Asymptotic Time Behavior of Correlation Functions. II. Kinetic and Potential Terms

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On the basis of the mode-coupling theory we obtain the long-time behavior $\sim t^{-d/2}$ for the kinetic, potential, and cross-terms in the Green–Kubo integrands, expressed completely in terms of transport coefficients and thermodynamic quantities. All two-mode amplitudes are explicitly evaluated in terms of measurable quantities such as specific heats, thermal expansion coefficients, etc.

1. INTRODUCTION

We propose to analyze the long-time behavior[1] of equilibrium current-current correlation functions by means of the mode-coupling theory, a brief account of which has been given in a previous publication.[2] The results derived here are, besides those in Ref. 2, also presented by Pomeau[3] and Kawasaki.[4] Our results agree by and large with the findings of these authors; where they differ, we believe ours to be correct. The reason we return to these results is that no coherent and complete derivation has been published. Moreover, we believe to have simplified the computations considerably and thus to have added to the transparency of the fairly involved amplitudes for the long-time tails. In paper I[4] of this series we have analyzed the kinetic parts of the correlation functions. A systematic extension of the basic ideas of paper I to more general currents leads to the so-called mode-coupling

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formula as a valid description for the dominant long-time behavior of the current–current correlation functions. Since the mode-coupling theories are of general importance beyond the specific goal of these papers, a separate paper is devoted to this subject.[5] The present paper will be of a more technical nature, in the sense that we calculate the long-time behavior of the correlation functions and evaluate the corresponding coefficients, starting from the assumption that the mode-coupling formula adequately describes these quantities for long times.

The correlation functions considered are of the form

$$C_i(t) = \lim_{V \to \infty} V^{-1} \langle J_i(0) J_i(t) \rangle$$

and the time integrals of the functions $C_i(t)$ are the Green–Kubo expressions for the transport coefficients. The currents J_i are the so-called projected or subtracted currents, which are related to the microscopic energy and momentum currents. In general, these currents consist of purely kinetic contributions J_i^K as well as potential contributions J_i^Φ, which involve the intermolecular potential. In paper I we have discussed the correlation functions $C_i^{KK}(t)$, in which the currents are replaced by their kinetic parts, and we have shown that these functions behave as $t^{-d/2}$ for long times, where d is the dimensionality of the system. In this paper these results are extended to include potential contributions, for which the same long-time behavior will be obtained.

In the remaining part of this section we introduce the relevant quantities; Section 2 presents the mode-coupling formula, and Section 3 is devoted to the explicit evaluation of the dominant long-time contributions.

The Green–Kubo formulas express the transport coefficients as time integrals over the corresponding correlation functions, and we use normalizations such that

$$\lambda = (\beta/T) \int_0^\infty dt\, C_\lambda(t); \qquad D_l = (\beta/\rho) \int_0^\infty dt\, C_l(t)$$

$$\zeta = \beta \int_0^\infty dt\, C_\zeta(t); \qquad \eta = \beta \int_0^\infty dt\, C_\eta(t) \tag{1}$$

Here, λ is the heat conductivity, D_l is the longitudinal diffusivity, ζ is the bulk viscosity, η is the shear viscosity, and $T = (k_B\beta)^{-1}$ is the temperature; the mass density $\rho = mn$, where n is the equilibrium number density and m is the mass of a fluid particle. The longitudinal diffusivity and both viscosities are not independent transport coefficients, but are related[4] to each other as

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[4] The quantities η, ζ, and ρD_l are related to the components of the viscosity tensor $\eta_{\alpha\beta\gamma\delta}$, which is an isotropic tensor of rank four and where $(\alpha, \beta, \gamma, \delta)$ label the d Cartesian components $(x, y, z, \ldots)$. This can be seen by inspecting the expressions for the currents

$\rho D_l = 2d^{-1}(d-1)\eta + \zeta$. The current–current correlation functions are defined as

$$C_i(t) = \lim_{V \to \infty} V^{-1}\langle J_i(0)J_i(t)\rangle \tag{2}$$

Here, $\langle \cdots \rangle$ denotes an average over a grand canonical ensemble, characterized by the parameters β and $\nu = \beta\mu$, where μ is the chemical potential; V is the volume of the system, and the thermodynamic limit is taken as the final step. The label i takes the values λ, l, ζ, and η, and $J_i(t)$ is the projected current at time t with initial value $J_i(0)$, defined as

$$J_\lambda = J_x^e - hJ_x^n \tag{3a}$$

$$J_l = J_{xx} - pV - \left(\frac{\partial p}{\partial e}\right)_n (H - \langle H \rangle) - \left(\frac{\partial p}{\partial n}\right)_e (N - \langle N \rangle) \tag{3b}$$

$$J_\zeta = d^{-1}\sum_x J_l = d^{-1}J_{\alpha\alpha} - pV - \left(\frac{\partial p}{\partial e}\right)_n (H - \langle H \rangle) - \left(\frac{\partial p}{\partial n}\right)_e (N - \langle N \rangle) \tag{3c}$$

$$J_\eta = J_{xy} \tag{3d}$$

with

$$J_x^e = \sum_{i=1}^{N} e_i v_{ix} - \frac{1}{2}\sum_{i\neq j}^{N} r_{ij,x}\frac{\partial \Phi(r_{ij})}{\partial r_{ij,\alpha}} v_{i\alpha} \tag{4a}$$

$$J_x^n = \sum_{i=1}^{N} v_{ix} \tag{4b}$$

$$J_{\alpha\beta} = \sum_{i=1}^{N} m v_{i\alpha}v_{i\beta} - \frac{1}{2}\sum_{i\neq j}^{N} r_{ij,\alpha}\frac{\partial \Phi(r_{ij})}{\partial r_{ij,\beta}} \tag{4c}$$

Cartesian components in a d-dimensional system are denoted by the subscripts $(\alpha, \beta) = (x, y, z,...)$ and the summation convention is used for repeated Greek indices. The equilibrium enthalpy per particle is $h = (e + p)/n$, where

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J_η, J_ζ, and J_l. Such a tensor contains only two independent constants. If we choose them as η and ζ, we have

$$\eta_{\alpha\beta\gamma\delta} = \eta(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} - 2d^{-1}\delta_{\alpha\beta}\delta_{\gamma\delta}) + \zeta\delta_{\alpha\beta}\delta_{\gamma\delta}$$

From this relation it follows that

$$\eta = \eta_{xyxy} = [(d+2)(d-1)]^{-1}(\eta_{\alpha\beta\alpha\beta} - d^{-1}\eta_{\alpha\alpha\beta\beta})$$
$$\zeta = d^{-2}\eta_{\alpha\alpha\beta\beta}; \qquad \rho D_l = \eta_{xxxx} = 2d^{-1}(d-1)\eta + \zeta$$

where the summation convention is used for repeated Greek indices. For the correlation functions a similar relation holds, i.e.,

$$C_l(t) = 2d^{-1}(d-1)C_\eta(t) + C_\zeta(t)$$

e and p are, respectively, the energy density and pressure in thermal equilibrium. The Hamiltonian of the system of N identical fluid particles is given by

$$H = \sum_{i=1}^{N} \tfrac{1}{2}mv_i^2 + \tfrac{1}{2}\sum_{i \neq j}^{N}\sum \Phi(r_{ij}) \tag{5}$$

where v_i is the velocity of the ith particle and r_i is its position, with $r_{ij} = r_i - r_j$. The intermolecular potential is $\Phi(r)$. The microscopic energy e_i of particle i is defined as

$$e_i = \tfrac{1}{2}mv_i^2 + \tfrac{1}{2}\sum_{j(\neq i)} \Phi(r_{ij}) \tag{6}$$

The currents J_i can in general be separated into a kinetic part J_i^K and a potential part J_i^Φ, with

$$J_i = J_i^K + J_i^\Phi \tag{7}$$

where the kinetic projected currents are given by

$$J_\lambda^K = \sum_{i=1}^{N}\left(mv_i^2 - \frac{d+2}{2\beta}\right)v_{ix}; \qquad J_i^K = \sum_{i=1}^{N} m(v_{ix}^2 - d^{-1}v_i^2) \tag{8}$$

$$J_\zeta^K = 0; \qquad J_\eta^K = \sum_{i=1}^{N} mv_{ix}v_{iy}$$

The potential parts are given through (3), (7), and (8). The separation (7) of the currents induces a separation of the correlation functions

$$C_i(t) = C_i^{KK}(t) + 2C_i^{K\Phi}(t) + C_i^{\Phi\Phi}(t) \tag{9}$$

where

$$C_i^{AB}(t) = \lim_{V \to \infty} V^{-1}\langle J_i^A(0)J_i^B(t)\rangle \tag{10}$$

with A and B equal to K or Φ.

Although the concept of kinetic currents is clear, the subtracted terms in (8) may seem rather arbitrary, and therefore also the separation made in (7). The subtracted parts in (8) are in fact chosen such that the kinetic currents J_i^K satisfy the same set of orthogonality conditions (to be discussed below) as the full currents J_i; and so does J_i^Φ by virtue of (7). If one were to split J_i differently into a kinetic part and a potential part, the correlation functions $C_i^{AB}(t)$ would in general approach constants for long times, and the mode-coupling theory would not adequately describe the long-time behavior of $C_i^{AB}(t)$.

2. MODE-COUPLING FORMULA

We are interested in the long-time behavior of correlation functions. In paper I we have shown that the decay of the correlation functions for long times is basically governed by the decay of products of hydrodynamic fields, which are the macroscopic energy density, number density, and momentum density. The extension of this theory to include potential as well as kinetic parts of the currents J_i turns out to be equivalent to the mode-coupling theory. In this paper we will assume the validity of the mode-coupling theory as a description of the correlation functions for long times, and we show how the $t^{-d/2}$ tails can be obtained, as well as how the exact coefficients can be evaluated. The justification of this procedure is given elsewhere.[6]

Our *basic assumption* is therefore that for long times the dominant contributions to the correlation functions of projected currents are given by the mode-coupling formula,

$$C_i(t) \simeq \lim_{V \to \infty} V^{-1} \sum_{\mathbf{k}} \tfrac{1}{2} \sum_{\mu\nu} [(J_i, a_{\mathbf{k}}^{\mu} a_{-\mathbf{k}}^{\nu})]^2 \exp[(z_k^{\mu} + z_k^{\nu})t] \qquad (11)$$

The superscripts μ and ν label the $d + 2$ hydrodynamic modes H, $\sigma = \pm$, and ϵ_i $(i = 1,..., d - 1)$, where H indicates the heat mode, $\sigma = \pm$ labels the two sound modes, and ϵ_i $(i = 1,..., d - 1)$ labels $d - 1$ shear modes of the system. We have also introduced an inner product between two microscopic quantities $a_{\mathbf{k}}$ and $b_{\mathbf{q}}$ as

$$(a_{\mathbf{k}}, b_{\mathbf{q}}) = V^{-1}\langle a_{\mathbf{k}}^* b_{\mathbf{q}}\rangle \qquad (12)$$

where, due to translation invariance of the equilibrium state,

$$(a_{\mathbf{k}}, b_{\mathbf{q}}) = (a_{\mathbf{k}}, b_{\mathbf{k}}) \, \delta_{\mathbf{k},\mathbf{q}} \qquad (13)$$

Here $\delta_{\mathbf{k},\mathbf{q}}$ is a Kronecker delta function. In the thermodynamic limit the summation over the reciprocal lattice of wave vectors $\mathbf{k}$ may be replaced by an integration, which yields the mode-coupling formula, to be used later:

$$C_i(t) \simeq \frac{1}{2} \int \frac{d\mathbf{k}}{(2\pi)^d} \sum_{\mu\nu} [(J_i, a_{\mathbf{k}}^{\mu} a_{-\mathbf{k}}^{\nu})]^2 \exp[(z_k^{\mu} + z_k^{\nu})t] \qquad (14a)$$

or for its *AB* part

$$C_i^{AB}(t) \simeq \frac{1}{2} \int \frac{d\mathbf{k}}{(2\pi)^d} \sum_{\mu\nu} (J_i^A, a_{\mathbf{k}}^{\mu} a_{-\mathbf{k}}^{\nu})(a_{-\mathbf{k}}^{\nu} a_{\mathbf{k}}^{\mu}, J_i^B) \exp[(z_k^{\mu} + z_k^{\nu})t] \qquad (14b)$$

The quantities $a_{\mathbf{k}}^{\mu}$ are the hydrodynamic modes, which for small values of the wave number $\mathbf{k}$ are given as (orthonormal) linear combinations of (the Fourier components of) the deviations from their equilibrium values of the

microscopic energy density e_k, the number density n_k, and the momentum density g_k, given by

$$e_k = \sum_{i=1}^{N} e_i \exp(-i\mathbf{k}\cdot\mathbf{r}_i) - \delta_{k,0}\langle H\rangle$$

$$n_k = \sum_{i=1}^{N} \exp(-i\mathbf{k}\cdot\mathbf{r}_i) - \delta_{k,0}\langle N\rangle \qquad (15)$$

$$g_k = \sum_{i=1}^{N} m\mathbf{v}_i \exp(-i\mathbf{k}\cdot\mathbf{r}_i)$$

The hydrodynamic modes themselves are

$$a_k^{H} = (n/k_B C_p)^{1/2} s_k \qquad (16a)$$

$$a_k^{\sigma} = (\beta/2\rho)^{1/2}(c_0^{-1}p_k + \sigma\hat{\mathbf{k}}\cdot g_k) \qquad (16b)$$

$$a_k^{\epsilon_i} = (\beta/\rho)^{1/2}\hat{\mathbf{k}}_\perp^{i}\cdot g_k \qquad (16c)$$

where $C_p = T(\partial s/\partial T)_p$ is the specific heat per particle at constant pressure and s is the equilibrium entropy per particle. The adiabatic sound velocity $c_0 = [(\partial p/\partial\rho)_s]^{1/2}$. The unit vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}_\perp^{i}$ ($i = 1,..., d-1$) form a set of d mutually orthogonal vectors. The quantities s_k and p_k are referred to as the entropy fluctuation per particle and the pressure fluctuation, respectively, and for small $\mathbf{k}$ are defined by

$$s_k = (1/nT)(e_k - hn_k) \qquad (17a)$$

$$p_k = (\partial p/\partial e)_n e_k + (\partial p/\partial n)_e n_k \qquad (17b)$$

For small values of k the hydrodynamic modes are approximately orthonormal, i.e.,

$$(a_k^{\mu}, a_k^{\nu}) = \delta_{\mu\nu} \qquad (18)$$

where terms of order k^2 have been neglected. In order to verify (18), one needs certain fluctuation formulas which are discussed in the appendix.

The hydrodynamic eigenvalues z_k^{μ} in (14) for small values of k are given by

$$z_k^{H} = -D_T k^2 \qquad (19a)$$

$$z_k^{\sigma} = -i\sigma c_0 k - \tfrac{1}{2}\Gamma_s k^2 \qquad (19b)$$

$$z_k^{\epsilon_i} = -\nu k^2 \qquad (19c)$$

where $D_T = \lambda/nC_p$ is the thermal diffusivity, $\Gamma_s = (\gamma - 1)D_T + D_l$ is the sound damping constant with $\gamma = C_p/C_v$, and $\nu = \eta/\rho$ is the kinematic viscosity.

We note that the mode-coupling formula may only be applied to

correlation functions (2) and (10), in which the currents are orthogonal to the hydrodynamic variables and the unit functions, so that

$$(J_i, a_{\mathbf{k}}{}^\mu) = 0, \qquad (J_i, 1) = 0 \tag{20}$$

and we have similar relations for $J_i{}^K$ and $J_i{}^\Phi$ separately.

The names *projected* or *subtracted* currents refer to property (20), indicating that currents J_i in (3) are obtained from the energy current $J_x{}^e$ and the momentum current J_{xx} by projecting out the component along hydrodynamic variables. Similar remarks apply to the kinetic currents in (8) and potential currents. One may again verify these orthogonality relations (20) by means of the fluctuation formulas of the appendix. If the currents do not satisfy (20), the mode-coupling formula (14) does not describe the dominant long-time behavior of the correlation functions.

3. ASYMPTOTIC TIME BEHAVIOR

Let us consider the long-time behavior of the separate two mode contributions in (14). Since the two-mode amplitudes $(J_i, a_{\mathbf{k}}{}^\mu a^\nu{}_{-\mathbf{k}})$ depend only on the directions of $\mathbf{k}$ and not on its magnitude [see Eq. (16)], the angular integrations can be carried out separately. We are then left with two types of contributions to (14), which behave quite differently for large times. For the cases $(\mu\nu) = (\epsilon_i\epsilon_j)$, $(\epsilon_i H)$, (HH) and $(\sigma, -\sigma)$, taking $(\epsilon_i\epsilon_j)$ as a typical example, we have

$$\int dk\, k^{d-1} \exp(-2\nu k^2 t) \simeq t^{-d/2} \tag{21}$$

For the cases $(\mu\nu) = (\epsilon_i\sigma)$, $(H\sigma)$, where $(\sigma\sigma)$ is taken as a representative example, we have

$$\sum_\sigma \int dk\, k^{d-1} \exp(-2i\sigma c_0 kt - \Gamma_s k^2 t) \simeq \begin{cases} t^{-d}, & d \text{ even} \\ t^{-1/2} \exp(-c_0{}^2 t/\Gamma_s), & d \text{ odd} \end{cases} \tag{22}$$

Hence, the first type of contribution (21) dominates for long times, and we need only consider it further.

In the remaining part of this section we will evaluate explicitly the coefficients of the $t^{-d/2}$ tails for each of the correlation functions. The first step will be to calculate the two-mode amplitude $(J_i, a_{\mathbf{k}}{}^\mu a^\nu{}_{-\mathbf{k}})$ for the separate cases by means of Eqs. (3), (8), (12), and (16). In order to do so we need some fluctuation formulas, which have been calculated in the appendix. For the currents J_λ and $J_\lambda{}^K$ we deduce from Eqs. (A.24)–(A.26) of the appendix

$$(J_\lambda, a_{\mathbf{k}}^{\epsilon_i} a^H_{-\mathbf{k}}) = \beta^{-1}(TC_p/m)^{1/2}\hat{k}^i_{\perp x} \tag{23a}$$

$$(J_\lambda, a_{\mathbf{k}}^\sigma a^{-\sigma}_{-\mathbf{k}}) = \beta^{-1}c_0\sigma\hat{k}_x \tag{23b}$$

and

$$(J_\lambda{}^K, a_{\mathbf{k}}^{\epsilon i} a_{-\mathbf{k}}^H) = (C_p{}^0/C_p)(J_\lambda, a_{\mathbf{k}}^{\epsilon i} a_{-\mathbf{k}}^H) \tag{24a}$$

$$(J_\lambda{}^K, a_{\mathbf{k}}^\sigma a_{-\mathbf{k}}^{-\sigma}) = (\alpha T C_p{}^0/C_p)(J_\lambda, a_{\mathbf{k}}^\sigma a_{-\mathbf{k}}^{-\sigma}) \tag{24b}$$

where $C_p{}^0 = \frac{1}{2}(d + 2)k_B$ is the ideal gas value of the specific heat per particle at constant pressure, and $\alpha = -n^{-1}(\partial T/\partial n)_p$ is the thermal expansion coefficient.

For the longitudinal momentum current J_l the following results are obtained by virtue of (A.15), (A.18), and (A.21):

$$(J_l, a_{\mathbf{k}}^{\epsilon i} a_{-\mathbf{k}}^{\epsilon j}) = 2\beta^{-1}\left(\hat{k}_{\perp x}^i \hat{k}_{\perp x}^j - \frac{\gamma - 1}{2\alpha T}\delta_{ij}\right) \tag{25a}$$

$$(J_l, a_{\mathbf{k}}^H a_{-\mathbf{k}}^H) = \beta^{-1}(\gamma - 1)\left[1 - \frac{1}{\alpha C_p}\left(\frac{\partial C_p}{\partial T}\right)_p + \frac{1}{\alpha^2}\left(\frac{\partial \alpha}{\partial T}\right)_p\right] \tag{25b}$$

$$(J_l, a_{\mathbf{k}}^\sigma a_{-\mathbf{k}}^{-\sigma}) = \beta^{-1}\left[\hat{k}_x{}^2 - \frac{\gamma - 1}{\alpha T} + \frac{n}{c_0}\left(\frac{\partial c_0}{\partial n}\right)_s\right] \tag{25c}$$

and for the kinetic part $J_l{}^K$, Eqs. (A.22) yield

$$(J_l{}^K, a_{\mathbf{k}}^{\epsilon i} a_{-\mathbf{k}}^{\epsilon j}) = 2\beta^{-1}(\hat{k}_{\perp x}^i \hat{k}_{\perp x}^i - d^{-1}\delta_{ij}) \tag{26a}$$

$$(J_l{}^K, a_{\mathbf{k}}^H a_{-\mathbf{k}}^H) = 0 \tag{26b}$$

$$(J_l{}^K, a_{\mathbf{k}}^\sigma a_{-\mathbf{k}}^{-\sigma}) = \beta^{-1}(\hat{k}_x{}^2 - d^{-1}) \tag{26c}$$

Here $\hat{k}_x$ and $\hat{k}_{\perp x}^i$ are the x components of the unit vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}_\perp^i$, and we used the relation $\hat{\mathbf{k}}_\perp^i \cdot \hat{\mathbf{k}}_\perp^j = \delta_{ij}$. The amplitudes for the bulk viscosity current J_ζ follow from (25) and (26) by summing over all Cartesian components and dividing the result by d. Since the kinetic part of the projected bulk viscosity current vanishes identically, the full projected current coincides with its potential part. The results are

$$(J_\zeta, a_{\mathbf{k}}^{\epsilon i} a_{-\mathbf{k}}^{\epsilon j}) = \beta^{-1}\delta_{ij}\left(\frac{2}{d} - \frac{\gamma - 1}{\alpha T}\right) \tag{27a}$$

$$(J_\zeta, a_{\mathbf{k}}^H a_{-\mathbf{k}}^H) = (J_l, a_{\mathbf{k}}^H a_{-\mathbf{k}}^H) \tag{27b}$$

$$(J_\zeta, a_{\mathbf{k}}^\sigma a_{-\mathbf{k}}^{-\sigma}) = \beta^{-1}\left[\frac{1}{d} - \frac{\gamma - 1}{\alpha T} + \frac{n}{c_0}\left(\frac{\partial c_0}{\partial n}\right)_s\right] \tag{27c}$$

In the case of the shear viscosity currents the amplitudes of J_η and $J_\eta{}^K$ along two hydrodynamic modes coincide, as follows from (A.23), and the result is

$$(J_\eta, a_{\mathbf{k}}^{\epsilon i} a_{-\mathbf{k}}^{\epsilon j}) = (J_\eta{}^K, a_{\mathbf{k}}^{\epsilon i} a_{-\mathbf{k}}^{\epsilon j}) = \beta^{-1}(\hat{k}_{\perp x}^i \hat{k}_{1y}^j + \hat{k}_{1y}^i \hat{k}_{1x}^j) \tag{28a}$$

$$(J_\eta, a_{\mathbf{k}}^\sigma a_{-\mathbf{k}}^{-\sigma}) = (J_\eta{}^K, a_{\mathbf{k}}^\sigma a_{-\mathbf{k}}^{-\sigma}) = \beta^{-1}\hat{k}_x \hat{k}_y \tag{28b}$$

As the final step we have to perform the angular averages in (14). Two properties of the unit vectors ($\hat{\mathbf{k}}$, $\hat{\mathbf{k}}_\perp^i$) simplify the calculations considerably. The first is that the d Cartesian components of the vectors $\hat{\mathbf{k}}$, $\hat{\mathbf{k}}_\perp^1$,..., $\hat{\mathbf{k}}_\perp^{d-1}$ form the rows of a $d \times d$ matrix, which describes the orthogonal transformation for the coordinate system with basis vectors ($\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$,...) to a new coordinate system with basis vectors ($\hat{\mathbf{k}}$, $\hat{\mathbf{k}}_\perp^1$, $\hat{\mathbf{k}}_\perp^2$,...). These rows are mutually orthogonal such that

$$\hat{k}_\alpha \hat{k}_\beta + \sum_{i=1}^{d-1} \hat{k}_{\perp\alpha}^i \hat{k}_{\perp\beta}^i = \delta_{\alpha\beta} \tag{29}$$

The second property, which has been derived in paper I, reads

$$\int d\hat{\mathbf{k}}\; \hat{k}_\alpha \hat{k}_\beta \hat{k}_\gamma \hat{k}_\delta = \frac{\Omega_d}{d(d+2)} (\delta_{\alpha\beta}\,\delta_{\gamma\delta} + \delta_{\alpha\gamma}\,\delta_{\beta\delta} + \delta_{\alpha\delta}\,\delta_{\beta\gamma}) \tag{30a}$$

$$\int d\hat{\mathbf{k}}\; \hat{k}_\alpha \hat{k}_\beta = \frac{\Omega_d}{d}\,\delta_{\alpha\beta} \tag{30b}$$

where $\Omega_d = \int d\hat{\mathbf{k}}$ is the surface of a d-dimensional sphere with radius equal to unity. If we use in addition the relation for $\Delta > 0$,

$$\frac{\Omega_d}{(2\pi)^d} \int dk\; k^{d-1} \exp(-\Delta k^2 t) = \int \frac{d\hat{\mathbf{k}}}{(2\pi)^d} \exp(-\Delta k^2 t) = (4\pi\Delta t)^{-d/2} \tag{31}$$

we find the following results for the long-time behavior of the correlation functions:

$$C_\lambda(t) \simeq \left(\frac{K_{\epsilon H}}{(\nu + D_T)^{d/2}} + \frac{K_{+-}}{\Gamma_s^{d/2}} \right) \left(\frac{1}{4\pi t} \right)^{d/2} \tag{32a}$$

$$C_i(t) \simeq \left(\frac{N_{\epsilon\epsilon}}{(2\nu)^{d/2}} + \frac{N_{+-}}{(\Gamma_s)^{d/2}} + \frac{N_{HH}}{(2D_T)^{d/2}} \right) \left(\frac{1}{4\pi t} \right)^{d/2} \tag{32b}$$

$$C_\zeta(t) \simeq \left(\frac{M_{\epsilon\epsilon}}{(2\nu)^{d/2}} + \frac{M_{+-}}{(\Gamma_s)^{d/2}} + \frac{M_{HH}}{(2D_T)^{d/2}} \right) \left(\frac{1}{4\pi t} \right)^{d/2} \tag{32c}$$

$$C_n(t) \simeq \left(\frac{L_{\epsilon\epsilon}}{(2\nu)^{d/2}} + \frac{L_{+-}}{(\Gamma_s)^{d/2}} \right) \left(\frac{1}{4\pi t} \right)^{d/2} \tag{32d}$$

where we have kept the notation consistent with Ref. 2, and where

$$K_{\epsilon H} = \frac{TC_p}{\beta^2 m} \frac{d-1}{d}; \qquad K_{+-} = \frac{c_0^2}{\beta^2} \frac{1}{d} \tag{33a}$$

$$L_{\epsilon\epsilon} = \frac{1}{\beta^2} \frac{d^2-2}{d(d+2)}; \qquad L_{+-} = \frac{1}{\beta^2} \frac{1}{d(d+2)} \tag{33b}$$

$$M_{\epsilon\epsilon} = \frac{2}{\beta^2}(d-1)\left(\frac{1}{d} - \frac{\gamma - 1}{2\alpha T}\right)^2$$

$$M_{+-} = \frac{1}{\beta^2}\left[\frac{1}{d} - \frac{\gamma - 1}{\alpha T} + \frac{n}{c_0}\left(\frac{\partial c_0}{\partial n}\right)_s\right]^2$$

$$M_{HH} = \frac{1}{\beta^2}\frac{1}{2}(\gamma - 1)^2\left[1 - \frac{1}{\alpha C_p}\left(\frac{\partial C_p}{\partial T}\right)_p + \frac{1}{\alpha^2}\left(\frac{\partial \alpha}{\partial T}\right)_p\right]^2 \tag{33c}$$

$$N_{\epsilon\epsilon} = \frac{2}{\beta^2}\left(\frac{d-1}{d}\right)\left[\frac{d+1}{d+2} - \frac{\gamma - 1}{\alpha T} + d\left(\frac{\gamma - 1}{2\alpha T}\right)^2\right]$$

$$N_{+-} = \frac{1}{\beta^2}\left\{\frac{3}{d(d+2)} - \frac{2}{d}\left[\frac{\gamma - 1}{\alpha T} - \frac{n}{c_0}\left(\frac{\partial c_0}{\partial n}\right)_s\right] + \left[\frac{\gamma - 1}{\alpha T} - \frac{n}{c_0}\left(\frac{\partial c_0}{\partial n}\right)_s\right]^2\right\}$$

$$N_{HH} = M_{HH} \tag{33d}$$

The asymptotic form of the AB parts of the correlation functions with A and B equal to K or ϕ can be obtained from (14b) and (23)–(28) in a similar way. The results can be cast in the same form as (32) by attaching superscripts AB to the quantities C, K, N, M, and L. We shall list the results for K^{AB}, N^{AB}, M^{AB}, and L^{AB}:

$$K_{\epsilon H}^{KK} = \left(\frac{C_p^0}{C_p}\right)^2 K_{\epsilon H}; \qquad\qquad K_{+-}^{KK} = \left(\frac{\alpha T C_p^0}{C_p}\right)^2 K_{+-} \tag{34a}$$

$$K_{\epsilon H}^{K\phi} = \frac{C_p^0}{C_p}\left(\frac{C_p^0}{C_p} - 1\right) K_{\epsilon H}; \qquad\qquad K_{+-}^{K\phi} = \frac{\alpha T C_p^0}{C_p}\left(\frac{\alpha T C_p^0}{C_p} - 1\right) K_{+-} \tag{34b}$$

$$K_{\epsilon H}^{\phi\phi} = \left(\frac{C_p^0}{C_p} - 1\right)^2 K_{\epsilon H}; \qquad\qquad K_{+-}^{\phi\phi} = \left(\frac{\alpha T C_p^0}{C_p} - 1\right)^2 K_{+-} \tag{34c}$$

$$N_{\epsilon\epsilon}^{KK} = \frac{2}{\beta^2}\frac{(d-1)(d^2 - 2)}{d^2(d+2)}; \qquad N_{+-}^{KK} = \frac{2}{\beta^2}\frac{(d-1)}{d^2(d+2)}; \quad N_{HH}^{KK} = 0 \tag{35a}$$

$$N_{\epsilon\epsilon}^{K\phi} = N_{+-}^{K\phi} = N_{HH}^{K\phi} = 0 \tag{35b}$$

$$N_{\epsilon\epsilon}^{\phi\phi} = \frac{2}{\beta^2}(d-1)\left(\frac{1}{d} - \frac{\gamma - 1}{2\alpha T}\right)^2; \quad N_{+-}^{\phi\phi} = M_{+-}; \quad N_{HH}^{\phi\phi} = M_{HH} \tag{35c}$$

$$M_{\epsilon\epsilon}^{KK} = M_{+-}^{KK} = M_{HH}^{KK} = 0 \tag{36a}$$

$$M_{\epsilon\epsilon}^{K\phi} = M_{+-}^{K\phi} = M_{HH}^{K\phi} = 0 \tag{36b}$$

$$M_{\epsilon\epsilon}^{\phi\phi} = M_{\epsilon\epsilon}; \qquad M_{+-}^{\phi\phi} = M_{+-}; \qquad M_{HH}^{\phi\phi} = M_{HH} \tag{36c}$$

$$L_{\epsilon\epsilon}^{KK} = L_{\epsilon\epsilon}; \qquad L_{+-}^{KK} = L_{+-} \tag{37a}$$

$$L_{\epsilon\epsilon}^{K\phi} = L_{+-}^{K\phi} = 0 \tag{37b}$$

$$L_{\epsilon\epsilon}^{\phi\phi} = L_{+-}^{\phi\phi} = 0 \tag{37c}$$

The results (34a) and (37a) for $C_\lambda^{KK}(t)$ and $C_\eta^{KK}(t)$, respectively, have already been obtained in paper I.

APPENDIX. FLUCTUATION FORMULAS

In this appendix we will consider fluctuation formulas in the grand canonical ensemble, characterized by the parameters $\beta = 1/k_B T$ and $\nu = \beta\mu$, where μ is the chemical potential. The fluctuation formulas considered will be in general of the form (a_k, b_k) and $(J_i, a_k b_{-k})$, and we only need to know such quantities for small values of the wave number $|\mathbf{k}|$. The inner product is defined in (12), and a_k and b_k are Fourier components of microscopic densities, such as given in (15). We observe that all Fourier components of microscopic variables for $\mathbf{k} \neq 0$ satisfy the relations $(1, a_k) = 0$ due to translational invariance, and from the definitions (15) it follows that this property remains valid in the limit for small $\mathbf{k}$. For the quantities of interest we have

$$e_0 = H - \langle H \rangle; \qquad n_0 = N - \langle N \rangle$$
$$g_0 = \sum_{i=1}^{N} \mathbf{v}_i; \qquad j_{0xx} = J_{xx} - \langle J_{xx} \rangle \tag{A.1}$$

or in general

$$a_0 = A - \langle A \rangle \tag{A.2}$$

For later convenience we have included in (A.1) the longitudinal momentum current j_{0xx}, of which J_l in (3b) is the corresponding projected current.

From the fluctuation formulas in the grand canonical ensemble, we deduce immediately that for any of the a_0's in (A.1) and (A.2) the following relations are fulfilled:

$$(a_0, e_0) = \lim_{V \to \infty} V^{-1} \langle (A - \langle A \rangle)(H - \langle H \rangle) \rangle = -(\partial a / \partial \beta)_\nu,$$
$$(a_0, n_0) = (\partial a / \partial \nu)_\beta \tag{A.3}$$

where $a = V^{-1} \langle A \rangle$. Similarly, we find from (A.3) and (17)

$$(a_0, s_0) = -\frac{1}{nT} \left[\left(\frac{\partial a}{\partial \beta} \right)_\nu + h \left(\frac{\partial a}{\partial \nu} \right)_p \right] = -\frac{1}{nT} \left(\frac{\partial a}{\partial \beta} \right)_p$$
$$(a_0, p_0) = -\left(\frac{\partial p}{\partial e} \right)_n \left(\frac{\partial a}{\partial \beta} \right)_\nu + \left(\frac{\partial p}{\partial n} \right)_e \left(\frac{\partial a}{\partial \nu} \right)_\beta = \frac{n}{\beta} \left(\frac{\partial a}{\partial n} \right)_s \tag{A.4}$$

In order to obtain these formulas one needs the thermodynamic identities

$$ds = (1/nT)(de - h\,dn) \tag{A.5a}$$
$$dp = -(nh/\beta)\,d\beta + (n/\beta)\,d\nu \tag{A.5b}$$

where n, e, p, and s are, respectively, the values of the number density, the energy density, the pressure, and the entropy per particle in thermal equilibrium, and $h = (e + p)/n$ is the enthalpy per particle. From (A.5) one may obtain the relations used to derive (A.4),

$$h = \left(\frac{\partial v}{\partial \beta}\right)_p = \left(\frac{\partial e}{\partial n}\right)_s \qquad (A.6a)$$

$$\left(\frac{\partial p}{\partial e}\right)_n = -\frac{n}{\beta}\left(\frac{\partial \beta}{\partial n}\right)_s; \qquad \left(\frac{\partial p}{\partial n}\right)_e = \frac{n}{\beta}\left(\frac{\partial v}{\partial n}\right)_s \qquad (A.6b)$$

The first relation (A.6a) follows directly from (A.5). The equalities (A.6b) may be obtained from (A.5), (A.6a), and the Maxwell relation $(\partial\beta/\partial n)_e = -(\partial v/\partial e)_n$. In cases where a_0 is a linear combination of the quantities in (A.1) with coefficients depending on thermodynamic parameters, we can apply (A.4) to each of the terms separately. This yields, e.g.,

$$(s_0, s_0) = \left(\frac{1}{nT}\right)^2\left[-\left(\frac{\partial e}{\partial \beta}\right)_p + h\left(\frac{\partial n}{\partial \beta}\right)_p\right] = -\frac{1}{nT}\left(\frac{\partial s}{\partial \beta}\right)_p = \frac{k_B C_p}{n} \qquad (A.7a)$$

$$(s_0, p_0) = \frac{1}{nT}\frac{n}{\beta}\left[\left(\frac{\partial e}{\partial n}\right)_s - h\left(\frac{\partial n}{\partial n}\right)_s\right] = \frac{n}{\beta}\left(\frac{\partial s}{\partial n}\right)_s = 0 \qquad (A.7b)$$

$$(p_0, p_0) = \frac{n}{\beta}\left[\left(\frac{\partial p}{\partial e}\right)_n\left(\frac{\partial e}{\partial n}\right)_s + \left(\frac{\partial p}{\partial n}\right)_e\left(\frac{\partial n}{\partial n}\right)_s\right] = \frac{n}{\beta}\left(\frac{\partial p}{\partial n}\right)_s = \frac{\rho c_0^2}{\beta} \qquad (A.7c)$$

where C_p is the specific heat per particle at constant pressure and c_0 is the adiabatic sound velocity.

These fluctuation expressions suffice to verify the orthonormality relations (18). Equations (20) can be verified if one uses in addition the virial theorem of equilibrium statistical mechanics in the form $\langle J_{xx}\rangle = pV$ and the relations

$$(J_x{}^e, g_{0x}) = h(J_x{}^n, g_{0x}) = nh/\beta \qquad (A.8)$$

which follow by direct computation.

In the body of the paper we also need fluctuation expressions of the form $(J_i, a_k{}^\mu a^\nu_{-k})$ for small values of k. Here, $a_k{}^\mu$ and $a_k{}^\nu$ are linear combinations of the hydrodynamic variables (15). In order to evaluate such objects we consider first fluctuation expressions $(a_0, b_0 c_0)$, where a_0, b_0, and c_0 are any of the quantities in (A.1). In this case we find directly

$$(a_0, b_0 e_0) = -(\partial/\partial\beta)_v(a_0, b_0)$$
$$(a_0, b_0 n_0) = (\partial/\partial v)_\beta(a_0, b_0) \qquad (A.9)$$

Next, we consider $(a_0, s_0 s_0)$ and replace both s_0's by its definition (17a) and apply (A.4) and (A.9); the result is

$$(a_0, s_0 s_0) = -\left(\frac{1}{nT}\right)^2\left[\left(\frac{\partial}{\partial\beta}\right)_p (a_0, e_0) - h\left(\frac{\partial}{\partial\beta}\right)_p (a_0, n_0)\right] \quad \text{(A.10a)}$$

$$= \left(\frac{1}{nT}\right)^2\left[\left(\frac{\partial^2 a}{\partial\beta^2}\right)_p - \left(\frac{\partial a}{\partial v}\right)_\beta\left(\frac{\partial^2 v}{\partial\beta^2}\right)_p\right] \quad \text{(A.10b)}$$

In the transition from (A.10a) to (A.10b) we have written

$$h\left(\frac{\partial}{\partial\beta}\right)_p (a_0, n_0) = \left(\frac{\partial}{\partial\beta}\right)_p\left[\left(\frac{\partial a}{\partial v}\right)_\beta\left(\frac{\partial v}{\partial\beta}\right)_p\right] + \left(\frac{\partial a}{\partial v}\right)_\beta\left(\frac{\partial^2 v}{\partial\beta^2}\right)_p \quad \text{(A.11)}$$

by virtue of (A.6a) and (A.3), and recombined the remaining terms into (A.10b). In the case of $(a_0, p_0 p_0)$ we proceed similarly; i.e., we replace one p_0 by its definition (17b), and use (A.4), (A.6), and (A.9) to obtain

$$(a_0, p_0 p_0) = \left(\frac{n}{\beta}\right)^2\left[\left(\frac{\partial v}{\partial n}\right)_s\left(\frac{\partial}{\partial n}\right)_s (a_0, n_0) - \left(\frac{\partial\beta}{\partial n}\right)_s\left(\frac{\partial}{\partial n}\right)_s (a_0, e_0)\right]$$

$$= \left(\frac{n}{\beta}\right)^2\left[\left(\frac{\partial^2 a}{\partial n^2}\right)_s - \left(\frac{\partial a}{\partial v}\right)_\beta\left(\frac{\partial^2 v}{\partial n^2}\right)_s - \left(\frac{\partial a}{\partial\beta}\right)_v\left(\frac{\partial^2\beta}{\partial n^2}\right)_s\right] \quad \text{(A.12)}$$

Fluctuation formulas of the form $(a_0, s_0 p_0)$ can be evaluated similarly, but are not needed in our theory.

Let us now use the above results for the computation of $(J_l, a_0^\mu b_0^\nu)$, and start with J_l, which is given by, according to (3b) and (A.1),

$$J_l = j_{0xx} - (\partial p/\partial e)_n e_0 - (\partial p/\partial n)_e n_0 \quad \text{(A.13)}$$

By applying (A.10) and (A.12) to the linear combination (A.13) and by observing that the terms in (A.10) and (A.12) containing only first derivatives of a cancel exactly, we find

$$(J_l, s_0 s_0) = \left(\frac{1}{n\beta}\right)^2\left[\left(\frac{\partial^2 p}{\partial T^2}\right)_p - \left(\frac{\partial p}{\partial e}\right)_n\left(\frac{\partial^2 e}{\partial T^2}\right)_p - \left(\frac{\partial p}{\partial n}\right)_e\left(\frac{\partial^2 n}{\partial T^2}\right)_p\right] \quad \text{(A.14a)}$$

$$(J_l, p_0 p_0) = \left(\frac{n}{\beta}\right)^2\left[\left(\frac{\partial^2 p}{\partial n^2}\right)_s - \left(\frac{\partial p}{\partial e}\right)_n\left(\frac{\partial^2 e}{\partial n^2}\right)_s - \left(\frac{\partial p}{\partial n}\right)_e\left(\frac{\partial^2 n}{\partial n^2}\right)_s\right] \quad \text{(A.14b)}$$

Equation (A.14a) may be simplified by writing it as

$$(J_l, s_0 s_0) = -\left(\frac{1}{n\beta}\right)^2\left(\frac{\partial p}{\partial e}\right)_n\left(\frac{\partial n}{\partial T}\right)_p\left[\left(\frac{\partial}{\partial T}\right)_p\left(\frac{\partial e}{\partial n}\right)_p\right]$$

$$= \frac{k_B C_p}{n}\frac{\gamma - 1}{\beta}\left[1 - \frac{1}{\alpha C_p}\left(\frac{\partial C_p}{\partial T}\right)_p + \frac{1}{\alpha^2}\left(\frac{\partial\alpha}{\partial T}\right)_p\right] \quad \text{(A.15)}$$

where we have used the thermodynamic relations

$$dh = T\,ds + n^{-1}\,dp; \qquad \left(\frac{\partial p}{\partial e}\right)_n = \frac{\gamma - 1}{\alpha T}$$

$$\left(\frac{\partial e}{\partial n}\right)_p = h + n\left(\frac{\partial h}{\partial n}\right)_p = h - \frac{C_p}{\alpha} \tag{A.16}$$

Here, $\alpha = -n^{-1}(\partial n/\partial T)_p$ is the thermal expansion coefficient; $C_p = T(\partial s/\partial T)_p = (\partial h/\partial T)_p$ and $\gamma = C_p/C_v$ is the ratio of the specific heats. Next, we reexpress (A.14b) by using the relations

$$\left(\frac{\partial^2 p}{\partial n^2}\right)_s = \left(\frac{\partial}{\partial n}\right)_s mc_0{}^2 = 2mc_0\left(\frac{\partial c_0}{\partial n}\right)_s$$

$$\left(\frac{\partial^2 e}{\partial n^2}\right)_s = \left(\frac{\partial h}{\partial n}\right)_s = \frac{1}{n}\left(\frac{\partial p}{\partial n}\right)_s = \frac{mc_0{}^2}{n} = \frac{C_p(\gamma - 1)}{n\alpha^2 T} \tag{A.17}$$

which yields

$$(J_l, p_0 p_0) = \frac{2\rho c_0{}^2}{\beta^2}\left[\frac{n}{c_0}\left(\frac{\partial c_0}{\partial n}\right)_s - \frac{\gamma - 1}{2\alpha T}\right] \tag{A.18}$$

A next set of fluctuation expressions $(J_l, a_0{}^\mu a_0{}^\nu)$ are those in which $a_0{}^\mu$ and $a_0{}^\nu$ involve the momentum density $g_{0\alpha}$, where α denotes a Cartesian component.

In this case we write

$$(J_l, g_{0\alpha}g_{0\beta}) = (\{J_l - \langle J_l\rangle_v\}, g_{0\alpha}g_{0\beta}) + (\langle J_l\rangle_v, g_{0\alpha}g_{0\beta}) \tag{A.19}$$

where $\langle\cdots\rangle_v$ indicates that the velocity average is carried out. If we carry out the remaining velocity integrations in the second term on the right-hand side of (A.19), we find

$$(\langle J_l\rangle_v, g_{0\alpha}g_{0\beta}) = \beta^{-1}m(\langle J_l\rangle_v, n_0)\,\delta_{\alpha\beta} = \beta^{-1}m(J_l, n_0)\,\delta_{\alpha\beta} = 0 \tag{A.20}$$

The inner product vanishes since the projected currents are orthogonal to the hydrodynamic variables, as can be verified explicitly from (3b). In the first term on the right-hand side of (A.19) only those terms in $(J_l - \langle J_l\rangle_v)$ survive that depend on velocity variables. After carrying out some obvious simplifications, we find

$$(J_l, g_{0\alpha}g_{0\beta}) = \rho\,\delta_{\alpha x}\,\delta_{\beta x}\,\langle(mv_x{}^2 - \beta^{-1})mv_x{}^2\rangle_v$$

$$- \left(\frac{\partial p}{\partial e}\right)_n \rho\,\delta_{\alpha\beta}\left\langle\left(\frac{1}{2}mv^2 - \frac{d}{2}\beta^{-1}\right)\frac{1}{d}mv^2\right\rangle_v$$

$$= \frac{2\rho}{\beta^2}\left(\delta_{\alpha x}\,\delta_{\beta x} - \frac{\gamma - 1}{2\alpha T}\,\delta_{\alpha\beta}\right) \tag{A.21}$$

where d is the dimensionality of the system and (A.16) has been used for $(\partial p/\partial e)_n$. The corresponding fluctuation expressions for the kinetic part J_l^K of J_l, as given in (8), can be written down directly from the previous results (A.14) and (A.21) by replacing $(\partial p/\partial e)_n$ by $(2/d)$ and $(\partial p/\partial n)_e$ by 0, and furthermore, by replacing in $(\partial^2 a/\partial T^2)_p$ and $(\partial^2 a/\partial n^2)_s$ the quantity a ($=p$ or e) by its ideal gas values a^0 ($=n/\beta$ or $dn/2\beta$). The result is

$$(J_l^K, s_0 s_0) = (J_l^K, p_0 p_0) = 0$$
$$(J_l^K, g_{0\alpha} g_{0\beta}) = (2\rho/\beta^2)[\delta_{\alpha x}\, \delta_{\beta x} - (1/d)\, \delta_{\alpha\beta}]$$

(A.22)

Consider now fluctuation expressions involving J_η and $J_\eta{}^K$, as given in (3e) and (8). The only nonvanishing expression can be easily obtained by directly carrying out the velocity integrations, yielding

$$(J_\eta, g_{0\alpha} g_{0\beta}) = (J_\eta^K, g_{0\alpha} g_{0\beta}) = (\rho/\beta)(\delta_{\alpha x}\, \delta_{\beta y} + \delta_{\alpha y}\, \delta_{\beta x})$$

(A.23)

The only nonvanishing fluctuation formulas involving $J_\lambda = J_x{}^e - h J_x{}^n$ are $(J_\lambda, g_{0x} s_0)$ and $(J_\lambda, g_{0x} p_0)$. By applying (A.9) with a_0 and b_0 respectively equal to $J_x{}^e$ and g_{0x} or equal to $J_x{}^n$ and g_{0x}, we find in a manner analogous to (A.4)

$$(J_\lambda, g_{0x} s_0) = -\frac{1}{nT}\left(\frac{\partial}{\partial\beta}\right)_p (J_x{}^e, g_{0x}) + \frac{h}{nT}\left(\frac{\partial}{\partial\beta}\right)_p (J_x{}^n, g_{0x})$$ (A.24a)

$$= -\frac{1}{\beta T}\left(\frac{\partial h}{\partial\beta}\right)_p = \beta^{-2} C_p$$ (A.24b)

and

$$(J_\lambda, g_{0x} p_0) = \frac{n}{\beta}\left(\frac{\partial}{\partial n}\right)_s (J_x{}^e, g_{0x}) - \frac{nh}{\beta}\left(\frac{\partial}{\partial n}\right)_s (J_x{}^n, g_{0x})$$ (A.25a)

$$= \left(\frac{n}{\beta}\right)^2 \left(\frac{\partial h}{\partial n}\right)_s = \frac{nC_p(\gamma - 1)}{\beta^2 \alpha^2 T} = \frac{\rho c_0^2}{\beta^2}$$ (A.25b)

In the second equalities of (A.24) and (A.25) we have used (A.8), (A.16), and (A.17). The fluctuation expressions for $J_\lambda{}^K$ defined in (8) follow directly from (A.24b) and (A.25b) by replacing h by its ideal gas value $h^0 = \frac{1}{2}(d + 2)\beta^{-1}$. The results are

$$(J_\lambda^K, g_{0x} s_0) = \beta^{-2} C_p{}^0 = (C_p{}^0/C_p)(J_\lambda, g_{0x} s_0)$$ (A.26a)

$$(J_\lambda^K, g_{0x} p_0) = nC_p{}^0(\gamma - 1)/\beta^2\alpha = \alpha T(C_p{}^0/C_p)(J_\lambda, g_{0x} p_0)$$ (A.26b)

where we have introduced $C_p{}^0 = \frac{1}{2}(d + 2)k_B$ and used the relation $(\partial T/\partial n)_s = (T/n)(\partial p/\partial e)_n = (\gamma - 1)/\alpha n$.

REFERENCES

1. B. J. Alder and T. E. Wainwright, *Phys. Rev. A* **1**:18 (1970).
2. M. H. Ernst, E. H. Hauge, and J. M. J. van Leeuwen, *Phys. Lett.* **34A**:419 (1971).
3. Y. Pomeau, *Phys. Rev. A* **5**:2569 (1972); **7**:1134 (1973); *Phys. Lett.* **38A**:245 (1972).
4. K. Kawasaki, *Prog. Theor. Phys. Japan* **45**:1691 (1971); **46**:1299 (1971).
5. M. H. Ernst, E. H. Hauge, and J. M. J. van Leeuwen, *Phys. Rev. A* **4**:2055 (1971).
6. M. H. Ernst, E. H. Hauge, and J. M. J. van Leeuwen, this issue, following paper.