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Critical Temperature for Two-Dimensional Ising Ferromagnets with Quenched Bond Disorder[1]

R. Fisch[2,3]

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The configuration-averaged free energy of a quenched, random bond Ising model on a square lattice which contains an equal mixture of two types of ferromagnetic bonds J_1 and J_2 is shown to obey the same duality relation as the ordered rectangular model with the same two bond strengths. If the random system has a single, sharp critical point, the critical temperature T_c must be identical to that of the ordered system, i.e., $\sinh(2J_1/kT_c)\sinh(2J_2/kT_c) = 1$. Since $p_c^{(B)} = \frac{1}{2}$, we can take $J_2 \to 0$ and use Bergstresser-type inequalities to obtain $(\partial/\partial p)\exp(-2J_1/kT_c)|_{p=p_c^+} = 1$, in agreement with Bergstresser's rigorous result for the diluted ferromagnet near the percolation threshold.

KEY WORDS: Random Ising model; duality relation; critical point; percolation.

1. INTRODUCTION

In this paper, we show that a quenched, random Ising spin system containing an equal mixture of two types of ferromagnetic bonds on a square lattice has essentially the same properties under the Kramers–Wannier–Onsager transformation[1,2] as the ordered anisotropic system with the same two bond strengths. By the usual argument, this implies that if the random system has only one critical point, it must be at the same temperature as that of the corresponding ordered system.

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[2] Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania.

[3] Present address: Department of Physics, Princeton University, Princeton, New Jersey 08540.

2. DUALITY RELATION

We begin with the standard Ising Hamiltonian:

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij} S_i S_j \tag{1}$$

where $S_i = \pm 1$, and the $\{J_{ij}\}$ are uncorrelated random variables. We will use the probability distribution

$$P(J) = p\,\delta(J - J_1) + (1 - p)\,\delta(J - J_2) \tag{2}$$

where $0 \leqslant p \leqslant 1$ and $P(J)$ is the same for all J_{ij}. We would like to know how the behavior of the system varies as a function of the parameters in $P(J)$.

The partition function is given by

$$\begin{aligned} Z &= \operatorname*{Tr}_{\{S_i\}=\pm 1} \exp(-\mathcal{H}/kT) \\ &= \operatorname*{Tr}_{\{S_i\}=\pm 1} \prod_{\langle ij\rangle} (\cosh K_{ij})(1 + S_i S_j \tanh K_{ij}) \end{aligned} \tag{3}$$

where $K_{ij} = J_{ij}/kT$. Upon performing the duality transformation, we obtain

$$Z = \operatorname*{Tr}_{\{\mu_i\}=\pm 1} \prod_{\langle ij\rangle} 2^{-1/2}(\exp K_{ij})[1 + \mu_i \mu_j \exp(-2K_{ij})] \tag{4}$$

where the $\{\mu_i\}$ are the usual plaquette variables. The duality transformation is valid for an arbitrary bond distribution and temperature.

In order to proceed further, we must put restrictions on $P(J)$. If we require J_1 and J_2 to be positive (i.e., ferromagnetic), then we can define $K_1{}^*$ and $K_2{}^*$ by

$$\tanh K_1{}^* = \exp(-2K_1) \tag{5a}$$

$$\tanh K_2{}^* = \exp(-2K_2) \tag{5b}$$

As is well known, these relations are symmetric with respect to K_i and $K_i{}^*$. Thus, if $K_2 = K_1{}^*$, then also $K_1 = K_2{}^*$.

Now we restrict ourselves to the case of an equal mixture of J_1 and J_2, i.e., $p = \frac{1}{2}$. Then $P(J)$ becomes invariant under the interchange of J_1 and J_2, so that all configuration-averaged functions must be symmetric with respect to J_1 and J_2. Moreover, the set of all lattices that contain an equal mixture of J_1 and J_2 is mapped onto itself under the topological duality transformation. This invariance, plus Eq. (5), is sufficient to determine the critical point, assuming the existence of a single, sharp phase transition. The existence of the sharp transition for this system has not been proven rigorously, but it is strongly implied by the renormalization group calculations of Harris and Lubensky[(3)] and Khmelnitsky.[(4)] They found a stable fixed point corresponding to a second-order phase transition for this model.

Assuming that the critical point indeed exists, then T_c satisfies the relation

$$\sinh(2J_1/kT_c)\sinh(2J_2/kT_c) = 1 \tag{6}$$

This, of course, is the same relation for T_c found by Onsager[2] for the rectangular Ising model. In fact, there is an entire series of ordered, topologically self-dual Ising models that contain equal mixtures of J_1 and J_2, all of which share the same T_c. The limiting cases are the rectangular lattice and a centered 2×2 alternating bond structure.

3. DILUTE FERROMAGNET

Since the bond percolation concentration $p_c = \frac{1}{2}$ for the square lattice, it is particularly interesting to take the limit $J_2 \to 0$. For small J_2 we can approximate Eq. (6) by

$$\lim_{J_2 \to 0^+, p = p_c} [\exp(2J_1/kT_c)]J_2/kT_c = 1 \tag{7}$$

which can be written as

$$(d/dJ_2)[kT_c \exp(-2J_1/kT_c)]|_{J_2=0^+, p=p_c} = 1 \tag{8}$$

Bergstresser[5] has recently proven inequalities for the dilute ferromagnet, which can be extended to the random ferromagnet ($J_2 > 0$) in a straightforward manner. These inequalities give us the relation

$$\frac{kT_c}{2}\frac{dkT_c}{dJ_2}\bigg|_{J_2=0^+, p=p_c^+} = (1 - p_c)\frac{dkT_c}{dp}\bigg|_{J_2=0^+, p=p_c^+} \tag{9}$$

Combining Eqs. (8) and (9), we find

$$\left(\exp\frac{-2J_1}{kT_c}\right)\left(1 + \frac{2J_1}{kT_c}\right)\frac{dkT_c}{dp}\bigg|_{J_2=0^+, p=p_c^+} = \frac{kT_c}{2(1 - p_c)} \tag{10}$$

Neglecting 1 in comparison with $2J_1/kT_c$, we get

$$\frac{d}{dp}\exp\frac{-2J_1}{kT_c}\bigg|_{J_2=0^+, p=p_c^+} = \frac{1}{2(1 - p_c)} = 1 \tag{11}$$

This agrees with Bergstresser's rigorous result,[5] thus supporting our assumption of the existence of a sharp phase transition.

4. SUMMARY AND CONCLUSIONS

We have shown that it is apparently possible to use generalized duality relations to locate critical temperatures for certain random Ising ferromagnets in two dimensions. This is not sufficient to prove that the phase transition is continuous, as predicted by renormalization group calculations. However, we can say that the upper and lower critical exponents for the specific heat (and the magnetic susceptibility) will be equal if they exist. In addition, we wish to mention that Monte Carlo data[6] for a related random, two-dimensional Ising model contain hints that the symmetry between high and low temperatures which is implicit in a duality relation may be a property possessed by a wide class of random, two-dimensional Ising models, rather than being restricted to the equal-mixture case explicitly considered here.

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