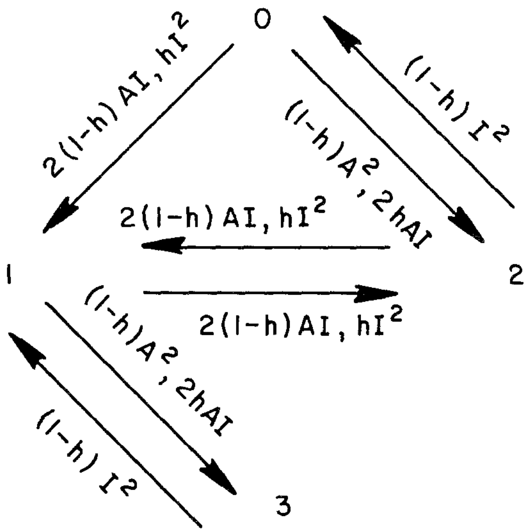
$$z(\mathbf{r}) \to z(\mathbf{r}) + 1 \tag{2}$$

For simplicity, consider the one-dimensional case. Generalization to higher dimensions is obvious. More importantly, the essential physics, including values of all the exponents, is independent of dimension in the mean field theory. In one dimension, $z_c = 1$. If on a particular site the variable assumes the value z, we say that this site is in state z. If $z > z_c$, the site is then "active." A state 0 will change to state 1 during the next time step with probability $1 - h$ if one of its two neighbors is active, and with probability h if none is active. Schematically we can represent the process $0 \to 1$ in a "reaction equation"

$$0 \xrightarrow{2(1-h)AI, hI^2} 1 \tag{3}$$

where A and I denote the active states (2, 3, etc.) and inactive states (0 or 1), respectively. Let P_z denote the fraction of sites in state z. In mean field approximation, P_z can be viewed as the probability that a site is in state z. The "reaction rate" of Eq. (3) can then be written as

$2(1-h)\,P_0 P_A P_I + hP_0 P_I^2$. We are interested in the critical region where j and h are small. This means that $P_A = j$ is small. To second order in P_2, the rate of Eq. (3) is $2P_0P_2(1-P_2)+2P_0P_3+hP_0-4hP_0P_2$. Generalizing the above argument to the other states, we get the following "reaction diagram":



The stationarity condition implies that the rate of flow into a state should be equal to the rate of flow out of the state. To second order in P_2, this leads to the following set of equations:

$$P_2(1-P_2)^2 - hP_2 = 2P_0P_2(1-P_2) + 2P_0P_3 + hP_0 + P_0P_2^2 - 2hP_0P_2 \tag{4a}$$

$$\begin{aligned}2P_0P_2(1-P_2) + 2P_0P_3 + hP_0 + 2P_2^2 + P_3 + hP_2 - 4hP_0P_2 \\ = 2P_1P_2(1-P_2) + 2P_1P_3 + hP_1 + P_1P_2^2 - 2hP_1P_2\end{aligned} \tag{4b}$$

$$2hP_0P_2 - 4hP_1P_2 + 2P_1P_2(1-P_2) + 2P_1P_3 + hP_1 + P_0P_2^2 = P_2 \tag{4c}$$

$$P_1P_2^2 + 2hP_1P_2 = P_3 \tag{4d}$$

The average slope θ is simply

$$\theta = 0\times P_0 + 1\times P_1 + 2\times P_2 + 3\times P_3 \tag{4e}$$

Equations (4) are our mean field equations for the self-organized critical phenomena.

When the field $h=0$, it is easy to see from Eqs. (4a) and (4b) that $P_0 = 1/2 + O(P_2)$ and $P_1 = 1/2 + O(P_2)$. Thus, at the critical point

$P_0 = P_1 = 1/2$ and $\theta_c = 1/2$. It can be shown that in the d-dimensional case at the critical point $P_0 = P_1 = P_2 = \cdots = P_{2d-1} = 1/2d$ and $\theta_c = d - 1/2$.

From Eqs. (4) we can derive an equation for the order parameter $j = P_2 + P_3 \approx P_2$. To the lowest interesting order

$$4P_2^2 + (1 - 2\theta + 2h)\, P_2 - h\theta = 0 \tag{5}$$

If $h = 0$, then $P_2 = 0$, or

$$P_2 = (1/2)(\theta - \theta_c), \qquad \text{so} \quad \beta = 1 \tag{6}$$

Differentiating Eq. (5) with respect to h and then setting $h = 0$, we get

$$8P_2\chi + 2P_2 + (1 - 2\theta)\,\chi - \theta = 0 \tag{7}$$

where $\chi = (dP_2/dh)_{h=0}$. For $\theta < \theta_c = 1/2$, $P_2 = 0$, and Eq. (7) gives

$$\chi = (1/4)(\theta_c - \theta)^{-1}, \qquad \gamma = 1 \tag{8a}$$

For $\theta > \theta_c$, $P_2 = (1/2)(\theta - \theta_c)$ and Eq. (7) gives

$$\chi = (1/4)(\theta - \theta_c)^{-1}, \qquad \gamma' = 1 \tag{8b}$$

At the critical point, $\theta = 1/2$ and Eq. (5) becomes

$$4P_2^2 + 2hP_2 - h/2 = 0$$

which gives (as $h \to 0$)

$$P_2 = (h/8)^{1/2}, \qquad \delta = 2 \tag{9}$$

We can also study the relaxation of the order parameter below the critical point. Recall that Eqs. (4) are the equalities for the "reaction rates." Hence,

$$dP_2/dt = \text{lhs of Eq. (4c)} - \text{rhs of Eq. (4c)}$$

Setting $h = 0$ and dropping all the terms of order higher than P_2, we have

$$dP_2/dt = 2P_1P_2 - P_2 = -(1 - 2\theta)\,P_2$$

where Eq. (4e) was used. Thus, the order parameter relaxes exponentially

$$P_2(t) = P_2(0)\exp(-t/t_{co}) \tag{10}$$

The relaxation time diverges at the critical point:

$$t_{co} = (\theta_c - \theta)^{-1}/2 \tag{11}$$

In ref. 2, it was shown that the order parameter relaxes as

$$j(t) \approx t^{\varphi - 1} \exp[-(t/t_{co})^{D(\tau - 2)/z}] \tag{12}$$

where φ is the power spectrum exponent, $t_{co} \approx (\theta_c - \theta)^{-z/D\sigma}$, D is the fractal dimension, τ is the cluster size distribution exponent, and z is the dynamical exponent. Comparing Eq. (12) with Eqs. (10) and (11), we have

$$\varphi = 1, \qquad z/D\sigma = 1, \qquad D(\tau - 2)/z = 1 \tag{13}$$

In ref. 2 we also derived a set of scaling relations that are valid in any dimensions:

$$\gamma = (3 - \tau)/\sigma, \qquad D = 1/\sigma \upsilon, \qquad \gamma/\upsilon = 2 \tag{14}$$

where $1/\sigma$ and υ are the exponents for "cutoff" cluster size and correlation length, respectively. From Eqs. (6), (8), (9), (13), and (14) we get the mean field values for all the exponents:

$$\beta = 1,\ \gamma = 1,\ \delta = 2,\ D = 4,\ \tau = 5/2,\ z = 2,\ \upsilon = 1/2,\ \sigma = 1/2,\ \varphi = 1 \tag{15}$$

It is very suggestive that these values are identical with the mean field exponents for percolation. However, direct numerical simulations for low-dimensional systems clearly indicate that the self-organized critical phenomena are not in the same universality class as percolation.[2]

In summary, we have constructed a dynamical mean field theory for the self-organized critical phenomena. It may help us to understand further the mechanisms for this new kind of critical phenomena. We can solve the mean field theory exactly for several exponents, and derive the remaining exponents from scaling relations. It is very interesting to note that the power spectrum exponent φ is 1. This gives us a hint on why "$1/f$" noise is usually so close to $1/f$. We point out that the theory presented here may not be *the* mean field theory for all the self-organized critical phenomena. One may get different mean field exponents for some other models. Furthermore, models with the same mean field exponents may easily have different "real" exponents and belong to different universality classes. It is far from clear how many different universality classes there are for the new phenomena. Symmetry will presumably play an important role here, like in any other critical phenomenon. The dynamics studied in this paper (Eqs. (1) and (2)) is both translational and rotational invariant. Models with different symmetries may give different exponents.[1],[3]

ACKNOWLEDGMENTS

This work was supported in part by the Institute for Theoretical Physics, Santa Barbara, California, under NSF grant PHY82-17853 supplemented by NASA, and by the Division of Materials Science, U. S. Department of Energy, under contract DE-AC02-76CH00016.

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