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On the Existence of the Steady State in the Stochastic Volterra–Lotka Model[1]

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Nicolis and Prigogine have shown that the Volterra–Lotka model taken as a Markovian stochastic process does not have a steady state, by considering the Fokker–Planck-type equation for small fluctuations. Here we use the exact master equation to show that the only steady state in the model is the trivial one.

KEY WORDS: Volterra–Lotka model; steady state.

In making a correspondence between a deterministic equation and its stochastic version in terms of Markovian birth and death processes, there is considerable arbitrariness. This arises from the different possibilities for the identification of birth and death terms in the deterministic equation and the option for including spontaneous birth. The choice depends on the particular physical process to which the model is to be applied. Usually, with the proper identification of the birth and death terms, the general results of the stochastic model, such as the existence of the steady state, time behavior of averages, etc., resemble those of the deterministic equation.

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Thus if the Verhulst equation

$$\dot{n} = \lambda n\left(1 - \frac{n}{N}\right) \tag{1}$$

is written as

$$\dot{n} = \lambda n\left(1 + \frac{1}{N}\right) - \frac{\lambda}{N} n(n - 1) \tag{2}$$

and the first term is identified as the birth term and the second as the death term, then in the stochastic version one obtains a steady-state distribution of probabilities such that $\langle n \rangle = N + 1$ (or $N + 2$ if spontaneous birth is assumed). For large N this result is not much different from the asymptotic value of $n = N$, obtained from the deterministic equation. If, on the other hand, birth and death terms are identified directly from (1), the stochastic model yields total extinction as the only steady-state distribution of probabilities.

We have the Volterra–Lotka equation for a prey–predator system

$$\dot{X} = K_1 X - \beta XY, \qquad \dot{Y} = -K_2 Y + \beta XY \tag{3}$$

where X is the number of prey and Y is the number of predators. This set of equations has a steady state $X_0 = K_2/\beta$ and $Y_0 = K_1/\beta$. Thus one might expect that the corresponding stochastic model also has a steady-state distribution of probabilities similar to the above.

We make the same choice of birth and death terms as in Refs. 1 and 2

$$\begin{aligned} P(X \to X + 1) &\propto K_1 X \\ P(X \to X - 1,\ Y \to Y + 1) &\propto \beta XY \\ P(Y \to Y - 1) &\propto K_2 Y \end{aligned} \tag{4}$$

P is the probability for the process in the parentheses to occur. With this, we get the master equation for the joint probability $P(X, Y, t)$ for finding X of the prey and Y of the predator at time t:

$$\begin{aligned} \dot{P}(X, Y, t) = {} & K_1(X - 1)P(X - 1, Y, t) - K_1 XP(X, Y, t) \\ & + K_2(Y + 1)P(X, Y + 1, t) - K_2 YP(X, Y, t) \\ & + \beta(X + 1)(Y - 1)P(X + 1, Y - 1, t) - \beta XYP(X, Y, t) \end{aligned} \tag{5}$$

In Refs. 1 and 2 Nicolis and Prigogine approximated this master equation by a Fokker–Planck type of equation for small fluctuations about the steady state and showed that in fact there is no possibility of a steady state. We shall prove this directly from the master equation, without any approximations.

We define

$$A(X, t) = \sum_{Y=0}^{\infty} P(X, Y, t) \qquad \text{and} \qquad B(Y, t) = \sum_{X=0}^{\infty} P(X, Y, t)$$

$A(X, t)$ is the probability of finding X of the prey at time t, irrespective of the number of predators, and $B(Y, t)$ is defined analogously for the predators. The equations of evolution for $A(X, t)$ and $B(Y, t)$ can be obtained directly from (5):

$$\dot{A}(X, t) = K_1(X - 1)A(X - 1, t) - K_1 XA(X, t) + \beta(X + 1) \sum_{Y=0}^{\infty} YP(X + 1, Y, t) - \beta X \sum_{Y=0}^{\infty} YP(X, Y, t) \quad (6)$$

$$\dot{B}(Y, t) = K_2(Y + 1)B(Y + 1, t) - K_2 YB(Y, t) + \beta(Y - 1) \sum_{X=0}^{\infty} XP(X, Y - 1, t) - \beta Y \sum_{X=0}^{\infty} XP(X, Y, t) \quad (7)$$

Equations (6) and (7) play the central role in the following proof.

Suppose there is a steady-state distribution of probabilities; then

$$\dot{B}(Y, t) = 0 \qquad \text{for} \quad Y = 0, 1, 2, \ldots$$

$$\dot{A}(X, t) = 0 \qquad \text{for} \quad X = 0, 1, 2, \ldots$$

Now we consider

$$0 = \dot{B}(0, t) = K_2(0 + 1)B(0 + 1, t) - K_2 \cdot 0 \cdot B(0, t) + \beta(0 - 1) \sum_{X=0}^{\infty} XP(X, 0 - 1, t) - \beta \cdot 0 \cdot \sum_{X=0}^{\infty} XP(X, 0, t) = K_2 B(1, t)$$

Hence

$$B(1, t) = \sum_{X=0}^{\infty} P(X, 1, t) = 0 \quad (8)$$

Since all the probabilities are positive,

$$P(X, 1, t) = 0 \qquad \text{for} \quad X = 0, 1, 2, \ldots \quad (9)$$

Next we consider

$$0 = \dot{B}(1, t) = K_2 \cdot 2 \cdot B(2, t) - K_2 \cdot 1 \cdot B(1, t) + \beta(1 - 1) \sum_{X=0}^{\infty} XP(X, 0, t) - \beta \cdot 1 \sum_{X=0}^{\infty} P(X, 1, t)$$

Since from (8) we have

$$P(X, 1, t) = 0, \qquad B(1, t) = 0$$

we get $B(2, t) = \sum_{X=0}^{\infty} XP(X, 1, t) = 0$ and due to the positivity of probabilities

$$P(X, 2, t) = 0 \qquad \text{for all} \quad X = 0, 1, \ldots$$

Proceeding this way, we get

$$P(X, Y, t) = 0 \qquad \text{for all} \quad X = 0, 1, \ldots \quad \text{and} \quad Y = 1, 2, \ldots$$

Thus the only possible nonzero probabilities left are $P(X, 0, t)$. We will show that these also vanish unless $X = 0$, i.e., the only nonzero probability in the steady state is $P(0, 0, t)$, which must equal unity.

To show this, we consider Eq. (6) for $X = 1$:

$$0 = A(1, t) = K_1(1 - 1)A(1 - 1, t) - K_1 \cdot 1 \cdot A(1, t)$$
$$+ \beta(1 + 1 \sum_{Y=0}^{\infty} YP(2, Y, t) - \beta \cdot 1 \cdot \sum_{Y=0}^{\infty} YP(1, Y, t)$$

The last two terms vanish because it has been shown that $P(2, Y, t) = P(1, Y, t) = 0$ for $Y = 1, 2, 3, \ldots$ and for $Y = 0$, they are weighted by zero. Hence $A(1, t) = \sum_{Y=0}^{\infty} P(1, Y, t) = 0$ and by the positivity of probabilities, $P(1, Y, t) = 0$ for $Y = 0, 1, 2$. We continue in the same way to show that

$$P(X, Y, t) = 0 \qquad \text{for all} \quad X = 1, 2, 3, \ldots \quad \text{and} \quad Y = 0, 1, 2, 3, \ldots$$

Hence the only nonzero probability left in the steady state is $P(0, 0, t) = 1$.

CONCLUSION

We conclude that the result of Nicolis and Prigogine obtained from the consideration of the approximate Fokker–Planck equation is rigorously true—the stochastic Volterra–Lotka equation does not have a steady-state solution other than the one in which both the prey and predators are extinct. It is quite possible but unlikely that a different identification of birth and death terms will give nontrivial asymptotic steady-state distributions.

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