WAVE-FLOW THEORY FOR A THIN VISCOUS LIQUID LAYER

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Izv. AN SSSR. Mekhanika Zhidkosti i Gaza, Vol. 3, No. 2, pp. 20–25, 1968

On the basis of a simplified system of equations we study the process of development and stability of wave flows in a thin layer of a viscous liquid. Any unstable disturbance of the laminar flow grows and leads to the establishment of the wave regime. The time to establish the flow changes little for large flow rates, but increases sharply with reduction of the flow rate. Given the same amplitudes of the initial disturbances, the optimum regimes which provide the greatest flow rate in a layer of given average thickness develop more rapidly than the other regimes. All the wave regimes are unstable to disturbances in the form of traveling waves. With moderate flow rates, the optimum regimes will be most stable to near-by disturbances.

Strictly periodic wave flows in a thin layer of a viscous liquid under the influence of the gravity force were calculated in [1]. Various flow wave regimes which differ in wavelength can theoretically be established for a given liquid flow rate. In particular, there is a wavelength for which the flowing layer exhibits minimum average thickness (and maximum flow rate for a given average thickness). These optimum regimes correspond closely to the experimental data [2]; however, with regard to calculation technique these regimes are no different from the others. In the following we make a comparison of the wave regimes on the basis of the nature of their development and stability.

1. Let us consider the following system of equations:

$$\frac{\partial h}{\partial \tau}+\frac{\partial}{\partial \xi}(q-zh)=0,$$

$$h^2\frac{\partial q}{\partial \tau}-(zh-{}^{12}/_5\,q)\,h\frac{\partial q}{\partial \xi}-$$

$$-{}^6/_5 q^2\frac{\partial h}{\partial \xi}-Gh^3\frac{\partial^3 h}{\partial \xi^3}-Hh^3+Eq=0. \qquad (1.1)$$

It is derived from certain simplifications of the complete system of equations and boundary conditions which describe the flow in the liquid layer; the functions $h(\tau,\xi)$, $q(\tau,\xi)$ coincide approximately with the thickness of the flowing layer and the liquid flow rate. In this sense we can assume that system (1.1) describes the flow of a viscous liquid in a thin layer, and we can treat its solutions as the liquid flow regimes. The solution of (1.1) is determined by four parameters

$$R_0=ga_0^3\nu^{-2},\quad R=3V_0a_0\nu^{-1},\quad n,\quad z,$$

in terms of which the equation coefficients are expressed as follows:

$$G=\frac{9\gamma n^2R_0^{1/3}}{R^2},\qquad H=\frac{9R_0}{R^2n},\qquad E=\frac{9}{Rn},$$

$$\gamma=\frac{\sigma}{\rho\nu^{4/3}g^{1/3}},\quad \tau=\frac{nV_0t}{a_0},\quad \xi=\frac{(x-zV_0t)n}{a_0}.$$

Here τ is a dimensionless variable, t is time, ξ is the longitudinal coordinate in a system traveling with the phase wave velocity; a_0 and V_0 are the average layer thickness and characteristic velocity; g, ν, and σ denote acceleration of gravity, kinematic viscosity, and surface tension. Two of the four parameters can be considered independent and specified arbitrarily to a certain degree, while the other two are found from the solution.

Steady-state solutions of (1.1) of two types are known:

$$h=1,\quad q=1,\qquad (R=R_0), \qquad (1.2)$$

$$h=1+\rho\sin\xi+h_{20}\sin 2\xi+h_{21}\cos 2\xi+\ldots,$$

$$q=1+z(h-1). \qquad (1.3)$$

The first solution corresponds to laminar flow in the layer, the second corresponds to periodic wave flow. It is shown in [1] that there is a two-parameter region in which wave solutions (1.3) exist. If we take R_0 and n as the independent parameters for the case of water flow at 15° C, this region in the R_0,n plane will lie between the n = 0 axis and line 1 in Fig. 1. The equation of this line is easily derived by means of the limiting form of the relations [1] for small amplitudes

$$n^5=R_0^{2/3}(\gamma A)^{-1},\quad A^2=27\gamma R_0^{-11/3}. \qquad (1.4)$$

The optimum regimes are realized for values of n_* lying on line 2 in Fig. 1.

Transition from one steady solution to another may occur as a result of loss of stability. Let us study the question of the stability of these steady solutions, and also the process of transition from one solution to the other. We use the method of [1] for the study. Let us construct the solution of (1.1) in the form

$$h=h_0+h_{10}\sin\xi+h_{11}\cos\xi+\ldots,$$

$$q=q_0+q_{10}\sin\xi+q_{11}\cos\xi+\ldots, \qquad (1.5)$$

where h_0, q_0, h_{lk}, and q_{lk} may depend on τ and ξ. The first equation is linear; therefore by equating the coefficients of its harmonics to zero we easily obtain the equations

$$\frac{\partial h_0}{\partial\tau}-(q_0'-zh_0')=0,$$

$$\frac{\partial h_{l0}}{\partial\tau}-[q_{l0}'-lq_{l1}-z(h_{l0}'-lh_{l1})]=0,$$

$$\frac{\partial h_{l1}}{\partial\tau}-[q_{l1}'+lq_{l0}-z(h_{l1}'+lh_{l0})]=0,$$

$$(l=1,2,\ldots). \qquad (1.6)$$

The difficulties are associated with the solution of the second nonlinear equation. After substituting (1.5) into this equation, we group terms of the same order in $\sin\xi$, $\cos\xi$ and reduce the result of the substitution to the form of a Fourier series. By equating the series coefficients to zero we obtain the equations for h_{lk}, q_{lk}. In first approximation, when the first three harmonics are reduced to the normal form, we have

$$\left(1+\frac{h_{10}^2+h_{11}^2}{2}\right)\frac{\partial q_0}{\partial \tau}+h_{10}\frac{\partial q_{10}}{\partial \tau}+h_{11}\frac{\partial q_{11}}{\partial \tau}= = S_0+\frac{S_{20}+S_{21}}{2},$$

$$(2h_{10}+h_{11}h_{20}-h_{10}h_{21})\frac{\partial q_0}{\partial \tau}+ +\left(1+\frac{3h_{10}^2+h_{11}^2}{4}-h_{21}\right)\frac{\partial q_{10}}{\partial \tau}+ +\frac{h_{10}h_{11}+2h_{20}}{2}\frac{\partial q_{11}}{\partial \tau}-h_{10}\frac{\partial q_{21}}{\partial \tau}+ +h_{11}\frac{\partial q_{20}}{\partial \tau}=\frac{S_{30}}{4}+S_{10},$$

$$(2h_{11}+h_{10}h_{20}+h_{11}h_{21})\frac{\partial q_0}{\partial \tau}+\frac{h_{10}h_{11}+2h_{20}}{2}\frac{\partial q_{10}}{\partial \tau}+ +h_{11}\frac{\partial q_{21}}{\partial \tau}+\left(1+\frac{h_{10}^2+3h_{11}^2}{4}+h_{21}\right)\frac{\partial q_{11}}{\partial \tau}+ +h_{10}\frac{\partial q_{20}}{\partial \tau}=\frac{S_{31}}{4}+S_{11},$$

$$(h_{10}h_{11}+2h_{20})\frac{\partial q_0}{\partial \tau}+h_{11}\frac{\partial q_{10}}{\partial \tau}+ +h_{10}\frac{\partial q_{11}}{\partial \tau}+\frac{\partial q_{20}}{\partial \tau}=\frac{S_{22}}{2},$$

$$\left(\frac{h_{11}^2-h_{10}^2}{2}+2h_{21}\right)\frac{\partial q_0}{\partial \tau}-h_0\frac{\partial q_{10}}{\partial \tau}+ +h_{11}\frac{\partial q_{11}}{\partial \tau}+\frac{\partial q_{21}}{\partial \tau}=\frac{S_{21}-S_{20}}{2}. \qquad (1.7)$$

Here

$$S_0=-Eq_0+Lh_0q_0'+{}^6/_5q_0^2h_0'+Mh_0^3$$

$$S_{1k}=(2Fh_{1k}-{}^{12}/_5h_0q_{1k})q_0'+Lh_0(q_{1k}'-\varepsilon q_{1m})-Eq_{1k}+3Mh_0^2h_{1k}+ +Gh_0^3(h_{1k}'''-3\varepsilon h_{1m}''-3h_{1k}'+\varepsilon h_{1m})+{}^6/_5q_0[q_0(h_{1k}'-\varepsilon h_{1m})+2q_{1k}h_0'],$$

$$S_{2k}=(P_{1k}h_{1k}-2\varepsilon Fh_{21}+{}^{12}/_5\varepsilon h_0q_{21})q_0'-\varepsilon Lh_0(q_{21}'+2q_{20})+ +(2Fh_{1k}-{}^{12}/_5h_0q_{1k})(q_{1k}'-\varepsilon q_{1m})+3Mh_0(h_{1k}^2-\varepsilon h_0h_{21})+ +{}^6/_5[-\varepsilon q_0^2(h_{21}'+2h_{20})+2q_0q_{1k}(h_{1k}'-\varepsilon h_{1m})+(q_{1k}^2-2\varepsilon q_0q_{21})h_0']+ +3Gh_0^2h_{1k}(h_{1k}'''-3\varepsilon h_{1m}''-3h_{1k}'+\varepsilon h_{1m})+\varepsilon Eq_{21}- -\varepsilon Gh_0^3(h_{21}'''+6h_{20}''-12h_{21}'-8h_{20}),$$

$$S_{22}=(P_{10}h_{11}+P_{11}h_{10}+4Fh_{20}-{}^{24}/_5h_0q_{20})q_0'+2(Fh_{10}-{}^6/_5h_0q_{10})(q_{11}'+q_{10})+ +(2Fh_{11}-{}^{12}/_5h_0q_{11})(q_{10}'-q_{11})+2Lh_0(q_{20}'-2q_{21})-2Eq_{20}+ +{}^{12}/_5[q_0^2(h_{20}'-2h_{21})+q_0q_{10}(h_{11}+h_{10})+q_0q_{11}(h_{10}'-h_{11})+ +(q_{10}q_{11}+2q_0q_{20})h_0']+6Mh_0(h_{10}h_{11}+h_0h_{20})+ +3Gh_0^2h_{10}(h_{11}'''+3h_{10}''-3h_{11}'-h_{10})+ +3Gh_0^2h_{11}(h_{10}'''-3h_{11}''-3h_{10}'+h_{11})+2Gh_0^3(h_{20}'''-6h_{21}''- -12h_{20}'+8h_{21}),$$

$$S_{3k}=4[-\varepsilon N_{10}h_{2m}+N_{11}h_{2k}+{}^6/_5(\varepsilon h_{10}q_{2m}-h_{11}q_{2k})]q_0'+ +(P_{10}h_{11}+P_{11}h_{10}+4Fh_{20}-{}^{24}/_5h_0q_{20})(q_{1m}'+\varepsilon q_{1k})+ +(P_{1m}h_{1m}-4\varepsilon Fh_{21}+{}^{24}/_5\varepsilon h_0q_{21}+3P_{1k}h_{1k})(q_{1k}'-\varepsilon q_{1m})+ +4(Fh_{11}-{}^6/_5q_{11}h_0)(q_{2k}'-2\varepsilon q_{2m})+4\varepsilon(-Fh_{10}+{}^6/_5h_0q_{10})(q_{2m}'+2\varepsilon q_{2k})+ +{}^6/_5[-4\varepsilon q_0q_{10}(h_{2m}'+2\varepsilon h_{2k})+4q_0q_{11}(h_{2k}'-2\varepsilon h_{2m})- -4(\varepsilon q_{10}q_{2m}-q_{11}q_{2k})h_0'+2(h_{1m}'+\varepsilon h_{1k})(q_{10}q_{11}+2q_0q_{20})+ +(h_{1k}'-\varepsilon h_{1m})(q_{1m}^2+3q_{1k}^2-4\varepsilon q_0q_{21})]+ +M(3h_{1k}^3+3h_{1k}h_{1m}^2-12\varepsilon h_0h_{10}h_{2m}+12h_0h_{11}h_{2k})+ +3Gh_0[2(h_{10}h_{11}+h_0h_{20})(h_{1m}'''+3\varepsilon h_{1k}''-3h_{1m}'-\varepsilon h_{1k})+ +(h_{1m}^2+3h_{1k}^2-2\varepsilon h_0h_{21})(h_{1k}'''-3\varepsilon h_{1m}''-3h_{1k}'+\varepsilon h_{1m})- -2\varepsilon h_0h_{1k}(h_{21}'''+6h_{20}''-12h_{21}'-8h_{20})+2h_0h_{1m}(h_{20}'''-6h_{21}''- -12h_{20}'+8h_{21})],$$

$$F=zh_0-{}^6/_5q_0,\quad L=zh_0-{}^{12}/_5q_0,\quad M=Gh_0'''+H,$$
$$N_{1k}=zh_{1k}-{}^6/_5q_{1k},\quad P_{1k}=zh_{1k}-{}^{12}/_5q_{1k}.$$

In these expressions the prime denotes partial differentiation with respect to ξ, $\varepsilon = (-1)^k$, the subscript k may take values of zero and one, and the subscript m, correspondingly, may take values of one and zero.

Equations (1.6) and (1.7) form a closed system for the ten unknown coefficients of expansions (1.5).

If we take h_{lk} = const, q_{lk} = const, then (1.5) will correspond to the steady wave solution, and system (1.6), (1.7) defines this solution in first approxima-

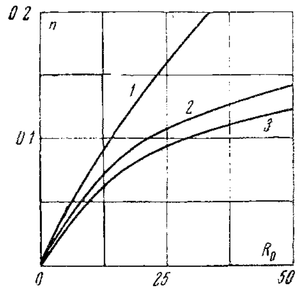

Fig. 1

tion. Comparison with the second approximation and exact numerical calculations show that for each R_0 there is an interval n, including n_*, in which the first approximation is sufficiently accurate. We shall consider only such n, and in cases in which h_{lk}, q_{lk} may depend on τ, ξ we assume that they do not exceed significantly the corresponding quantities for the steady solutions.

2. Assume that at the instant $\tau = 0$ there arises a disturbance of the steady solution (1.2) which is periodic with respect to ξ; then it may be considered periodic during the entire subsequent development time and it may be described with the aid of system (1.6), (1.7), in which h_{lk}, q_{lk} depend only on time.

The disturbance parameters can be found on the basis of linear theory. We take for convenience that $q_0 = 1$ and set

$$h=1+\rho_0e^{\omega\tau}\sin\xi,$$
$$q=1+\rho_0e^{\omega\tau}(z\sin\xi+\omega\cos\xi). \qquad (2.1)$$

We substitute (2.1) into the linearized equations (1.1) and after equating the expressions with $\sin\xi$, $\cos\xi$ to zero we obtain

$$\omega^2+H\omega-(z^2-{}^{12}/_5z+{}^6/_5-G)=0,$$

$$H(3 - z) - 2(z - {}^6/_5)\omega = 0. \qquad (2.2)$$

From this we find the disturbance amplitude growth index ω and the ratio z of the phase velocity to the characteristic velocity. The regions of growing and

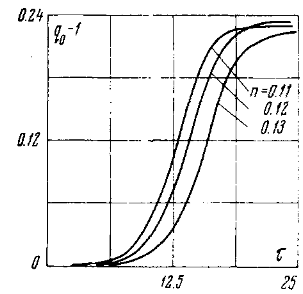

Fig. 2

decaying disturbances are separated by the line $\omega = 0$, on which, in accordance with (2.2), z = 3 and G = 3. This implies that the considered steady solution is always unstable [3,4] and that disturbance (2.1) will be increasing for $n < n_+$, where

$$n_+^2 = R_0^{5/3} / 3\gamma. \qquad (2.3)$$

It is easy to see that (2.3) coincides with Eq. (1.4) for the boundary curve separating the region of the existence of wave solutions in Fig. 1. Consequently, the wave regimes exist when there is instability of the laminar flow.

According to linear theory, the most rapidly growing disturbances are those represented by the points of line 3 in Fig. 1. The corresponding values of n differ markedly from n_* for the optimum regimes. Therefore, from the point of view of linear theory, the optimum regimes are not different from the other regimes with regard to rate of development.

Now let us turn to the nonlinear development of disturbance (2.1) with a small initial amplitude ρ_0. We take the expressions (2.1) as the initial conditions for h and q and calculate the development of this disturbance in the course of time with the aid of (1.6) and (1.7). These calculations were made for various ρ_0, R_0, and n and showed the following.

For any ρ_0 (sufficiently small so that in the development process h_{lk} and q_{lk} do not markedly exceed the steady values) disturbance (2.1) grows and leads to transition of the first steady solution into the second (wave) solution. The nature of the development is shown in Fig. 2, in which as a function of dimensionless time τ there is shown the flow rate increment q − 1 for $R_0 = 32.56$; $\rho_0 = 10^{-2}$, and three values of n,

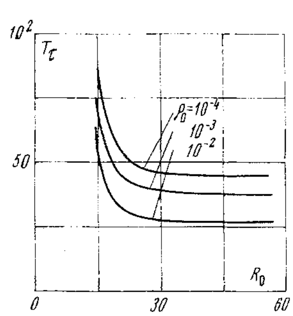

Fig. 3

of which the middle value equals n_*, and the other two differ from it by $\Delta n = \pm 0.01$. The curves of the disturbance growth with time have two segments. Up to some value of τ the disturbance is small and then increases sharply and approaches the wave regime. With change in the initial amplitude ρ_0 there is a change only of the slow growth segment, but the general nature of the growth is retained.

The behavior of the curves in Fig. 2 is such that we can find with sufficient definiteness the regime establishment time T_τ. The variation of the time to establish the optimum regimes with R_0 is shown in Fig. 3. It changes little for large values of R_0 and increases markedly for values less than $R_0 \approx 15$.

As a result, for small flow rates noticeable changes associated with the development of the incipient disturbance will not be observed for long periods of time. Linear theory [5] also leads to a similar conclusion concerning the development of small disturbances. A marked slowing of the wave regime establishment process for small flow rates has also been observed in experiments. The distance from the liquid inlet to the formation of wave flow was measured in [6]. The curve of this distance versus flow rate has the same form as the curve in Fig. 2.

In [2] the wave regimes were not observed at all for small flow rates.

Figure 4 shows the development curves for the same disturbances as in Fig. 2, but as a function of the physical time t. In the segment with slow changes the conclusions of linear theory are retained and the disturbance with $n < n_*$ grows more rapidly and that with $n > n_*$ grows more slowly than that with $n = n_*$.

However, in the exit segment, where the disturbance amplitudes increase sharply, the disturbance corresponding to the optimum regime begins to develop more rapidly than the others. Therefore, for a given average layer thickness and the same initial disturbance amplitudes

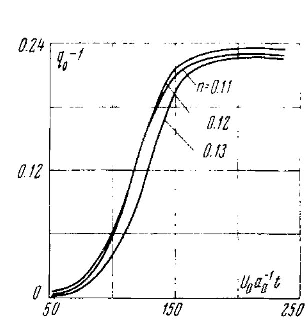

Fig. 4

the optimum regime will be established more rapidly than the others. This property is a result of the nonlinear nature of the development and is retained for all R_0.

3. The study of wave flow stability involves tedious calculations. We consider only the solution sequence and the final results without presenting the intermediate arguments. It is convenient to characterize a given wave flow by the numbers R and n. The development of a small disturbance of the wave regime may be described in the form (1.5), where h_{lk}, and q_{lk} are the functions τ and ξ, defined by (1.6) and (1.7). We write this solution in the form

$$h_{lk} = h_{0lk} + h_{1lk}, \qquad q_{lk} = q_{0lk} + q_{1lk}, \qquad (3.1)$$

where h_{0lk} and q_{0lk} is the wave solution and h_{1lk} and q_{1lk} are small disturbances.

We substitute (3.1) into (1.6) and (1.7) and drop terms of second order relative to h_{1lk} and q_{1lk}; as a result we obtain a homogeneous system of ten equations with constant coefficients for h_{1lk} and q_{1lk}. We consider the particular solutions of this system in the form of traveling waves

$$f_{1lk} = a_{1lk} e^{i(b\xi - i\omega_1\tau)}. \qquad (3.2)$$

Here f_{1lk} denotes any of the unknown quantitites h_{1lk}, q_{1lk}; here a_{1lk} is the corresponding initial amplitude. The quantity b is considered as given; it characterizes the ratio of the wavelengths: for b < 1 the disturbance wavelength is greater and for b > 1 it is less than the wavelength of the basic flow. To determine a_{1lk} from the linearized system we obtain a system of algebraic equations of tenth order; in the case of a nontrivial solution its determinant must vanish. This leads to a characteristic equation for ω_1 which has ten possible solu-

Values of characteristic equation roots corresponding to growing disturbances of wave flow for R = 30 ($n = n_* = 0.1093$)

$b = 0.6$	$b = 0.75$	$b = 1.0005$	$b = 1.25$	$b = 1.40$	$b = 1.55$
0.5601	0.4784	$0.6287 \cdot 10^{-3}$	0.3466	0.4196	0.4709
$-0.3856 \cdot 10^{-1}$	−0.1663	$-0.5868 \cdot 10^{-3}$	0.1590	0.1572	0.1079
0.5049	0.5250	0.3690	0.3389	0.4052	0.5473
0.1974	$-0.4034 \cdot 10^{-1}$	−0.4412	−0.6874	−0.8570	−1.030
$0.3518 \cdot 10^{-1}$	0.2930	0.2920	2.133	2.793	3.498
−0.7825	−0.9277	$0.6529 \cdot 10^{-1}$	−1.524	−1.751	−1.965
4.197	4.955	1.114	7.843	8.813	9.830
−2.255	−2.512	−1.135	−3.313	−3.542	−3.767
		6.335			
		−2.922			

tions. The imaginary part ω_i of each root determines the disturbance frequency and the real part ω_r determines the amplitude growth rate; the disturbance will grow if $\omega_r > 0$ and will decay if $\omega_r < 0$. Using the definition of τ and ξ, we transform (3.2) to the form

$$f_{11k} = a_{11k} \exp\left(\frac{n}{a_0} V_0 \omega_r t\right) \times$$

$$\times \exp\left\{ i \left[bx - \left(b - \frac{\omega_i}{z} \right) V_0 z t \right] \frac{n}{a_0} \right\}. \qquad (3.3)$$

This implies that the disturbance time growth rate is $(nV_0/a_0)\omega_r$, and the ratio of the disturbance frequency to the basic flow frequency is $b - \omega_i/z$.

The characteristic equation was solved for various flow rates, and for each flow rate we examined the optimum regime ($n = n_*$) and two nearby regimes ($n = n_* \pm 0.01$). For any b in each of the considered versions, among the roots of the characteristic equation, there are roots with a positive real part ω_r. This means that the wave flow, just as laminar flow, is unstable with regard to disturbances in the form of traveling waves, and a disturbance—periodic with respect to x—imposed on the wave flow will always grow. At the same time it is carried downstream with the velocity $(z - \omega_i b^{-1}) V_0$. The number of growing disturbances with the same b in the calculations reached five in accordance with the number of discretely located roots ω_i of the characteristic equation. The parameters of these disturbances (propagation velocity and wavelength) differ from the parameters of the basic wave flow by finite values. For a given wavelength the most rapidly growing disturbance is that which differs most from the wave disturbance. For example, the table shows the roots of the characteristic equation with $\omega_r > 0$ for R = 30 (ω_r is the upper number, ω_i is the lower). A similar root distribution holds for other values of R; it is also retained for large b; however, we have in mind primarily the interval $1/2 < b < 2$, in which the assumptions made are best justified.

As $b \to 1$, among the roots of the characteristic equation there appears a decreasing root for which $|\omega_i| \to 0$. It describes a disturbance which is close to the basic wave flow in its parameters. Such a disturbance is of particular interest from the point of view of realizing wave flow experimentally. Calculations show that the close disturbance decays if $b < 1$ and grows if $b > 1$. Figure 5 shows the growth indices for the close distrubances for R = 3. The disturbances of the optimum regime, which is most stable with respect to the close disturbances, grow most slowly. This property is clearly evidenced for moderate flow rates and becomes weaker for large and small flow rates. For $R \geq 50$ the optimum regimes do not exhibit the maximum stability property. We note that in the experiments of [2] clearly evidenced wave flows of the considered type were observed only up to $R \approx 55$.

Thus the optimum liquid wave flow regimes are identified among the other possible regimes by the fact that they develop more rapidly from small disturbances of the laminar flow and, in a definite interval of flow rates, are more stable to nearby disturbances. These properties

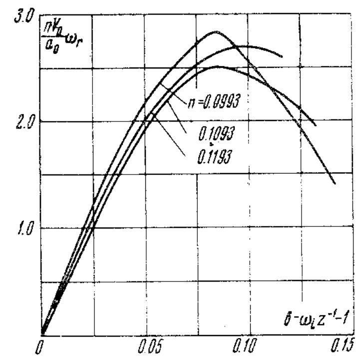

Fig. 5

must obviously have some effect in the experimental realization of these flows.

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12 July 1967

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