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The Charge Fluctuations in Classical Coulomb Systems

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We study the asymptotic behavior of the charge fluctuations $\langle(Q_\Lambda - \langle Q_\Lambda\rangle)^2\rangle$ in infinite classical systems of charged particles, and show, under certain clustering assumptions, that if the charge fluctuations are not extensive, then they are necessarily of the order of the surface $|\partial\Lambda|$. Moreover, when the canonical sum rules that are typical for equilibrium states of particles interacting with long-range forces hold true, we prove a central limit theorem for the normalized charge variable $|\partial\Lambda|^{-1/2}(Q_\Lambda - \langle Q_\Lambda\rangle)$ in two and three dimensions. In one dimension, the probability distribution of the charge itself converges. The latter case is illustrated by the example of the one-dimensional Coulomb gas.

KEY WORDS: Classical Coulomb systems; long-range force; charge fluctuations; canonical sum rules; central limit theorem; one-dimensional Coulomb gas.

1. INTRODUCTION

Gibbs states of classical systems of particles interacting with long-range forces (in particular, Coulomb systems) have specific properties which are not present in equilibrium states of particles interacting with short-range forces. At a very heuristic level, the reason for the occurrence of these new properties in Coulomb systems can be understood with the help of the following argument based on Gauss' law. Let us consider the total charge Q_Λ carried by the particles located in some region Λ; Q_Λ is related to the electric field $E(x)$ in the system by Gauss' law $Q_\Lambda = \int_{\partial\Lambda} E(x)\cdot ds$. We divide the surface $\partial\Lambda$ of Λ into N cells Δ_n, $n = 1,\dots, N$ of size $|\Delta_n| = \omega$, i.e., $|\partial\Lambda| = N\omega$. Then we have approximately $\int_{\partial\Lambda} E(x)\cdot ds = \omega\sum_{n=1}^{N} E_n$, where E_n is the projection of the electric field in the direction normal to the surface of the cell Δ_n. If the average electric field in the system is zero, $\langle E_n\rangle = 0$, the system is

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also locally neutral: $\langle Q_\Lambda \rangle = \omega \sum_{n=1}^{N} \langle E_n \rangle = 0$. If we assume, moreover, that the electric field random variables E_n are approximately statistically independent, we can apply the law of large numbers to conclude that the charge fluctuations behave as $|\Lambda| \to \infty$ like

$$\langle (Q_\Lambda - \langle Q_\Lambda \rangle)^2 \rangle = \langle Q_\Lambda{}^2 \rangle \simeq \omega^2 \sum_{n,m=1}^{N} \langle E_n E_m \rangle \simeq N \simeq |\partial\Lambda|$$

Thus the charge fluctuations, being of the order of the surface, are not extensive with Λ, as would be the case for the fluctuations of usual macroscopic quantities (outside of critical points). Clearly this fact has far-reaching consequences on the structure of Gibbs states, and the purpose of this paper is to study the properties of equilibrium states of charged particles in the case where the charge fluctuations are not extensive.

This work completes and extends results given in Ref. 1. It is deduced in Ref. 1 from suitably defined equilibrium equations that equilibrium states of infinite systems of particles interacting with long-range forces must possess new characteristic properties, provided that the state has an integrable clustering. Among these properties the most important are the neutrality and the hierarchy of canonical sum rules, the latter implying the nonextensivity of the charge fluctuations (see Ref. 1, Sections 4 and 5). In this work, the relation between the charge fluctuations and these canonical sum rules is further investigated, and a solvable model, the one-dimensional Coulomb gas, exhibiting all the features of the general theory is given as an illustration.

As in Ref. 1, we will always assume that we are given an equilibrium state of the infinitely extended system. The charge fluctuations are then those of the total charge of finite regions in this infinite system. We do not consider here the problem of constructing such infinite states by means of thermodynamic limits of finite-volume Gibbs states, nor do we study global charge fluctuations including charges at the boundary of the system. We include in our treatment both translation-invariant states and possible periodic states that are only invariant under some discrete subgroup of the translations. Furthermore, we emphasize that this analysis is not limited to the strict Coulomb force, but applies to all classical systems of charged particles having the features specific to long-range forces discussed in Ref. 1.

In Section 3 we establish under certain mild clustering assumptions that if the charge fluctuations are not extensive, they must be of the order of the surface. It is interesting to note that this result does not depend on the explicit form of the interparticle forces (contrary to the argument involving the Gauss law), but appears here as a general geometrical fact in the formalism of statistical mechanics. We study in Section 4 the implications of the canonical sum rules for the asymptotic behavior of the full probability distribution of the charge. The first sum rule with slightly stronger clustering assumptions than integrability implies the surface behavior of the charge fluctuations. Moreover, in three or two space dimensions, higher order sum rules together with similar clustering assumptions enable us to prove a central limit theorem for the properly normalized charge observable $|\partial\Lambda|^{-1/2}(Q_\Lambda - \langle Q_\Lambda\rangle)$. In one space dimension, the charge Q_Λ itself has a limiting distribution as $|\Lambda| \to \infty$. Since the sum rules are expected to hold in Coulomb systems as soon as screening occurs (see Ref. 1, Section 5), all the features of charge fluctuations will be true in such systems (with the possible exception of critical values of thermodynamic parameters). This is in fact the main point in this paper.

We reexamine briefly in Section 5 the one-dimensional Coulomb gas that has been treated by Edwards and Lenard,[2] with the purpose of illustrating the preceding results by an explicit example. In this respect, we supplement the work of Ref. 2 on two points: the existence of a limiting charge distribution and the validity of the sum rules.

Finally, we add that Lieb and Lebowitz have conjectured in their proof of the thermodynamic limit of the quantum Coulomb system that the charge fluctuations should also be nonextensive in the quantum case.[3] The study of the quantum charge fluctuations will be the subject of subsequent investigations.

2. GENERAL SETTING AND NOTATIONS

We consider a classical system of N kinds of particles with charges $\sigma \in \Sigma$, where Σ is a finite subset of R. We denote the charge σ and the position x of a particle by $q = (\sigma, x)$, $\sigma \in \Sigma$, $x = \{x_\alpha,\ \alpha = 1,\ldots, \nu\} \in R^\nu$, and we write

$$\int_\Lambda dq = \int_\Lambda dx \sum_\sigma, \qquad \Lambda \subset R^\nu$$

The state ρ of the system is given in terms of its correlation functions $\rho^{(n)}(q_1,\ldots, q_n) = \rho^{(n)}(x_1\sigma_1;\ldots; x_n\sigma_n) = \rho^{(n)}_{\sigma_1\cdots\sigma_n}(x_1,\ldots, x_n)$. These correlation functions are understood to describe an equilibrium state of the infinitely extended system (i.e., they are obtained as thermodynamic limits of those of finite Gibbs states, or as solutions of appropriate equilibrium equations[1]). Λ will always denote a finite region of this infinite system. The $\rho^{(n)}(q_1,\ldots, q_n)$ are positive and symmetric under the permutations of the arguments $q_1,\ldots, q_n$. Moreover, we shall assume throughout this paper that as functions of $x_1,\ldots, x_n$, they are continuous and bounded in $R^{n\nu}$.

We will be interested in two types of equilibrium states, the translation-invariant states (R^ν-invariant states) and the nontrivially periodic states, invariant under a discrete subgroup $\mathscr{T}$ of the group of translations ($\mathscr{T}$-invariant states). For an R^ν-invariant state we write simply the two-point functions as $\rho^{(2)}(\sigma_1 x_1; \sigma_2 x_2) = \rho^{(2)}_{\sigma_1\sigma_2}(x_1 - x_2)$.

We characterize a discrete subgroup $\mathscr{T}$ of the translations by ν fundamental vectors e_α, $\alpha = 1,\dots,\nu$, i.e.,

$$\mathscr{T} = \left\{ a_n \middle| a_n = \sum_{\alpha=1}^{\nu} n_\alpha e_\alpha, \quad n_\alpha \in Z, \quad \alpha = 1,\dots,\nu \right\}$$

and we denote by Ω the fundamental cell based on the vectors e_α. For a $\mathscr{T}$-invariant state, we have

$$\rho^{(k)}_{\sigma_1\cdots\sigma_k}(x_1 + a_n,\dots, x_k + a_n) = \rho^{(k)}_{\sigma_1\cdots\sigma_k}(x_1,\dots, x_k)$$

for all a_n in $\mathscr{T}$.

The total charge of the particles with coordinates and charges $q_j = (x_j\sigma_j)$ located in the finite region Λ is the observable

$$Q_\Lambda = \sum_j \sigma_j \chi_\Lambda(x_j) \tag{1}$$

with

$$\chi_\Lambda(x) = \begin{cases} 1 & x \in \Lambda \\ 0 & x \notin \Lambda \end{cases}$$

We denote by $\langle Q_\Lambda \rangle$ the average charge of the region Λ (in the state ρ of the infinitely extended system) and by $\langle Q_\Lambda{}^n \rangle$, $n = 2, 3,\dots$, the higher moments of the probability distribution of the charge in Λ. In particular $\langle Q_\Lambda \rangle$ and $\langle Q_\Lambda{}^2 \rangle$ are expressed in terms of the one- and two-point correlation functions by

$$\langle Q_\Lambda \rangle = \sum_\sigma \int_\Lambda \sigma \rho^{(1)}_\sigma(x)\, dx \tag{2}$$

$$\langle Q_\Lambda{}^2 \rangle = \sum_{\sigma_1\sigma_2} \int_\Lambda dx_1 \int_\Lambda dx_2\, \rho^{(2)}_{\sigma_1\sigma_2}(x_1, x_2) + \sum_\sigma \int_\Lambda dx\, \sigma^2 \rho^{(1)}_\sigma(x) \tag{3}$$

We denote by $|\Lambda|$ (resp. $|\partial\Lambda|$) the volume (resp. the surface) of a finite region Λ (resp. of its boundary $\partial\Lambda$). If there is an underlying lattice $\mathscr{T}$, we set $[\Lambda] = \sum_{a_j \in \mathscr{T}} \chi_\Lambda(a_j)$, the number of lattice points in Λ. Clearly $|\Lambda|$ differs from $|\Omega|[\Lambda]$ by the volume of cells intersecting $\partial\Lambda$.

If d is the maximal diameter of Ω, we have in any case

$$||\Lambda| - |\Omega|[\Lambda]| \leqslant |\Lambda^d| \tag{4}$$

where Λ^d is the set of points within a distance less than or equal to d to $\partial\Lambda$.

In order to investigate the behavior of the probability distribution of the charge for large Λ, we specify in what sense sequences of the region Λ converge to R^ν.

We assume always that $\Lambda \to R^\nu$ in the sense of van Hove [i.e., $\lim_{\Lambda \to R^\nu} (|\Lambda^h|/|\Lambda|) = 0$, where Λ^h is the set of points that are within a distance less than or equal to h to $\partial\Lambda$]. In particular we shall consider sequences

$$\Lambda_\lambda = \lambda\Lambda_0 = \{\lambda x | x \in \Lambda_0\}, \qquad \lambda > 0 \tag{5}$$

that are dilatations of a fixed region Λ_0, where the boundary $\partial\Lambda_0$ of Λ_0 is supposed to be a piecewise differentiable manifold.

In connection with $\mathscr{T}$-invariant states, we introduce the union of cells $\tilde{\Omega}$ (invariant under space inversion)

$$\tilde{\Omega} = \left\{ x | x = \sum_\alpha x_\alpha e_\alpha, \quad |x_\alpha| \leqslant 1 \right\}$$

and the sequence of regions Λ_k that are the union of cells obtained as dilatations of $\tilde{\Omega}$,

$$\Lambda_k = \{x | x = ky, \quad y \in \tilde{\Omega}\}, \qquad k = 1, 2, \ldots \tag{6}$$

3. ASYMPTOTIC BEHAVIOR OF THE CHARGE FLUCTUATIONS

In this section, we examine the asymptotic behavior of charge fluctuations $\langle (Q_\Lambda - \langle Q_\Lambda \rangle)^2 \rangle = \langle Q_\Lambda{}^2 \rangle - \langle Q_\Lambda \rangle^2$ as $\Lambda \to R^\nu$. We show that if the charge fluctuations are not extensive with Λ, then, under certain clustering assumptions, they are necessarily of the order of $|\partial\Lambda|$. This is the content of Propositions 1 and 2 below. In Proposition 1, we derive an integral relation between the one-point and the two-point correlation functions [Eqs. (8) and (9)] which has to hold when the charge fluctuations are not extensive. On the basis of this relation and of a geometrical lemma, we establish in Proposition 2 that $\langle (Q_\Lambda - \langle Q_\Lambda \rangle)^2 \rangle$ behaves as $|\partial\Lambda|$, $\Lambda \to R^\nu$.

By the very definition of the correlation functions, we get from (2) and (3)

$$\begin{aligned} \langle (Q_\Lambda - \langle Q_\Lambda \rangle)^2 \rangle &= \int_\Lambda dx \sum_\sigma \sigma^2 \rho^{(1)}(\sigma x) \\ &\quad + \int_\Lambda dx_1 \int_\Lambda dx_2 \sum_{\sigma_1 \sigma_2} \sigma_1 \sigma_2 \rho_T^{(2)}(\sigma_1 x_1; \sigma_2 x_2) \\ &= \int_\Lambda dx \sum_\sigma \sigma^2 \rho^{(1)}(\sigma x) - \int_\Lambda dx_1 \int_\Lambda dx_2 f(x_1, x_2) \end{aligned} \tag{7}$$

where we have set for brevity

$$f(x_1, x_2) = -\sum_{\sigma_1 \sigma_2} \sigma_1 \sigma_2 \rho_T^{(2)}(\sigma_1 x_1; \sigma_2 x_2) \tag{8}$$

and $\rho_T^{(2)}(\sigma_1 x_1; \sigma_2 x_2)$ is the two-point truncated correlation function defined in the usual way.

Proposition 1. (i) Let ρ be an R^ν-invariant state with $\mathscr{L}^1$ clustering two-point functions

$$\int |\rho^{(2)}_{T\sigma_1\sigma_2}(x)|\, dx < \infty$$

and Λ a sequence of regions converging to R^ν in the sense of van Hove. Then $\lim_{\Lambda\to R^\nu}[\langle(Q_\Lambda - \langle Q_\Lambda\rangle)^2\rangle/|\Lambda|] = 0$ if and only if

$$\sum_\sigma \sigma^2\rho^{(1)}_\sigma = \int dx\, f(x) \tag{9}$$

(ii) Let ρ be a $\mathscr{T}$-invariant state with $\mathscr{L}^1$ clustering two-point function

$$\sup_{x_1} \int |\rho^{(2)}_T(\sigma_1 x_1; \sigma_2 x_2)|\, dx_2 < \infty$$

and Λ_k the sequence of regions defined in (6). Then

$$\lim_{\Lambda_k\to R^\nu} \frac{\langle(Q_{\Lambda_k} - \langle Q_{\Lambda_k}\rangle)^2\rangle}{|\Lambda_k|} = 0$$

if and only if

$$\int_\Omega dx \sum_\sigma \sigma^2\rho^{(1)}_\sigma(x) = \int_\Omega dx \int dy\, f(x, y) \tag{10}$$

Proposition 2. (i) Let ρ be an R^ν-invariant state with

$$\int |\rho^{(2)}_{T\sigma_1\sigma_2}(x)|\, |x|\, dx < \infty$$

and $\Lambda_\lambda = \lambda\Lambda_0$ a sequence of dilated regions (5). If

$$\lim_{\Lambda_\lambda\to R^\nu} \frac{\langle(Q_{\Lambda_\lambda} - \langle Q_{\Lambda_\lambda}\rangle)^2\rangle}{|\Lambda_\lambda|} = 0$$

then

$$\lim_{\Lambda_\lambda\to R^\nu} \frac{\langle(Q_{\Lambda_\lambda} - \langle Q_{\Lambda_\lambda}\rangle)^2\rangle}{|\partial\Lambda_\lambda|} = \int f(y)\gamma(y)\, dy \equiv d \tag{11}$$

where $\gamma(y)$ is defined in Lemma 1(α_2).

(ii) Let ρ be a $\mathscr{T}$-invariant state with

$$\sup_x \int |\rho^{(2)}_T(\sigma_1 x; \sigma_2 y)|\, |y|\, dy < \infty$$

and Λ_k the sequence of regions (6). If

$$\lim_{\Lambda_k\to R^\nu} \frac{\langle(Q_{\Lambda_k} - \langle Q_{\Lambda_k}\rangle)^2\rangle}{|\Lambda_k|} = 0$$

then

$$\lim_{\Lambda_k \to R^\nu} \frac{\langle (Q_{\Lambda_k} - \langle Q_{\Lambda_k} \rangle)^2 \rangle}{|\partial \Lambda_k|} = \frac{1}{|\Omega|} \int_\Omega dx \int dy\, f(x, y) \bar{\gamma}(y) \equiv \bar{d} \tag{12}$$

where $\bar{\gamma}(y)$ is defined in Lemma 1 (β_2). (Proposition 2 holds in one dimension with $|\partial\Lambda| = 1$.)

The proofs of Propositions 1 and 2 depend on the following lemma, which is of a purely geometrical nature. For any finite region Λ and $y \in R^\nu$, we define

$$\gamma_\Lambda(y) = |\Lambda \cap (R^\nu \backslash \Lambda - y)| \tag{13}$$

and in the case where we have an underlying lattice $\mathscr{T}$,

$$\bar{\gamma}_\Lambda(y) = |\Omega| [\Lambda \cap (R^\nu \backslash \Lambda - y)] \tag{14}$$

$(\Lambda + y) = \{x + y | x \in \Lambda\}$ is the translate of Λ.

Lemma 1. (α_1) If $\Lambda \to R^\nu$ in the sense of van Hove, then

$$\lim_{\Lambda \to R^\nu} \frac{1}{|\Lambda|} \gamma_\Lambda(y) = 0$$

(α_2) If Λ is a sequence of dilated regions (5), then

$$\lim_{\Lambda \to R^\nu} \frac{1}{|\partial\Lambda|} \gamma_\Lambda(y) = \frac{1}{2|\partial\Lambda_0|} \int_{\partial\Lambda_0} |y \cdot ds| \equiv \gamma(y) \tag{15}$$

Let Λ_k be the sequence of regions (6); then

(β_1) $$\lim_{\Lambda_k \to R^\nu} \frac{1}{|\Lambda_k|} \bar{\gamma}_{\Lambda_k}(y) = 0$$

(β_2) $$\lim_{\Lambda_k \to R^\nu} \frac{1}{|\partial\Lambda_k|} \bar{\gamma}_{\Lambda_k}(y) = \frac{|\Omega|}{|\partial\Omega|} \sum_\alpha [|y_\alpha|] = \bar{\gamma}(y) \tag{16}$$

where y_α are the components of the vector y in the (not necessarily orthonormal) basis formed by the e_α, $\alpha = 1,\ldots,\nu$ (i.e., $y = \sum_{\alpha=1}^{\nu} y_\alpha e_\alpha$), and $[a]$ is the integer part of a.

The lemma is proved in the Appendix.

Remarks. 1. Coulomb systems have to be locally neutral, i.e., $\langle Q_\Lambda \rangle = 0$ (see Ref. 1, Section 4). However, neutrality has not been assumed in Propositions 1 and 2.

2. We see that under the conditions of Propositions 1 and 2, the integrals (9) and (10) involving $f(x)$ have to be positive, and the integrals (11) and (12)

have to be nonnegative. This is a screening effect. Indeed we get from (8) for an R^ν-invariant neutral state (i.e., $\sum_\sigma \sigma\rho_\sigma^{(1)} = 0$)

$$f(x_1, x_2) = \sum_{\substack{\sigma_1,\sigma_2\\ \sigma_1\sigma_2<0}} |\sigma_1\sigma_2|\rho^{(2)}_{\sigma_1\sigma_2}(x_1 - x_2) - \sum_{\substack{\sigma_1,\sigma_2\\ \sigma_1\sigma_2>0}} \sigma_1\sigma_2\rho^{(2)}_{\sigma_1\sigma_2}(x_1 - x_2)$$

Positivity conditions on $f(x_1, x_2)$ indicate that in the neighborhood of a given charge, it is more likely to find charges of the opposite sign than charges of the same sign. It is even expected (and proven in certain models[2]) that $f(x_1, x_2)$ is pointwise positive. If this is so, d and $\bar{d}$ in (11) and (12) are strictly positive. In any case, we shall assume in the rest of the paper that d and $\bar{d}$ are strictly positive, so that the charge fluctuations do not grow more slowly than $|\partial\Lambda|$.

3. Proposition 1 holds under the assumption of $\mathscr{L}^1$ clustering. However, in Proposition 2, a stronger clustering condition is needed, i.e., $|x_2|\rho_T^{(2)}(\sigma_1 x_1;\ \sigma_2 x_2) \in \mathscr{L}^1(dx_2)$. There might exist a critical temperature for which the latter condition is violated, leading to charge fluctuations of larger order than $|\partial\Lambda|$ (although not extensive).

4. In the case of a translation- and rotation-invariant state, and under the conditions of Proposition 2(i), we have

$$\lim_{\Lambda_\lambda \to R^\nu} \frac{\langle (Q_{\Lambda_\lambda} - \langle Q_{\Lambda_\lambda}\rangle)^2\rangle}{|\partial\Lambda_\lambda|} = C_\nu \int |y| f(y)\, dy$$

$$C_\nu = \begin{cases} 1, & \nu = 1 \\ 1/\pi, & \nu = 2 \\ 1/4, & \nu = 3 \end{cases}$$

the limit being independent of the shape of the sequence of regions Λ_λ. Indeed by the invariance of $f(y)$ under rotations, we can replace $\gamma(y)$ in (11) by its average over rotations R and we find with (15)

$$\frac{\int \gamma(Ry)\, d\mu(R)}{\int d\mu(R)} = C_\nu |y|$$

Proof of Proposition 1. (i) If ρ is translation invariant, the second term of (7) can be written as

$$\frac{1}{|\Lambda|}\int_\Lambda dx \int_\Lambda dy\, f(y - x)$$

$$= \frac{1}{|\Lambda|}\int_\Lambda dx \int dy\, f(y - x)\chi_\Lambda(y)$$

$$= \int dy\, f(y) \frac{1}{|\Lambda|}\int_\Lambda dx\, \chi_\Lambda(x + y) = \int dy\, f(y) - \int dy\, f(y) \frac{1}{|\Lambda|}\gamma_\Lambda(y) \tag{17}$$

since by (13),

$$\int_{\Lambda} dx\, \chi_{\Lambda}(x + y) = |\Lambda \cap (\Lambda - y)| = |\Lambda| - \gamma_{\Lambda}(y)$$

We have $|\gamma_{\Lambda}(y)|/|\Lambda| \leqslant 1$ and $\lim_{\Lambda \to R^{\nu}}[\gamma_{\Lambda}(y)/|\Lambda|] = 0$ by Lemma 1(α_1). The result follows by dominated convergence from (7) and (17) and by the fact that $\rho^{(1)}(\sigma x) = \rho_{\sigma}^{(1)}$ is constant.

(ii) If ρ is $\mathscr{T}$ invariant, we have

$$\frac{1}{|\Lambda_k|}\int_{\Lambda_k} dx \sum_{\sigma} \sigma^2 \rho_{\sigma}^{(1)}(x) = \frac{1}{|\Omega|}\int_{\Omega} dx \sum_{\sigma} \sigma^2 \rho_{\sigma}^{(1)}(x) \tag{18}$$

Moreover, we can write the second term of (7) as

$$\begin{aligned}
&\frac{1}{|\Lambda_k|}\int_{\Lambda_k} dx \int_{\Lambda_k} dy\, f(x, y) \\
&\quad = \frac{1}{[\Lambda_k]|\Omega|} \sum_{a_j \in \Lambda_k} \int_{\Omega} dx \int dy\, f(x + a_j, y)\chi_{\Lambda_k}(y) \\
&\quad = \frac{1}{|\Omega|}\int_{\Omega} dx \int dy\, f(x, y)\Big(\frac{1}{[\Lambda_k]} \sum_{a_j \in \Lambda_k} \chi_{\Lambda_k}(a_j + y)\Big) \\
&\quad = \frac{1}{|\Omega|}\int_{\Omega} dx \int dy\, f(x, y) - \frac{1}{|\Omega|}\int_{\Omega} dx \int dy\, f(x, y)\, \frac{1}{|\Lambda_k|}\, \bar{\gamma}_{\Lambda_k}(y)
\end{aligned} \tag{19}$$

since

$$\sum_{a_j \in \Lambda_k} \chi_{\Lambda_k}(a_j + y) = [\Lambda_k] - \bar{\gamma}_{\Lambda_k}(y)/|\Omega|$$

The result follows again by dominated convergence from Lemma 1(β_1) and (7), (18), (19). ∎

Proof of Proposition 2. (i) If the charge fluctuations are not normal, the relation (9) of Proposition 1 (i) holds true. Using (9) in (7), we get

$$\begin{aligned}
\frac{\langle (Q_{\Lambda} - \langle Q_{\Lambda}\rangle)^2\rangle}{|\partial\Lambda|} &= \frac{1}{|\partial\Lambda|}\Big(\int_{\Lambda} dx \int dy\, f(y) - \int_{\Lambda} dx \int_{\Lambda} dy\, f(y - x)\Big) \\
&= \frac{1}{|\partial\Lambda|}\int_{\Lambda} dx \int_{R^{\nu}\setminus\Lambda} dy\, f(y - x) \\
&= \int dy\, f(y)\, \frac{1}{|\partial\Lambda|}\, \gamma_{\Lambda}(y)
\end{aligned}$$

Since $(1/|\partial\Lambda|)\gamma_{\Lambda}(y)$ is uniformly bounded by (constant) $|y|$ and converges to $\gamma(y)$ [Lemma 1(α_2)], we obtain the result (11) by dominated convergence.

(ii) We know by Proposition 1(ii) that (10) holds. Using (10) and the periodicity, we have

$$\begin{aligned}\int_\Lambda dx \sum_\sigma \sigma^2 \rho^{(1)}(\sigma x) &= [\Lambda] \int_\Omega dx \sum_\sigma \sigma^2 \rho^{(1)}(\sigma x) \\ &= [\Lambda] \int_\Omega dx \int dy\, f(x, y) \\ &= \int_\Lambda dx \int dy\, f(x, y) \end{aligned} \tag{20}$$

Therefore (7) with (20) gives

$$\begin{aligned}\frac{\langle (Q_\Lambda - \langle Q_\Lambda \rangle)^2 \rangle}{|\partial\Lambda|} &= \frac{1}{|\partial\Lambda|} \int_\Lambda dx \int_{R^\nu \setminus \Lambda} dy\, f(x, y) \\ &= \frac{1}{|\partial\Lambda|} \sum_{a_j \in \Lambda} \int_\Omega dx \int dy\, f(x + a_j, y) \chi_{R^\nu \setminus \Lambda}(y) \\ &= \frac{1}{|\Omega|} \int_\Omega dx \int dy\, f(x, y) \frac{1}{|\partial\Lambda|} \bar{\gamma}_\Lambda(y)\end{aligned}$$

The result follows from Lemma 1(β_2) again by dominated convergence. ■

4. THE CANONICAL SUM RULES AND THE PROBABILITY DISTRIBUTION OF THE CHARGE

We have seen in Proposition 1 that the nonextensivity of the charge fluctuations is equivalent to the existence of certain constraints, i.e., Eqs. (9) and (10), that link the one-point and the two-point correlation functions. Using the definition (8), let us rewrite (9) and (10) explicitly in the form

$$\sum_{\sigma_1} \sigma_1 \left[\sigma_1 \rho_{\sigma_1}^{(1)} + \int dx \sum_\sigma \sigma \rho_{T\sigma_1\sigma}^{(2)}(x) \right] = 0 \tag{21}$$

$$\int_\Omega dx_1 \sum_{\sigma_1} \sigma_1 \left[\sigma_1 \rho^{(1)}(\sigma_1 x_1) + \int dx \sum_\sigma \sigma \rho_T^{(2)}(\sigma_1 x_1 ; \sigma x) \right] = 0 \tag{22}$$

Clearly Eqs. (21) and (22) are implied by the following relation between the one- and the two-point correlation functions:

$$\sigma_1 \rho^{(1)}(q_1) + \int dq\, \sigma \rho_T^{(2)}(q_1, q) = 0 \tag{23}$$

Equation (23) is called the first canonical sum rule and is the first member of a hierarchy of constraints that link the n-point to the $(n+1)$-point correlation function

$$\left(\sum_{j=1}^{n} \sigma_j\right) \rho^{(n)}(q_1,\ldots,q_n) + \int dq\, \sigma[\rho^{(n+1)}(q_1,\ldots,q_n q) - \rho^{(1)}(q)\rho^{(n)}(q_1,\ldots,q_n)] = 0, \qquad n = 1,2,\ldots \tag{24}$$

Equation (24) is called the nth canonical sum rule because it is identical to the relation that links the n and the $n+1$ correlation functions in a finite-volume canonical ensemble (see Section 5 of Ref. 1 for a discussion of this point). It is important to note that the sum rules are not true in general for infinite systems of particles interacting via short-range forces [obviously (24) is false for a gas of two kinds of noninteracting particles]. However, the sum rule (24) is expected to hold for infinite systems of charged particles interacting with long-range forces (like the Coulomb force) and it is in fact an essential characteristic of such systems. Indeed it can be shown that as soon as the state has some mild clustering properties ($\mathscr{L}^1$ clustering), the sum rule (24) is a consequence of the BBGKY hierarchy which defines the equilibrium states (see Proposition 8 of Ref. 1). Moreover, it will be checked explicitly in the next section that the sum rule (24) is true in the case of the one-dimensional Coulomb gas. In this section we study the implications of the canonical sum rules for the properties of the probability distribution of the charge. We have immediately the following result.

Proposition 3. Let ρ be an R^ν- or $\mathscr{T}$-invariant state. If the first sum rule holds and if

$$\sup_{x_2} \int |\rho_T^{(2)}(x_1\sigma_1; x_2\sigma_2)|\, |x_2|\, dx_2 < \infty \tag{25}$$

then the charge fluctuations $\langle (Q_\Lambda - \langle Q_\Lambda \rangle)^2 \rangle$ are of the order of the surface $|\partial\Lambda|$.

Proof. Indeed, (23) implies (9) and (10), and thus the charge fluctuations are not normal by Proposition 1. Therefore the results of Proposition 2 are true. ■

Corollary. The charge fluctuations $\langle (Q_\Lambda - \langle Q_\Lambda \rangle)^2 \rangle$ in R^ν- or $\mathscr{T}$-invariant equilibrium states of Coulomb systems are of the order of the surface $|\partial\Lambda|$ if the following clustering holds: for x_2 fixed

$$\rho_T^{(2)}(\sigma_1 x_1; \sigma_2 x_2) = O(1/|x_1|^{\nu+1+\epsilon}), \qquad \epsilon > 0$$

for x_3 fixed

$$\rho_T^{(3)}(\sigma_1 x_1; \sigma_2 x_2; \sigma_3 x_3) = O(1/|x_1|^{\nu+\epsilon})$$

uniformly in x_2, and

$$\lim_{|x_2|\to\infty} \int |\rho_T^{(3)}(\sigma_1 x_1; \sigma_2 x_2; \sigma_3 x_3)| \, dx_1 = 0$$

Proof. These clustering assumptions on the two- and three-point functions ensure by Proposition 8 of Ref. 1 that the first sum rule holds; thus Proposition 3 applies. ■

All these clustering properties are true for a class of one-dimensional Coulomb systems[2,4] (these systems have exponential clustering). In this case, $|\partial\Lambda| = 1$, and therefore the charge fluctuations converge to a finite limit.

In order to investigate the higher moments of the probability distribution of the charge, it is useful to introduce the new set of correlation functions $\hat{\rho}^{(n)}(q_1,..., q_n)$ giving the probability density for finding n particles, not necessarily different, at $x_1,..., x_n$.[5,6] To abbreviate the notation, we shall denote simply by Q the set of variables $(q_1,..., q_n)$ with $|Q| = n$, and

$$\rho^{(n)}(q_1,..., q_n) = \rho(Q), \quad \rho_T^{(n)}(q_1,..., q_n) = \rho_T(Q), \quad \text{etc.}$$

The correlations $\hat{\rho}^{(n)}(q_1,..., q_n)$ are defined by

$$\hat{\rho}(Q) = \sum_{\mathscr{P}(Q)} \left[\prod_j \Delta(Q_j^{\mathscr{P}})\right] \rho(Q^{\mathscr{P}}) \tag{26}$$

$\mathscr{P}(Q)$ is a partition $\bigcup_j Q_j^{\mathscr{P}} = Q$ of the set $Q = (q_1,..., q_n)$ into disjoint subsets $Q_j^{\mathscr{P}}, j = 1, 2,..., k, 1 \leqslant k \leqslant n$, and the sum in (26) runs on all such partitions. Here $\Delta(Q_j^{\mathscr{P}})$ is the product of Dirac functions identifying the arguments in $Q_j^{\mathscr{P}}$, i.e.,

$$\Delta(q_1,..., q_k) = \begin{cases} 1, & k = 1 \\ \delta_{q_1,q_2}\delta_{q_2,q_3} \cdots \delta_{q_{k-1},q_k}, & k \geqslant 2 \end{cases}$$

with

$$\delta_{q_1,q_2} = \delta_{\sigma_1,\sigma_2}\, \delta(x_1 - x_2)$$

$Q^{\mathscr{P}}$ is the subset of Q obtained by selecting one argument in each $Q_j^{\mathscr{P}}$.

Let $f_r(q) = f_r(\sigma x)$, $r = 1,..., n$, be continuous functions with compact support in the variable x, and let

$$F_r = \sum_j f_r(q_j), \qquad r = 1,..., n \tag{27}$$

be the corresponding local one-body observables. Then we have by the very definition of $\hat{\rho}(Q)$ that the average in the state ρ of the product $\prod_{r=1}^n F_r$ of one-body observables is given by

$$\left\langle \prod_{r=1}^n F_r \right\rangle = \int dq_1 \cdots dq_n\, \hat{\rho}^{(n)}(q_1,..., q_n) \prod_{r=1}^n f_r(q_r) \tag{28}$$

Together with the $\hat{\rho}(Q)$, we consider the corresponding truncated functions $\hat{\rho}_T(Q)$ related to the $\hat{\rho}(Q)$ in the usual way[6]

$$\hat{\rho}(Q) = \sum_{\mathscr{P}(Q)} \prod_j \hat{\rho}_T(Q_j^{\mathscr{P}}) \tag{29}$$

It follows from (26) and (29) that the $\hat{\rho}_T(Q)$ are related to the ordinary truncated correlation functions by the same formula as (26),[6]

$$\hat{\rho}_T(Q) = \sum_{\mathscr{P}(Q)} \prod_j \Delta(Q_j^{\mathscr{P}})\rho_T(Q^{\mathscr{P}}) \tag{30}$$

$\hat{\rho}_T^{(n)}(q_1,..., q_n)$ [resp. $\hat{\rho}^{(n)}(q_1,..., q_n)$] is a linear combination of $\rho_T^{(k)}$, $1 \leqslant k \leqslant n$ [resp. $\rho^{(k)}$] with arguments in subsets of $(q_1,..., q_n)$.

The next lemma gives four equivalent forms of the sum rules in terms of the different types of correlation functions. The proof of Lemma 2 can be found in the Appendix.

Lemma 2. Assume that $\int |\rho_T(Qq)|\, dq < \infty$, $|Q| = 1,..., n$; then the following relations are equivalent:

(i) $$\Big(\sum_j \sigma_j\Big)\rho(Q) + \int dq\, \sigma[\rho(Qq) - \rho(q)\rho(Q)] = 0, \qquad |Q| = 1,..., n \tag{31}$$

(ii) $$\Big(\sum_j \sigma_j\Big)\rho_T(Q) + \int dq\, \sigma \rho_T(Qq) = 0, \qquad |Q| = 1,..., n \tag{32}$$

(iii) $$\int dq\, \sigma[\hat{\rho}(Qq) - \rho(q)\hat{\rho}(Q)] = 0, \qquad |Q| = 1,..., n \tag{33}$$

(iv) $$\int dq\, \sigma \hat{\rho}_T(Qq) = 0, \qquad |Q| = 1,..., n \tag{34}$$

We are now ready to study the probability distribution of the charge $P_{Q_\Lambda}(s)$, which is the probability (in the state ρ) that the total charge carried by the particles in Λ is equal to s. Let us introduce the normalized charge observable $\bar{Q}_\Lambda = (Q_\Lambda - \langle Q_\Lambda \rangle)/|\partial\Lambda|^{1/2}$. We know by Proposition 3 that if the first sum rule holds [with the integrability conditions (25)], then $\langle \bar{Q}_\Lambda{}^2 \rangle$ has a limit as $\Lambda \to R^\nu$. One can expect that the higher order moments $\langle \bar{Q}_\Lambda{}^k \rangle$ of the normalized observable $\bar{Q}_\Lambda$ also converge when we have clustering and sum rules involving higher order correlation functions. The next two propositions show that this is indeed the case: in $\nu = 2$ or 3 space dimensions, the probability distribution $P_{\bar{Q}_\Lambda}(s)$ for the normalized charge observable $\bar{Q}_\Lambda$ converges to the Gaussian, whereas for one-dimensional (neutral) systems, $P_{\bar{Q}_\Lambda}$ converges to a discrete distribution.

Proposition 4. Let ρ be an R^ν- or $\mathcal{T}$-invariant state. Assume that the two-point truncated correlation functions have the integrability properties (25) and that the first sum rule holds (so that by Proposition 3, $\lim_{\Lambda\to R^\nu}\langle\bar{Q}_\Lambda{}^2\rangle = d$) and suppose $d > 0$ ($\bar{d} > 0$).

Then $P_{\bar{Q}_\Lambda}(s)$ converges to $(2\pi d)^{-1/2}\exp(-s^2/2d)$ as $\Lambda \to R^\nu$ if furthermore the higher order correlation functions have the properties listed below.

(a) $\mathcal{L}^1$ clustering for $n \geqslant 3$:

$$\sup_{x_1}\int dq_2 \cdots \int dq_n\, |\rho_T^{(n)}(\sigma_1 x_1, q_2,\ldots, q_n)| < \infty, \qquad n = 3, 4,\ldots$$

(b) Moreover, in dimension $\nu = 3$, the second sum rule holds.

In dimension $\nu = 2$:

(cl) $\sup_{x_1}\int dx_2 \int dx_3\, |x_2|\rho_T^{(3)}(\sigma_1 x_1, \sigma_2 x_2, \sigma_3 x_3) < \infty.$

(c2) The second and the third sum rules hold.

Proof. We give the proof for an R^ν-invariant state. The periodic state can be treated in exactly the same way. $\Lambda = \lambda\Lambda_0$ is a sequence of dilated regions (5).

Let $M_n(\bar{Q}_\Lambda)$ be the nth-order cumulant associated with the probability distribution $P_{\bar{Q}_\Lambda}(s)$.[2]

We have

$$M_1(\bar{Q}_\Lambda) = \langle\bar{Q}_\Lambda\rangle = 0$$
$$\lim_{\Lambda\to R^\nu} M_2(\bar{Q}_\Lambda) = \lim_{\Lambda\to R^\nu}\langle\bar{Q}_\Lambda{}^2\rangle = d > 0$$

We shall show that $\lim_{\Lambda\to R^\nu} M_n(\bar{Q}_\Lambda) = 0$ for $n \geqslant 3$. This will imply that the moments $\langle\bar{Q}_\Lambda{}^k\rangle$, $k = 1, 2,\ldots$, tend to those of the Gaussian, and by the theorem of convergence of moments, that $P_{\bar{Q}_\Lambda}(s)$ converges weakly to the Gaussian with covariance d.

The nth-order cumulant $M_n(F)$ of any single-particle local observable $F = \sum_j f(q_j)$ can be expressed in terms of the correlation functions $\hat{\rho}_T(q_1,\ldots, q_n)$ by

$$M_n(F) = \int dq_1 \cdots dq_n\, \hat{\rho}_T^{(n)}(q_1,\ldots, q_n) f(q_1) \cdots f(q_n)$$

Since $M_n(\bar{Q}_\Lambda) = (1/|\partial\Lambda|^{n/2}) M_n(Q_\Lambda)$ for $n \geqslant 2$ we have, with (1),

$$M_n(\bar{Q}_\Lambda) = \frac{1}{|\partial\Lambda|^{n/2}}\int_\Lambda dq_1 \cdots \int_\Lambda dq_n\, \sigma_1 \cdots \sigma_n \hat{\rho}_T^{(n)}(q_1,\ldots, q_n) \qquad (35)$$

[2] The cumulants are defined by the formal expansion

$$\ln\langle\exp(\alpha\bar{Q}_\Lambda)\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{n!} M_n(\bar{Q}_\Lambda)$$

and thus

$$|M_n(\bar{Q}_\Lambda)| \leqslant \frac{|\Lambda|}{|\partial\Lambda|^{n/2}} \left(\sup_\sigma \sigma\right)^n N \sup_{q_1} \int dq_2 \cdots dq_n \, |\hat{\rho}_T^{(n)}(q_1, q_2, \ldots, q_n)| \tag{36}$$

It follows from the linear relation between the $\hat{\rho}_T$ and the ρ_T and the structure of formula (30) that the $\mathscr{L}^1$ clustering implies

$$\sup_{q_1} \int dq_2 \cdots dq_n \, |\hat{\rho}_T^{(n)}(q_1, q_2, \ldots, q_n)| < \infty$$

Since for a sequence of dilated regions $\lambda\Lambda_0$

$$|\Lambda|/|\partial\Lambda|^{n/2} = \lambda^{\nu + (n/2)(1-\nu)} |\Lambda_0|/|\partial\Lambda_0|^{n/2} \tag{37}$$

we get from (36), (37), and the $\mathscr{L}^1$ clustering (a)

$$\lim_{\Lambda \to R^\nu} M_n(\bar{Q}_\Lambda) = 0 \qquad \text{if } n > 3 \text{ when } \nu = 3$$

$$\lim_{\Lambda \to R^\nu} M_n(\bar{Q}_\Lambda) = 0 \qquad \text{if } n > 4 \text{ when } \nu = 2$$

Thus it remains to examine $M_3(\bar{Q}_\Lambda)$ when $\nu = 3$ and $M_3(\bar{Q}_\Lambda)$ and $M_4(\bar{Q}_\Lambda)$ when $\nu = 2$.

Case $\nu = 3$. Using the second sum rule in the form (34), $\int dq_1 \, \sigma_1 \hat{\rho}_T^{(3)} (q_1, q_2, q_3) = 0$, we can write

$$\begin{aligned} |M_3(\bar{Q}_\Lambda)| &= \frac{1}{|\partial\Lambda|^{3/2}} \left| \int_\Lambda dq_1 \int_\Lambda dq_2 \int_\Lambda dq_3 \, \sigma_1\sigma_2\sigma_3 \hat{\rho}_T^{(3)}(q_1, q_2, q_3) \right| \\ &= \frac{1}{|\partial\Lambda|^{3/2}} \left| \int_{R^\nu \setminus \Lambda} dq_1 \int_\Lambda dq_2 \int_\Lambda dq_3 \, \sigma_1\sigma_2\sigma_3 \hat{\rho}_T^{(3)}(q_1, q_2, q_3) \right| \\ &\leqslant \frac{1}{|\partial\Lambda|^{3/2}} \int_{R^\nu \setminus \Lambda} dx_1 \int_\Lambda dx_2 \, g(x_1, x_2) \end{aligned} \tag{38}$$

with

$$g(x_1, x_2) = \sum_{\sigma_1\sigma_2} \int dq_3 \, |\sigma_1\sigma_2\sigma_3 \hat{\rho}_T^{(3)}(\sigma_1 x_1; \sigma_2 x_2; q_3)| \tag{39}$$

The $\mathscr{L}^1$ clustering assumption (a) for $n = 3$ implies that

$$\sup_{x_1} \int dx_2 \, |g(x_1, x_2)| < \infty$$

Therefore we can proceed as in the proof of Proposition 2 and apply Lemma $1(\alpha_1)$ to obtain

$$\lim_{\Lambda \to R^\nu} \frac{1}{|\Lambda|} \int_{R^\nu \setminus \Lambda} dx_1 \int_\Lambda dx_2 \, g(x_1, x_2) = \lim_{\Lambda \to R^\nu} \int dy \, g(y) \frac{1}{|\Lambda|} \gamma_\Lambda(y) = 0 \tag{40}$$

In view of the fact that $|\Lambda|/|\partial\Lambda|^{3/2} = |\Lambda_0|/|\partial\Lambda_0|^{3/2}$ = constant when $\nu = 3$, we conclude from (38) and (40) that $\lim_{\Lambda\to R^\nu} M_3(\bar{Q}_\Lambda) = 0$.

Case $\nu = 2$. We majorize $M_3(\bar{Q}_\Lambda)$ as in (38) with the same definition (39) for $g(x_1, x_2)$. Now we have from assumption (cl) that

$$\sup_{x_1} \int dx_2\, g(x_1, x_2)|x_2| < \infty$$

and therefore we find by Lemma $1(\alpha_2)$ that

$$\lim_{\Lambda\to R^\nu} \frac{1}{|\partial\Lambda|} \int_{R^\nu\setminus\Lambda} dx_1 \int_\Lambda dx_2\, g(x_1, x_2) = \int dy\, g(y) \frac{1}{|\partial\Lambda|} \gamma_\Lambda(y)$$

converges as $\Lambda \to R^\nu$. From this and (38) we have that $M_3(\bar{Q}_\Lambda) = O(1/|\partial\Lambda|^{1/2}) \to 0$ as $\Lambda \to R^\nu$.

For $M_4(\bar{Q}_\Lambda)$ we use the third sum rule, $\int dq_1\, \sigma_1 \hat{\rho}_T^{(4)}(q_1, q_2, q_3, q_4) = 0$, to write

$$|M_4(\bar{Q}_\Lambda)| = \frac{1}{|\partial\Lambda|^2} \left| \int_{R^\nu\setminus\Lambda} dq_1 \int_\Lambda dq_2 \int_\Lambda dq_3 \int_\Lambda dq_4\, \sigma_1\sigma_2\sigma_3\sigma_4 \hat{\rho}_T^{(4)}(q_1, q_2, q_3, q_4) \right|$$

$$\leqslant \frac{1}{|\partial\Lambda|^2} \int_{R^\nu\setminus\Lambda} dx_1 \int_\Lambda dx_2\, h(x_1, x_2)$$

with

$$h(x_1, x_2) = \sum_{\sigma_1\sigma_2} \int dq_3 \int dq_4\, |\sigma_1\sigma_2\sigma_3\sigma_4 \hat{\rho}_T^{(4)}(\sigma_1 x_1; \sigma_2 x_2; q_3, q_4)|$$

The $\mathscr{L}^1$ clustering for $n = 4$ and Lemma $1(\alpha_1)$ ensure again that

$$\lim_{\Lambda\to R^\nu} \frac{1}{|\Lambda|} \int_{R^\nu\setminus\Lambda} dx_1 \int_\Lambda dx_2\, h(x_1, x_2) = 0$$

From this and the fact that $|\Lambda|/|\partial\Lambda|^2 = |\Lambda_0|/|\partial\Lambda_0|^2$ = constant when $\nu = 2$ we get $\lim_{\Lambda\to R^\nu} M_4(\bar{Q}_\Lambda) = 0$. ■

Proposition 5. Let $\nu = 1$ (i.e., $|\partial\Lambda| = 1$ and $\bar{Q}_\Lambda = Q_\Lambda - \langle Q_\Lambda \rangle$) and ρ be an R-invariant state. If

$$\sup_{x_1} \int dx_2 \int dx_3 \cdots \int dx_n\, |x_2|\, |\rho_T^{(n)}(\sigma_1 x_1; \sigma_2 x_2; \ldots; \sigma_n x_n)| < \infty, \qquad n \geqslant 2$$

and all sum rules hold, then $P_{\bar{Q}_\Lambda}(s)$ converges weakly to a finite distribution $P(s)$ as $\Lambda \to R$.

Proof. Using the nth sum rule $\int dq_1\, \sigma_1 \hat{\rho}_T^{(n)}(q_1, \ldots, q_n) = 0$, we have for $n \geqslant 2$,

$$M_n(\bar{Q}_\Lambda) = -\int_{R\setminus\Lambda} dq_1 \int_\Lambda dq_2 \cdots \int_\Lambda dq_n\, \sigma_1\sigma_2 \cdots \sigma_n \hat{\rho}_T^{(n)}(q_1, \ldots, q_n)$$

$$= -\int_{R\setminus\Lambda} dx \int_\Lambda dy\, K_\Lambda^{(n)}(x, y) \tag{41}$$

with

$$K_\Lambda^{(n)}(x, y) = \sum_{\sigma_1\sigma_2} \int_\Lambda dq_3 \cdots \int_\Lambda dq_n\, \sigma_1\sigma_2 \cdots \sigma_n \hat{\rho}_T^{(n)}(\sigma_1 x; \sigma_2 y; q_3, \ldots, q_n) \tag{42}$$

We can write

$$\begin{aligned} M_n(\bar{Q}_\Lambda) &= -\int dx \int dy\, \chi_\Lambda(y)\chi_{R\setminus\Lambda}(x) K_\Lambda^{(n)}(x, y) \\ &= -\int dx \int dy\, \chi_\Lambda(x + y)\chi_{R\setminus\Lambda}(x) K_\Lambda^{(n)}(x, y + x) \\ &= -\int dy \int dx\, \chi_\Lambda(x + y)\chi_{R\setminus\Lambda}(x) K_{\Lambda - x}^{(n)}(0, y) \end{aligned} \tag{43}$$

We have made the change of integration variables $y \to y + x$ and used translation invariance.

The exchange of the x and y integrals in (43) is allowed, since with (42)

$$\begin{aligned} &\left| \int dx\, \chi_\Lambda(x + y)\chi_{R\setminus\Lambda}(x) K_{\Lambda - x}^{(n)}(0, y) \right| \\ &\quad \leqslant |y| (\sup_\sigma \sigma)^n \sum_{\sigma_1\sigma_2} \int dq_3 \cdots \int dq_n |\hat{\rho}_T^{(n)}(\sigma_1 0; \sigma_2 y; q_3, \ldots, q_n)| \end{aligned} \tag{44}$$

which is an integrable function of y by assumption.

Setting $\Lambda = [-a, a]$ and after the change of variable $x \to -x - a$, we have for $y \geqslant 0$ as a consequence of the $\mathscr{L}^1$ clustering

$$\begin{aligned} &\lim_{\Lambda \to R} \int dx\, \chi_\Lambda(x + y)\chi_{R\setminus\Lambda}(x) K_{\Lambda - x}^{(n)}(0, y) \\ &\quad = \lim_{a \to \infty} \int_0^y dx\, K_{[x, 2a + x]}^{(n)}(0, y) \\ &\quad = \int_0^y dx\, K_{[x, \infty]}^{(n)}(0, y) \end{aligned} \tag{45}$$

Similarly we obtain for $y \leqslant 0$ $(x \to -x + a)$

$$\lim_{\Lambda \to R} \int dx\, \chi_\Lambda(x + y)\chi_{R\setminus\Lambda}(x) K_{\Lambda - x}^{(n)}(0, y) = \int_y^0 dx\, K_{[-\infty, x]}^{(n)}(0, y) \tag{46}$$

From (43)–(46) we conclude by dominated convergence that all cumulants $M_n(\bar{Q}_\Lambda)$ have a limit as $\Lambda \to R$. So do all moments of the charge distribution, and therefore $P_{\bar{Q}_\Lambda}(s)$ converges weakly to a limit distribution $P(s)$.[3]

Remark 1. In Propositions 4 and 5 we have not made any assumption on the behavior of the average charge $\langle Q_\Lambda \rangle$ as $\Lambda \to R^\nu$. If $\lim_{\Lambda \to R^\nu}(\langle Q_\Lambda \rangle / |\partial\Lambda|^{1/2})$ converges as $\Lambda \to R^\nu$, we get immediately the following corollary:

Corollary 1. Assume that $\lim_{\Lambda \to R^\nu}(\langle Q_\Lambda \rangle / |\partial\Lambda|^{1/2}) = c < \infty$.

(i) When $\nu = 2, 3$, under the assumptions of Proposition 4,

$$\lim_{\Lambda \to R^\nu} P\left(\frac{\langle Q_\Lambda \rangle}{|\partial\Lambda|^{1/2}}, s \right) = (2\pi d)^{-1/2} \exp\left[\frac{-(s - c)^2}{2d} \right]$$

[3] We assume here that the set of limiting moments defines a unique probability distribution.

(ii) When $\nu = 1$, under the assumptions of Proposition 5, $\lim_{\Lambda \to R^\nu} P(Q_\Lambda, s)$ exists.

Remark 2. It is interesting to notice that in dimension $\nu = 3$ (resp. $\nu = 2$) one uses only the first two sum rules (resp. the first three sum rules) to obtain the convergences of the probability distribution, whereas for $\nu = 1$ all sum rules are needed. One gets a slightly stronger result if one knows that the state is invariant under charge conjugation.

A state is invariant under charge conjugation if both σ and $-\sigma$ belong to Σ and

$$\rho^{(n)}_{\sigma_1 \cdots \sigma_n}(x_1,\ldots, x_n) = \rho^{(n)}_{-\sigma_1 \cdots -\sigma_n}(x_1,\ldots, x_n)$$

Corollary 2. If the state is invariant under charge conjugation, assumptions (b) and (cl) are not needed in Proposition 4.

Proof. Indeed (b) (for $\nu = 3$) and (cl) (for $\nu = 2$) are used to show that $\lim_{\Lambda \to R^\nu} M_3(\bar{Q}_\Lambda) = 0$. But if the state is invariant under charge conjugation,

$$\langle Q_\Lambda{}^n \rangle = 0 \qquad \text{and} \qquad M_n(\bar{Q}_\Lambda) = 0 \qquad \text{for } n \text{ odd.} \quad \blacksquare$$

We see that in dimension $\nu = 3$ and for a state invariant under charge conjugation, we can conclude that the asymptotic charge distribution is Gaussian when we know that we have the $\mathscr{L}^1$ clustering (a) and that the normalized second-order moment converges to a nonzero limit (i.e., the situation of Proposition 2 with $d > 0$).

Remark 3. One-dimensional equilibrium Coulomb systems are known to be neutral and to have exponential clustering.[2,4] Therefore Proposition 5 and Corollary 1(ii) (with $c = 0$) apply and we conclude that such systems have a limiting charge distribution. Since the range of values of Q_Λ is discrete, the limiting distribution is also discrete. An explicit example of such a distribution will be given in the next section. Proposition 4 and Corollary 1(i) (with $c = 0$) will apply to equilibrium states of Coulomb systems in dimensions $\nu = 2$ and 3 as soon as screening occurs, i.e., if all clustering properties needed for the validity of Proposition 4 and of the sum rules (Proposition 8 of Ref. 1) hold true.

To conclude this section, we discuss the correlations of the charge with the local observables. Since the charge Q_Λ is a macroscopic observable, we expect that the probability distribution of Q_Λ is decorrelated from that of local observables as $\Lambda \to R^\nu$.

Proposition 6. Assume that $\int |\rho_T^{(r+1)}(q_1,\ldots, q_r, q)| \, dq < \infty$, $r = 1,\ldots, n$. The first n sum rules hold if and only if

$$\lim_{\Lambda \to R^\nu} (\langle F_1 \cdots F_r Q_\Lambda \rangle - \langle F_1 \cdots F_r \rangle \langle Q_\Lambda \rangle) = 0$$

for all local observables F_r of the form (27) and $r = 1, 2,\ldots, n$.

Proof. We have

$$
\begin{aligned}
0 &= \lim_{\Lambda \to R^\nu} (\langle F_1 \cdots F_r Q_\Lambda \rangle - \langle F_1 \cdots F_r \rangle \langle Q_\Lambda \rangle) \\
&= \lim_{\Lambda \to R^\nu} \int dq_1 \cdots dq_r f_1(q_1) \cdots f_r(q_r) \\
&\quad \times \int_\Lambda dq\, \sigma[\hat\rho(q_1,\ldots,q_r q) - \hat\rho(q_1,\ldots,q_r)\rho(q)] \\
&= \int dq_1 \cdots dq_r f_1(q_1) \cdots f_r(q_r) \\
&\quad \times \int dq\, \sigma[\hat\rho(q_1,\ldots,q_r q) - \hat\rho(q_1,\ldots,q_r)\rho(q)]
\end{aligned} \tag{47}
$$

because $\hat\rho(q_1,\ldots,q_r q) - \hat\rho(q_1,\ldots,q_r)\rho(q)$ is integrable by the $\mathscr{L}^1$-clustering assumption. Since (47) is true for all choices of the $f_j(q_j)$, the sum rules (33) hold for $r = 1,\ldots, n$. ■

For the higher moments of the charge distribution we have the following:

Proposition 7. Assume that $\int |\rho_T^{(n)}(q_1, q_2,\ldots, q_n)|\, dq_2 \cdots dq_n < \infty, n \geqslant 2$, and let F_j, $j = 1,\ldots, r$, be local observables; then:

(i) The following holds:

$$
\langle F_1 \cdots F_r Q_\Lambda{}^n \rangle - \langle F_1 \cdots F_r \rangle \langle Q_\Lambda{}^n \rangle = O\Big(\sup_{1 \leqslant k \leqslant n-1} |\langle Q_\Lambda{}^k \rangle|\Big) \tag{48}
$$

(ii) The sum rules hold if and only if

$$
\langle F_1 \cdots F_r Q_\Lambda{}^n \rangle - \langle F_1 \cdots F_r \rangle \langle Q_\Lambda{}^n \rangle = o\Big(\sup_{1 \leqslant k \leqslant n-1} |\langle Q_\Lambda{}^k \rangle|\Big), \qquad r = 1, 2, \ldots, n \tag{49}
$$

Proof. We have

$$
\begin{aligned}
&\langle F_1 \cdots F_r Q_\Lambda{}^n \rangle - \langle F_1 \cdots F_r \rangle \langle Q_\Lambda{}^n \rangle \\
&\quad = \int dq_1 \cdots dq_r f_1(q_1) \cdots f_r(q_r) \int_\Lambda d\bar q_1 \cdots \int_\Lambda d\bar q_n\, \bar\sigma_1 \cdots \bar\sigma_n \\
&\qquad \times [\hat\rho(q_1,\ldots, q_r, \bar q_1,\ldots, \bar q_n) - \hat\rho(q_1,\ldots, q_r)\hat\rho(\bar q_1,\ldots, \bar q_n)]
\end{aligned} \tag{50}
$$

Abbreviating $Q = q_1,\ldots, q_r$ and $\bar Q = \bar q_1,\ldots, \bar q_n$, it follows from the definition of the truncated functions that the integrand of (50) can be written as

$$
\hat\rho(Q\bar Q) - \hat\rho(Q)\hat\rho(\bar Q) = \sum_{k=1}^{n=1} \sum_{\substack{U \subset \bar Q \\ |U| = k}} R(Q\bar Q \backslash U)\hat\rho(U) \tag{51}
$$

where

$$
R(Q\bar Q\backslash U) = R(q_1,\ldots, q_r, \bar q_{j_1},\ldots, \bar q_{j_{n-k}}), \qquad \bar q_{j_s} \in U
$$

is a product of truncated functions $\hat\rho_T$ where the arguments $\bar q_{j_s}$, $s = 1,\ldots, n - k$, appear always in conjunction with at least one argument $q_i \in Q$.

Therefore the integrals

$$\int dq_1 \cdots dq_r f_1(q_1) \cdots f_r(q_r) \times \int_\Lambda d\bar{q}_{j_1} \cdots \int_\Lambda d\bar{q}_{j_{n-k}} \bar{\sigma}_{j_1} \cdots \bar{\sigma}_{j_{n-k}} R(q_1,\ldots, q_r, \bar{q}_{j_1},\ldots, \bar{q}_{j_{n-k}}) \tag{52}$$

are bounded uniformly with respect to Λ by the $\mathscr{L}^1$-clustering. From this, formula (51), and the fact that

$$\int_\Lambda du_1 \cdots \int_\Lambda du_k\, \tau_1 \cdots \tau_k \hat{\rho}(u_1,\ldots, u_k) = \langle Q_\Lambda{}^k \rangle$$

we get the estimate (48).

If in addition the sum rules (34) hold, the integral (52) tends to zero as $\Lambda \to R^\nu$ and we get (49).

Conversely, if (49) is true, the particular case $n = 1$ gives the sum rules by Proposition 6. ■

From Proposition 7, we obtain easily that in any dimension, the probability distribution for the charge decorrelates from that of local observables as $\Lambda \to R^\nu$.

Corollary. Let $P(F_1, s_1;\ldots; F_r, s_r; Q_\Lambda/|\partial\Lambda|^{1/2}, s)$ be the joint probability distribution for the local one-body observables $F_1,\ldots, F_r$ and the normalized charge $Q_\Lambda/|\partial\Lambda|^{1/2}$. Under the assumptions of Propositions 4 and 5 and if $\lim_{\Lambda\to R^\nu}(\langle Q_\Lambda\rangle/|\partial\Lambda|^{1/2})$ exists, then

$$\lim_{\Lambda\to R^\nu} P(F_1, s_1;\ldots; F_r, s_r; Q_\Lambda/|\partial\Lambda|^{1/2}, s) = P(F_1, s_1;\ldots; F_r, s_r) \lim_{\Lambda\to R^\nu} P(Q_\Lambda/|\partial\Lambda|^{1/2}, s)$$

Proof. We know that $(1/|\partial\Lambda|^{n/2})\langle Q_\Lambda{}^n\rangle$, $n = 1, 2,\ldots$, converges (Corollary 1 of Propositions 4 and 5). Therefore when $\nu = 3$ or 2,

$$\lim_{\Lambda\to R\nu} (1/|\partial\Lambda|^{n/2})\langle Q_\Lambda{}^k\rangle = 0,$$

$1 \leqslant k \leqslant n - 1$, and the result follows from Proposition 7(i). When $\nu = 1$, $\langle Q_\Lambda{}^n\rangle$ remains bounded as $\Lambda \to R^\nu$ and the result follows from Proposition 7(ii). ■

5. THE ONE-DIMENSIONAL COULOMB GAS

We illustrate the general theory by a solvable model, the one-dimensional Coulomb gas. We establish the convergence of the probability distribution of the charge by a direct computation and show the validity of the sum rules.

A general expression for the charge probability distribution $P(Q_\Lambda, s)$ can be given in terms of the family of density distributions of particles $\mu_\Lambda{}^n(q_1,\ldots, q_n)$

$$\mu_\Lambda{}^n(q_1,\ldots, q_n) = \frac{1}{n!}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\int_\Lambda d\bar{q}_1 \cdots \int_\Lambda d\bar{q}_p\, \rho^{(n+p)}(q_1,\ldots, q_n; \bar{q}_1,\ldots, \bar{q}_p) \tag{53}$$

$\mu_\Lambda{}^n(q_1 \cdots q_n)$ is the probability density for finding exactly n particles in Λ located at $x_1, \ldots, x_n$ with charges $\sigma_1, \ldots, \sigma_n$.[5] Then one has

$$P(Q_\Lambda, s) = \sum_{n=0}^{\infty}\left[\sum_{\sigma_1\cdots\sigma_n}\delta_{\sum_{j=1}^n \sigma_j, s}\int_\Lambda dx_1 \cdots \int_\Lambda dx_n\, \mu_\Lambda{}^n(\sigma_1 x_1,\ldots, \sigma_n x_n)\right] \tag{54}$$

We calculate $P(Q_\Lambda, s)$ for a two-component, one-dimensional Coulomb gas of particles of charge ± 1, using the method of functional integration introduced in Ref. 2, which we summarize briefly here (for details see Ref. 2).

The Hamiltonian of the neutral Coulomb gas of n particles is

$$\begin{aligned} H(q_1,\ldots, q_n) &= -e^2 \sum_{k<l}^{n} \sigma_k\sigma_l |x_k - x_l|, \qquad \sigma_j = \pm 1, \quad \sum_{j=1}^{n}\sigma_j = 0 \\ &= e^2 \sum_{k,l} \sigma_k\sigma_l \min(x_k, x_l) \end{aligned} \tag{55}$$

The main observation is that $H(q_1,\ldots, q_n)$ is the covariance of the Wiener integral. More precisely, one considers the space of Brownian paths $\varphi(x)$ and the family of joint probability distributions $R(\varphi_1 x_1,\ldots, \varphi_n x_n)$ for paths starting in φ_0 at $x = x_0 = 0$ to be found in $d\varphi_1 \cdots d\varphi_n$ around $\varphi_1,\ldots, \varphi_n$ at $x_1,\ldots, x_n$ averaged on initial values φ_0, $\varphi_0 \in [-\pi, \pi]$:

$$R(\varphi_1 x_1,\ldots, \varphi_n x_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi_0 \prod_{k=1}^{n} R(\varphi_k - \varphi_{k-1}, x_k - x_{k-1}) \tag{56}$$

$$R(\varphi, x) = \frac{1}{(4\pi\beta e^2 x)^{1/2}} \exp\left(\frac{-\varphi^2}{4\beta e^2 x}\right)$$

Then the Boltzmann factor is represented as

$$\exp[-\beta H(q_1,\ldots, q_n)] = \langle \exp[i\sigma_1\varphi(x_1)] \cdots \exp[i\sigma_n\varphi(x_n)]\rangle \tag{57}$$

where $\langle\cdots\rangle$ denotes expectations with respect to the probability measure defined by (56). All statistical mechanical quantities can be expressed as functional integrals of the form $\langle \exp[\int_0^x F(\varphi(y), y)\, dy]\rangle$, which can be evaluated by the Wiener–Kac formula

$$\left\langle \exp\left[\int_0^x F(\varphi(y), y)\, dy\right]\right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi_0 \int_{-\infty}^{+\infty} d\varphi\, U_x(\varphi, \varphi_0) \tag{58}$$

$U_x(\varphi, \varphi_0)$ is the kernel of the operator U_x solution of

$$dU_x/dx = \Gamma(x)U_x, \qquad U_{x=0} = I$$

where $\Gamma(x)$ is the differential operator

$$\Gamma(x) = \beta e^2\, d^2/d\varphi^2 + F(\varphi, x) \tag{59}$$

acting in $\mathscr{L}^2(R, d\varphi)$.

In the application, $F(\varphi, x)$ will be periodic of period 2π, and we can replace $U_x(\varphi, \varphi_0)$ in (58) by the periodic kernel

$$\hat{U}_x(\varphi, \varphi_0) = \sum_{n=-\infty}^{\infty} U_x(\varphi + 2\pi n, \varphi_0) \tag{60}$$

solution of

$$d\hat{U}_x/dx = \hat{\Gamma}(x)\hat{U}_x \tag{61}$$

where $\hat{\Gamma}(x)$ is now the differential operator (59) acting on $\mathscr{L}^2([-\pi, \pi], d\varphi)$ with periodic boundary conditions. With (60) we get finally

$$\left\langle \exp\left[\int_0^x F(\varphi(y), y)\, dy\right]\right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi_0 \int_{-\pi}^{\pi} d\varphi\, \hat{U}_x(\varphi, \varphi_0) \tag{62}$$

From (57) one finds easily that the grand canonical partition function $Z(L)$ for the two-component Coulomb gas in a finite interval $[0, L]$ is

$$\begin{aligned} Z(L) &= \sum_{n=0}^{\infty} \frac{z^n}{n!}\int_0^L dx_1 \cdots \int_0^L dx_n \sum_{\sigma_1\cdots\sigma_n} \exp[-\beta H(q_1,\dots, q_n)] \\ &= \left\langle \exp\left[\int_0^L 2z\cos\varphi(y)\, dy\right]\right\rangle \end{aligned} \tag{63}$$

and the corresponding finite-volume correlation functions are given by

$$\begin{aligned} \rho_L(q_1,\dots, q_n) = \frac{z^n}{Z(L)} \Big\langle &\exp[i\sigma_1\varphi(x_1)] \cdots \exp[i\sigma_n\varphi(x_n)] \\ &\times \exp\left[\int_0^L 2z\cos\varphi(y)\, dy\right]\Big\rangle \end{aligned} \tag{64}$$

From (64) and (53) we get the density distributions

$$\begin{aligned} \mu^n_{\Lambda,L}(q_1,\dots, q_n) = \frac{1}{n!}\frac{z^n}{Z(L)} \Big\langle &\exp[i\sigma_1\varphi(x_1)] \cdots \exp[i\sigma_n\varphi(x_n)] \\ &\times \exp\left[\int_0^L F_\Lambda(\varphi(y), y)\, dy\right]\Big\rangle \end{aligned} \tag{65}$$

with

$$F_\Lambda(\varphi, y) = \begin{cases} 2z\cos\varphi, & y \notin \Lambda \qquad \Lambda \subset [0, L] \\ 0, & y \in \Lambda \end{cases} \tag{66}$$

Finally, writing

$$\delta_{\sum\sigma_j,s} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha \exp\left[i\left(\sum_{j=1}^{n}\sigma_j - s\right)\alpha\right]$$

in (54), we have from (65) that the characteristic function $G_{\Lambda,L}(\alpha)$ of the charge probability distribution (in the finite system) is given by the following simple functional integral:

$$G_{\Lambda,L}(\alpha) = \frac{1}{Z(L)}\left\langle \exp\left[\int_0^L F_\Lambda{}^\alpha(\varphi(y), y)\, dy\right]\right\rangle \tag{67}$$

with

$$F_\Lambda{}^\alpha(\varphi, y) = \begin{cases} 2z\cos\varphi, & y \notin \Lambda \\ 2z\cos(\varphi + \alpha), & y \in \Lambda \end{cases}$$

$G_{\Lambda,L}(\alpha)$ can now be computed with the help of (62), i.e.,

$$G_{\Lambda,L}(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi_0 \int_{-\pi}^{\pi} d\varphi\, \hat{U}_L{}^\alpha(\varphi, \varphi_0) \Big/ \frac{1}{2\pi}\int_{-\pi}^{\pi} d\varphi_0 \int_{-\pi}^{\pi} d\varphi\, \hat{U}_L(\varphi, \varphi_0) \tag{68}$$

with

$$\frac{d\hat{U}_x}{dx} = \hat{\Gamma}\hat{U}_x, \qquad \hat{\Gamma} = \beta e^2 \frac{d^2}{d\varphi^2} + 2z\cos\varphi, \qquad \hat{U}_x = \exp(\hat{\Gamma}_x) \tag{69}$$

and

$$\frac{d\hat{U}_x{}^\alpha}{dx} = \begin{cases} \hat{\Gamma}\hat{U}_x{}^\alpha, & x \notin \Lambda \\ \hat{\Gamma}_\alpha \hat{U}_x{}^\alpha, & x \in \Lambda \end{cases} \qquad \hat{\Gamma}_\alpha = \beta e^2 \frac{d^2}{d\varphi^2} + 2z\cos(\varphi + \alpha) \tag{70}$$

When $\Lambda = [a, b]$ is some interval contained in $[0, L]$ the solution of (70) is

$$\hat{U}_L{}^\alpha = \exp[\hat{\Gamma}(L - b)]\exp[\hat{\Gamma}_\alpha(b - a)]\exp(\hat{\Gamma}a) \tag{71}$$

One knows that $\hat{\Gamma}$ defined in (69) has a maximal nondegenerate eigenvalue γ_0 with eigenvector $|Y\rangle$ (actually $Y(\varphi) \in \mathscr{L}^2([-\pi, \pi], d\varphi)$ is the fundamental solution of the Mathieu differential equation). Hence as $L \to \infty$, $\hat{U}_L$ behaves as

$$\hat{U}_L = e^{\gamma_0 L}|Y\rangle\langle Y| + o(e^{\gamma_0 L}) \tag{72}$$

With this we obtain from (68) and (71) $G_{\Lambda,L}(\alpha)$ in the thermodynamic limit (letting $L - b \to \infty$, $a \to \infty$, $b - a$ fixed)

$$G_\Lambda(\alpha) = \lim_{L\to\infty} G_{\Lambda,L}(\alpha) = \{\exp[-\gamma_0(b - a)]\}\langle Y|\exp[\hat{\Gamma}_\alpha(b - a)]|Y\rangle \tag{73}$$

and the charge probability distribution in the region Λ of the infinite system is

$$P(Q_\Lambda, s) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha\, G_\Lambda(\alpha) e^{-i\alpha s} \tag{74}$$

Proposition 8. In an equilibrium state of the one-dimensional, two-component Coulomb gas (as described above), we have the following:

(i) The charge probability distribution converges:

$$\lim_{\Lambda \to R} P(Q_\Lambda, s) = P(s) = \sum_{n=-\infty}^{\infty} c_{n+s}^2 c_n^2, \quad s \text{ integer} \tag{75}$$

(the c_n are the Fourier coefficients of the fundamental solution of the Mathieu equation).

(ii) The canonical sum rules hold true.

Proof. (i) $\hat{\Gamma}$ and $\hat{\Gamma}_\alpha$ [defined by (69) and (70)] are unitarily equivalent by the translation operator $\exp(ip\alpha)$, $p = -i\, d/d\varphi$. Therefore we can write $G_\Lambda(\alpha)$, (73), in the form

$$G_\Lambda(\alpha) = e^{-\gamma_0(b-a)} \langle Y_\alpha | \hat{U}_{b-a} | Y_\alpha \rangle \tag{76}$$

where $Y_\alpha(\varphi) = Y(\varphi - \alpha)$ is the translate of the fundamental eigenfunction $Y(\varphi)$. With this and (72) we have

$$\lim_{\Lambda = b-a \to \infty} G_\Lambda(\alpha) = |\langle Y_\alpha | Y \rangle|^2 \tag{77}$$

The pointwise convergence of the characteristic function implies the convergence of the probability distribution

$$\lim_{\Lambda \to \infty} P(Q_\Lambda, s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\alpha s} |\langle Y_\alpha | Y \rangle|^2 \, d\alpha \tag{78}$$

We obtain (75) from (78) and the Fourier series $Y(\varphi) = (1/\sqrt{\pi}) \sum_{n=-\infty}^{\infty} c_n e^{in\varphi}$ of $Y(\varphi)$.

(ii) We show that the joint probability for the charge and local observables factorizes as $\Lambda \to R$. Then the validity of the sum rules follows from Proposition 6. Let $F_u = \sum_j f_u(q_j)$, $u = 1, \ldots, r$, be local observables with $f_u(\sigma, x)$ continuous functions of compact support. The joint probability distribution for $F_1, \ldots, F_r$ and the charge Q_Λ is given by the generalization of (54),

$$\begin{aligned} &P(F_1, s_1, \ldots, F_r, s_r; Q_\Lambda, s) \\ &\quad = \sum_n \int_\Lambda dx_1 \cdots \int_\Lambda dx_n \sum_{\sigma_1 \cdots \sigma_n} \prod_{u=1}^{r} \delta\Big(\sum_j^n f_u(q_j) - s_u\Big) \\ &\quad \times \delta_{\sum_{j=1}^n \sigma_j, s} \mu_\Lambda{}^n(\sigma_1 x_1, \ldots, \sigma_n x_n) \end{aligned}$$

Let $[a_1, b_1]$ be an interval containing the union of all supports of the $f_u(\sigma, x)$ and $\Lambda = [a, b] \supset [a_1, b_1]$. The calculation of $P(F_1, s_1, \ldots, F_r, s_r; Q_\Lambda, s)$ can be carried out along the same lines as that of $P(Q_\Lambda, s)$ and we leave it to the reader.

The corresponding characteristic function (after the thermodynamic limit) is found to be

$$G_\Lambda(\alpha_1,\ldots,\alpha_r,\alpha) = e^{-\gamma_0(b-a)}\langle Y_\alpha|\hat{U}_{b-b_1}W_{\alpha_1\cdots\alpha_r}(b_1,a_1)\hat{U}_{a_1-a}|Y_\alpha\rangle \tag{79}$$

$W(x,y)$ is the solution of

$$\frac{d}{dx}W_{\alpha_1\cdots\alpha_r}(x,y) = \Gamma_{\alpha_1\cdots\alpha_r}(x)W_{\alpha_1\cdots\alpha_r}(x,y), \qquad W_{\alpha_1\cdots\alpha_r}(x,y)\Big|_{x=y} = I$$

$$\Gamma_{\alpha_1\cdots\alpha_r}(x) = \beta e^2\frac{d^2}{d\varphi^2} + z\left\{\exp i\left[\varphi + \sum_{u=1}^{r}\alpha_u f_u(+1,x)\right]\right.$$
$$\left. + \exp i\left[-\varphi + \sum_{u=1}^{r}\alpha_u f_u(-1,x)\right]\right\}$$

From (72) again, we find that $G_\Lambda(\alpha_1,\ldots,\alpha_r,\alpha)$ factorizes as $\Lambda\to\infty$

$$\lim_{b-a\to\infty} G_\Lambda(\alpha_1,\ldots,\alpha_r,\alpha) = |\langle Y_\alpha|Y\rangle|^2 G(\alpha_1,\ldots,\alpha_r)$$

where

$$G(\alpha_1,\ldots,\alpha_r) = e^{-\gamma_0(b_1-a_1)}\langle Y|W_{\alpha_1\cdots\alpha_r}(b_1,a_1)|Y\rangle$$

is the characteristic function of the joint probability of $F_1,\ldots,F_r$ and $|\langle Y_\alpha|Y\rangle|^2$ is the characteristic function (77) of the limiting charge distribution. We deduce also by an analysis of the structure of (79) that $G_\Lambda(\alpha_1,\ldots,\alpha_r,\alpha)$ is differentiable and that its derivatives with respect to $\alpha_1,\ldots,\alpha_r$ and α converge to the corresponding factorized quantities, which is equivalent to the asymptotic factorization of the moments. Then the sum rules follow from Proposition 6. ■

Remarks. 1. The result of Proposition 8 could be deduced from the fact that the set of correlation functions of the one-dimensional Coulomb gas satisfies the (generalized) BBGKY hierarchy[7] and that the clustering is exponentially fast. Then the validity of the sum rules follows from Proposition 8 of Ref. 1 and the convergence of the probability distribution of the charge from Proposition 5 of this paper.

2. We have the same results for the class of states of the one-dimensional Coulomb gas with different boundary charges constructed in Ref. 4 (θ-states). This can be shown directly as in Proposition 8, or deduced from the equilibrium equations and the exponential clustering as mentioned in the preceding remark.

3. It is easy to check that the fluctuations $\langle(N_\Lambda - \langle N_\Lambda\rangle)^2\rangle$ of the total particle number are extensive, with $N_\Lambda = N_\Lambda^+ + N_\Lambda^-$, N_Λ^+ and N_Λ^- being the numbers of particles of charge plus and minus in Λ. The fluctuations of

N_Λ^+ and N_Λ^- are also extensive since those of $Q_\Lambda = N_\Lambda^+ - N_\Lambda^-$ are $O(1)$. Therefore in a large interval Λ the fluctuations of N_Λ^+ and N_Λ^- are large, but strongly correlated to produce finite fluctuations for Q_Λ.

APPENDIX

Proof of Lemma 1

(α_1) Clearly $|\Lambda \cap (R^\nu \backslash \Lambda - y)| \leqslant |\Lambda^{|y|}|$, where $\Lambda^{|y|}$ is the set of points that are within a distance less than or equal to $|y|$ from the boundary of $|\partial\Lambda|$. Thus

$$\lim_{\Lambda\to R^\nu} \frac{1}{|\Lambda|} \gamma_\Lambda(y) \leqslant \lim_{\Lambda\to R^\nu} \frac{|\Lambda^{|y|}|}{|\Lambda|} = 0$$

by the van Hove property.

(β_1) For any $|\Lambda|$ with $|\Lambda| \geqslant |\Lambda^d|$ we have from (4)

$$|\Lambda| - |\Lambda^d| \leqslant |\Omega|[\Lambda] \leqslant |\Lambda| + |\Lambda^d|$$

Now $|(\Lambda \cap (R^\nu\backslash\Lambda - y))^d| \leqslant 2|\Lambda^d|$, and therefore

$$\frac{1}{|\Omega|[\Lambda_k]} \bar{\gamma}_{\Lambda_k}(y) \leqslant \frac{\gamma_{\Lambda_k}(y) + 2|\Lambda^d|}{|\Lambda_k| - |\Lambda^d|}$$

tends to zero as $|\Lambda_k| \to \infty$ as in (α_1).

(α_1) We have the scaling property $\gamma_{\lambda\Lambda}(\lambda y) = \lambda^\nu \gamma_\Lambda(y)$ or equivalently $\gamma_{\lambda\Lambda}(y) = \lambda^\nu\gamma_\Lambda(\lambda^{-1}y)$. Thus, setting $\lambda = (1/\epsilon)|y|$, y fixed, and $\Lambda = \lambda\Lambda_0$, $|\partial\Lambda| = \lambda^{\nu-1}|\partial\Lambda_0|$, we get

$$\lim_{\Lambda\to R^\nu} \frac{1}{|\partial\Lambda|} \gamma_\Lambda(y) = \lim_{\lambda\to\infty} \frac{\lambda\gamma_{\Lambda_0}(\lambda^{-1}y)}{|\partial\Lambda_0|}$$

$$= \frac{|y|}{|\partial\Lambda_0|} \lim_{\epsilon\to 0} \frac{1}{\epsilon} \gamma_{\Lambda_0}(\epsilon\hat{y}), \qquad \hat{y} = \frac{y}{|y|} \tag{A1}$$

One has (see Fig. 1)

$$\gamma_{\Lambda_0}(\epsilon\hat{y}) = \epsilon \int_{\partial(\Lambda_0 - \epsilon\hat{y})\cap\Lambda_0} |\hat{y}\cdot ds| + o(\epsilon) \tag{A2}$$

With (A2) and by the symmetry $\hat{y} \to -\hat{y}$, we obtain

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \gamma_\Lambda(\epsilon\hat{y}) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} [\gamma_\Lambda(\epsilon\hat{y}) + \gamma_\Lambda(-\epsilon\hat{y})] = \frac{1}{2}\int_{\partial\Lambda_0} |\hat{y}\cdot ds| \tag{A3}$$

(A3) and (A1) give the result. Moreover, (A2) shows that $\gamma_{\Lambda_0}(\epsilon\hat{y}) \leqslant \epsilon|\partial\Lambda_0| + o(\epsilon)$ and thus $(1/|\partial\Lambda|)\gamma_\Lambda(y) \leqslant |y| + o(1)$ uniformly with respect to Λ.

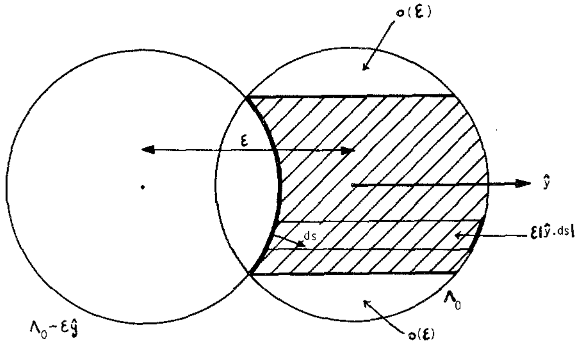

Fig. 1

(β_2) For the sequence of regions Λ_k defined in (6) we obtain by counting the number of cells (see Fig. 2)

$$\bar{\gamma}_{\Lambda_k}(y) = |\Omega|\{[|y_1|](2k)^{\nu-1} + \cdots + [|y_\nu|](2k)^{\nu-1} + O(k^{\nu-2})\} \qquad \text{(A4)}$$

and by dilatation

$$|\partial\Lambda_k| = k^{\nu-1}|\partial\tilde{\Omega}| = (2k)^{\nu-1}|\partial\Omega| \qquad \text{(A5)}$$

(A4) and (A5) give the result (16) and show that $(1/|\partial\Lambda_k|)\bar{\gamma}_{\Lambda_k}(y)$ is bounded by constant $|y|$ uniformly with respect to Λ_k. ■

Proof of Lemma 2. The equivalence between (i) and (ii) is proved in the Appendix of Ref. 1. The same proof, involving only the relation between the correlations and the truncated correlation functions, also establishes the equivalence between (iii) and (iv). We will obtain a proof of the lemma if we show that (ii) and (iv) are equivalent. We can write (30) in the form

$$\hat{\rho}_T(Qq) = \sum_{\mathscr{P}(Q)} \prod_j \Delta(Q_j^{\mathscr{P}})\rho_T(Q^{\mathscr{P}}q) + \sum_{\bar{\mathscr{P}}(Qq)} \prod_j \Delta(Q_j^{\bar{\mathscr{P}}})\rho_T[(Qq)^{\bar{\mathscr{P}}}] \qquad \text{(A6)}$$

where in the first sum $\mathscr{P}$ runs on all partitions of Q and in the second sum $\bar{\mathscr{P}}$

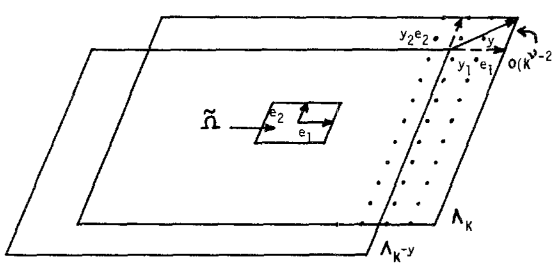

Fig. 2

runs on all partitions of Qq such that q occurs always in conjunction with some other argument $q_i \in Q$. We get from (A6)

$$\int dq\, \sigma\hat{\rho}_T(Qq) = \sum_{\mathscr{P}(Q)} \prod_j \Delta(Q_j^{\mathscr{P}}) \int dq\, \sigma\rho_T(Q^{\mathscr{P}}q) + \sum_{\bar{\mathscr{P}}(Qq)} \int dq\, \sigma \prod_j \Delta(Q_j^{\bar{\mathscr{P}}})\rho_T[(Qq)^{\bar{\mathscr{P}}}] \tag{A7}$$

In each term of the second sum of (A7), q appears certainly in a Dirac function δ_{q,q_i} with some $q_i \in Q$, and therefore the integral can be performed immediately.

Since the same partition $Q_1^{\mathscr{P}},..., Q_j^{\mathscr{P}},..., Q_k^{\mathscr{P}}$ of Q arises from the k partitions $\bar{\mathscr{P}}$ of Qq of the form $Q_1^{\mathscr{P}},..., Q_j^{\mathscr{P}}q,..., Q_k^{\mathscr{P}}$ where q occurs in conjunction successively with $Q_j^{\mathscr{P}}, j = 1,..., k$, we get

$$\begin{aligned}\int dq\, \sigma\hat{\rho}_T(Qq) &= \sum_{\mathscr{P}(Q)} \prod_j \Delta(Q_j^{\mathscr{P}})\left[\int dq\, \sigma\rho_T(Q^{\mathscr{P}}q) + \left(\sum_j \sigma_j^{\mathscr{P}}\right)\rho_T(Q^{\mathscr{P}})\right] \\ &= \int dq\, \sigma\rho_T(Qq) + \left(\sum_{j=1}^{n} \sigma_j\right)\rho_T(Q) \\ &\quad + \sum_{\mathscr{P}_1(Q)} \prod_j \Delta(Q_j^{\mathscr{P}_1})\left[\int dq\, \sigma\rho_T(Q^{\mathscr{P}_1}q) + \left(\sum_j \sigma_j^{\mathscr{P}_1}\right)\rho_T(Q^{\mathscr{P}_1})\right]\end{aligned} \tag{A8}$$

$\sum_j \sigma_j^{\mathscr{P}}$ is the total charge of the particles with coordinates belonging to $Q^{\mathscr{P}}$ and the sum in the second term of (A8) runs on all partitions $\mathscr{P}_1$ of Q into at most $n - 1$ subsets. Since $|Q^{\mathscr{P}_1}| \leqslant n - 1$, it is clear from (A8) that (ii) implies (iv). Conversely, if (ii) holds for $|Q| \leqslant n - 1$, then (iv) implies (ii) for $|Q| = n$. The equivalence of (ii) and (iv) being trivially seen in the case $n = 1$, the proof of the lemma is completed. ∎

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