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Critical Exponents and Large-Order Behavior of Perturbation Theory

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The principles of the recent calculations of critical exponents from three- and two-dimensional field theory are reviewed. They rely on the Callan–Symanzik equations, diagram calculations, and on the characterization of the asymptotic behavior of perturbation series at large order. We then present new results concerning the normalization of the large-order behavior.

KEY WORDS: Critical exponents; perturbation theory; asymptotic behavior; Callan–Symanzik equations.

1. INTRODUCTION

Field theoretic techniques applied to the problem of critical phenomena have recently been able to produce very accurate values for the three-dimensional critical indices.(1) In addition to all the previously available information, they involve recent progress concerning the quantitative characterization of the asymptotic orders of the perturbation series, when the order goes to infinity.(2) In this article we report some new results concerning this large-order problem, but it seemed to us that it might be useful at this stage to summarize the various logical steps combined in these 3D calculations. This article thus contains two very different parts. In the first, we expose without any derivation the set of principles underlying these calculations. In the second, we present the computation of a Fredholm determinant which occurs when one looks for the absolute normalization of the coefficients of very large orders of perturbation theory.

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2. PRINCIPLES OF THE THREE-DIMENSIONAL CALCULATIONS OF CRITICAL INDICES FROM FIELD THEORY

It is convenient to distinguish the following steps:

1. Use of Wilson's renormalization group theory.
2. Wilson's Feynman graph approach, which leads to the understanding that the Callan–Symanzik equations govern the scaling properties near T_c and may be used to generate the ϵ expansion.
3. Direct use of the Callan–Symanzik equations in three dimensions.
4. Systematic calculation of several orders of the perturbation series (up to six loops) in three dimensions.
5. The obtaining of the large-order behavior of the perturbation series by instantons. They govern the instability of the vacuum when the coupling constant becomes attractive and reveal the nature of the divergence of the series.
6. Summation techniques based on the large-order information, which lead to accurate estimates from the perturbation series though the series is divergent and the fixed-point value of the expansion parameter is of order unity.

2.1. Step 1

The general ideas of Wilson's renormalization group approach are presented in great detail in several books and articles[3] and need not be repeated here.

2.2. Step 2

Similarly, the basis of the use of Callan–Symanzik equations in the problem of critical phenomena has been exposed at length elsewhere.[4] However, it may be useful to restate here the following features. In the Landau–Ginzburg–Wilson theory, an N-component order parameter $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$ is introduced. Its spatial distribution in d-dimensional space $\boldsymbol{\phi}(x)$ is weighted by a probability measure

$$\exp(-S\{\phi\})$$

with

$$S\{\phi\} = \int d^d x \, \{\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2} r_0 \boldsymbol{\phi}^2 + \tfrac{1}{4} u_0 (\boldsymbol{\phi}^2)^2\} \tag{1}$$

In addition, if $\phi(x)$ is decomposed into Fourier components, the original lattice spacing a of the underlying magnetic model forbids variation of $\phi(x)$ with a wavelength smaller than a and therefore cuts off the wavenumbers at a value $\Lambda \sim 1/a$. The M-point correlation functions given by the expectation value [with the respect to the normalized weight $\exp(-S)$] $\langle\phi_{\alpha_1}(x_1) \cdots \phi_{\alpha_M}(x_M)\rangle$ depend on r_0, u_0, Λ. It is convenient to eliminate r_0 at the benefit of the correlation length ξ. Near the critical point ξ is very large and we are interested in correlation functions in the regime $|x_i - x_j| \gg a$, $\xi \gg a$. In addition, u_0 is a dimensional parameter proportional to a^{d-4} and thus it becomes very large in the limit of interest in less than four dimensions. It is therefore convenient to introduce a renormalized coupling constant g instead of u_0 and to modify the scale of the correlation functions. This may be done in the following way. Let

$$\delta(p_1 + \cdots + p_M)G_M(p_1, \alpha_1;\ldots; p_M, \alpha_M)$$
$$= \int \langle\phi_{\alpha_1}(x_1) \cdots \phi_{\alpha_M}(x_M)\rangle_c \exp[i(p_1x_1 + \cdots + p_Mx_M)]\, dx_1 \cdots dx_M \quad (2)$$

the M-point, connected (i.e., it contains only connected diagrams) correlation function in momentum space. We define the correlation length ξ and the field strength Z by parametrizing the behavior of G_2 near zero momentum by

$$G_2(p, \alpha; -p, \beta) = Z\frac{\delta_{\alpha\beta}}{1/\xi^2 + p^2 + O(p^4)} \quad (3)$$

and the dimensionless renormalized coupling constant g by

$$G_4(0, \alpha; 0, \beta; 0, \gamma; 0, \delta) = \xi^{4+d}2Z^2g(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \quad (4)$$

The renormalized correlation functions defined as

$$G_M^{(R)} = Z^{-M/2}G_M \quad (5)$$

may be expressed in terms of ξ and g (instead of u_0 and r_0) and they have a finite limit when the lattice spacing a vanishes.

One of the main advantages of this parametrization is that the dimensionless renormalized coupling g remains finite[4] also when the bare dimensionless coupling $u_0\xi^{4-d}$ goes to infinity.

These renormalized functions $G_M^{(R)}$ satisfy a first-order partial differential equation, first obtained by Callan and Symanzik[(6)] in the following way. Let us vary the temperature scale r_0, or equivalently the correlation length ξ with fixed values for u_0 and the lattice spacing a. The coupling constant g

[4] This has been rigorously proved for $N = 1$ in Ref. 5, using the Lebowitz inequality.

defined by (4) is modified by this variation. Therefore we perform the derivation

$$Z^{-M/2}\xi \frac{d}{d\xi}\bigg|_{u_0,a} Z^{M/2}G_M^{(R)}$$

and obtain from (5)

$$\left\{\xi \frac{\partial}{\partial \xi} + \xi \frac{\partial g}{\partial \xi}\bigg|_{u_0,a} \frac{\partial}{\partial g} + \frac{M}{2}\xi \frac{\partial}{\partial \xi}\bigg|_{u_0,a} \ln Z\right\} G_M^{(R)} = Z^{-M/2}\frac{\partial r_0}{\partial \xi}\bigg|_{u_0,a}\frac{\partial G_M}{\partial r_0}$$

or

$$\left\{\xi \frac{\partial}{\partial \xi} - \beta(g)\frac{\partial}{\partial g} - \frac{M}{2}\eta(g)\right\} G_M^{(R)} = \Delta G_M^{(R)} \tag{6}$$

in which we have defined

$$\beta(g) = -\xi \frac{\partial g}{\partial \xi}\bigg|_{u_0,a} \qquad \text{and} \qquad \eta(g) = -\xi \frac{\partial}{\partial \xi}\bigg|_{u_0,a} \ln Z \tag{7}$$

For momenta large compared with the inverse correlation length ξ^{-1} (but of course small compared to the inverse lattice spacing Λ, which has gone to infinity) the right-hand side is negligible compared to the left-hand side (technically this is true order by order in a double expansion in powers of g and $4 - d$). The reason is that $\partial G_M/\partial r_0$ contains only diagrams with one propagator squared which fall off more rapidly when the external momenta go to infinity. Therefore in this regime $a^{-1} \gg |p_i| \gg \xi^{-1}$, $G_M^{(R)}$ goes to $G_{M,\text{as}}^{(R)}$, which is a solution of

$$\left\{\xi \frac{\partial}{\partial \xi} - \beta(g)\frac{\partial}{\partial g} - \frac{M}{2}\eta(g)\right\} G_{M,\text{as}}^{(R)} = 0 \tag{8}$$

In a field theory g can be chosen arbitrarily. In this problem it is expressed in terms of r_0, u_0, and a. In the limit $u_0^{1/(4-d)}a \to 0$ one shows that

$$g \to g^*: \qquad \beta(g^*) = 0 \tag{9}$$

Consequently $G_{M,\text{as}}^{(R)}(\xi, g^*)$ is proportional to $\xi^{M\eta(g^*)/2}$. For the two-point function, noting that $G_2^{(R)}(p, \xi, g^*)$ is dimensionally of the form $(1/p^2)f(p\xi)$, this leads to

$$G_2^{(R)}(p, \xi) \underset{p\xi \gg 1}{\sim} C(p\xi)^{\eta(g^*)}/p^2 \tag{10}$$

from which one sees that $\eta(g^*)$ is the usual η exponent. All the scaling laws and properties have been derived from this procedure. It relies on the existence of a fixed point g^* such that $\beta(g^*) = 0$ and its stability requires

$\beta'(g^*) > 0$. It is a crucial and implicit assumption that the renormalized Green's functions are finite for $g = g^*$. Near four dimensions, if $d = 4 - \epsilon$,

$$\beta(g) = -\epsilon g + \frac{N+8}{6} g^2 + O(\epsilon g^2, \epsilon^3), \qquad \eta(g) = \frac{N+2}{72} g^2 + O(\epsilon g^2, \epsilon^3) \tag{11}$$

and thus there is a stable fixed point

$$g^* = \frac{6\epsilon}{N+8} + O(\epsilon^2), \qquad \eta(g^*) = \frac{N+2}{2(N+8)^2} \epsilon^2 + O(\epsilon^3) \tag{12}$$

Higher orders in this direction have led to the calculations of the critical exponents as ϵ series. It is now known that these series are divergent.[14] However, they give very reasonable values for $\epsilon = 1$, either if truncated after two or three terms, or if resummed adequately.

2.3. Callan–Symanzik Equations in Three Dimensions

The ϵ expansion has been introduced for two purposes. It allows one to define a critical theory by neglecting the right-hand side of Eq. (6). Second, it gives a small parameter for the fixed point g^* and thus for the critical exponents. The $1/N$ expansion has the same properties and allows one to define a critical theory directly in three dimensions. It has been proposed in Ref. 6 that the existence of a three-dimensional critical theory implies that the right-hand side of (6) should be globally negligible for large $p\xi$, even if it is not true order by order in g. This may be supported by the ϵ expansion, which together with the short distance expansion allows one to show that, to all orders in ϵ,

$$\Delta G_2^{(R)}/G_2^{(R)} \underset{p\to\infty}{\sim} Ap^{-1/\nu} + Bp^{-(1-\alpha)/\nu} \tag{13}$$

Therefore, extrapolating to three dimensions, as long as α remains smaller than one, the right-hand side of (6) can be globally neglected. The problem is reduced to a direct computation of $\beta(g)$ and $\eta(g)$, and then to the search of the fixed point g^* defined by $\beta(g^*) = 0$, $\beta'(g^*) > 0$, which is now a finite number, and finally to the computation of $\eta = \eta(g^*)$. Similar equations hold for ν and the other critical exponents.

2.4. Three-Dimensional Calculations

After the promising initial calculations of Ref. 7 a very extensive program of calculating the power series expansion of the functions $\beta(g)$, $\eta(g)$, and $\eta_4(g)$, from which one deduces η and ν, was initiated by Nickel[8] and extended in Ref. 9. All Feynman diagrams involving at most six loops, that is, a priori 18-dimensional integrals, have been computed, together with their symmetry factor, which depends on N, the dimension of the order parameter. In order to decrease the number of integrals for a given diagram, these authors used the fact that one can compute analytically in three dimensions any one-loop subgraph with an arbitrary number of lines leaving the loop. The technical details are presented in Ref. 10. The total number of diagrams is 1142. Similarly these authors have computed the same functions in two dimensions up to four loops. The direct use of these results to calculate critical exponents is not possible. The series are badly divergent and the fixed point is a finite number. However, the control of the nature of the divergence of the series turns out to be a powerful tool for handling this problem.

2.5. Large Order Behavior of Perturbation Series

Perturbation series in many problems of quantum mechanics or field theory are asymptotic but divergent for any value of the coupling constant. The origin in the complex coupling constant plane is an essential singularity and the first indication of this fact was given by Dyson,[11] who argued that in quantum electrodynamics, if one changes the sign of the fine structure constant α, the vacuum would become unstable since electron–positron pairs would be pulled out of the vacuum. This argument has recently been made quantitative and it can be applied to many problems.

Let us first discuss a very simple example from quantum mechanics. Consider the one-dimensional anharmonic oscillator

$$H = \tfrac{1}{2}(p^2 + x^2) + gx^4 \tag{14}$$

and imagine that we want to calculate the ground-state energy in perturbation

$$E(g) = \tfrac{1}{2} + \sum_1^\infty g^K E_K \tag{15}$$

The E_K have been computed up to $K = 150$ (Ref. 12); $E_1 = 0.75$, $E_2 = -2.625$, $E_3 = 20.8125$, $E_4 = -241.289,\ldots$, but E_{75} is already of order 10^{144}. The series (15) looks divergent. This, and its rate of divergence, can be shown in the following way. Let us express $E(g)$ as the zero-temperature limit of the free energy

$$E(g) = \frac{1}{2} + \lim_{\beta\to\infty} -\frac{1}{\beta}\log\frac{\operatorname{Tr} e^{-\beta H}}{\operatorname{Tr} e^{-\beta H_0}} \tag{16}$$

and express the partition function by the Feynman–Kac formula

$$F(g) = \frac{\operatorname{Tr} e^{-\beta H}}{\operatorname{Tr} e^{-\beta H_0}} \propto \int \mathscr{D}x(\tau)\exp\left\{-\int_0^\beta \left[\frac{1}{2}(\dot{x}^2 + x^2) + gx^4\right]\right\} dt \tag{17}$$

in which we integrate over all periodic paths such that $x(0) = x(\beta)$. The function $F(g)$ is analytic in the complex g plane cut along the negative real axis and for large g behaves like $g^{1/3}$. The coefficients of its expansion in powers of g, $F(g) = \sum F_K g^K$, may be expressed as the moments of the discontinuity of $F(g)$, i.e.,

$$F_K = \int_C \frac{dg}{2i\pi} g^{-(K+1)} F(g) \tag{18}$$

in which the contour C encloses the negative real axis. If we substitute the representation (17) of F, we obtain for F_K an integral over g and paths, which for large K can be evaluated by the saddle-point method. The saddle points are given by the conditions

$$\begin{pmatrix} \delta/\delta g \\ \delta/\delta x(t) \end{pmatrix} \left\{ K \log g + \int_0^\beta dt \left[\frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 + g x^4 \right] \right\} = 0$$

which are

$$\ddot{x}_c = x_c + 4 g_c x_c^3 \tag{19}$$

$$K/g_c = -\int_0^\beta dt\, x_c^4(t) \tag{20}$$

From the second equation we see that g_c is, as expected, negative, and through the rescaling

$$x_c = (-g_c)^{-1/2} y_c \tag{21}$$

these equations become

$$\ddot{y}_c = y_c - 4 y_c^3 \tag{22}$$

$$g_c = -(1/K) \int_0^\beta dt\, y_c^4 \tag{23}$$

which show that g_c is infinitesimal for K large. It is convenient to introduce a mechanical analog to depict Eq. (22), which represents the motion in time of a particle located at $y_c(t)$ moving in the potential $V = (-\frac{1}{2} y_c^2 + y_c^4)$ (Fig. 1). We look for periodic solutions, and an elementary calculation shows

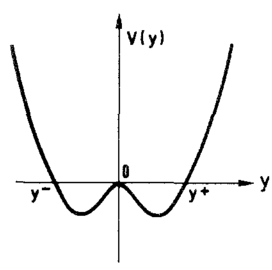

Fig. 1

that the leading contribution is given by the paths of period β of minimal action since

$$S(x_c, g_c) = K \log g_c + \int_0^\beta dt\,(\tfrac{1}{2}\dot{x}_c^2 + \tfrac{1}{2}x_c^2 + g_c x_c^4) \tag{24}$$

and

$$\exp[-S(x_c, g_c)] = [\exp(K \ln K - K)]/\left[-\int^\beta y_c^4\, dt\right]^K \tag{25}$$

A periodic path in this potential corresponds to a fixed energy, and in the large-β limit the lowest action corresponds to the periodic path that starts infinitesimally close to the origin, goes to y_+ (or y_-), and comes back:

$$y_c(t) = \pm\sqrt{\tfrac{1}{2}}\,[1/\cosh(t - t_0)] \tag{26}$$

$$\int_{-\infty}^{+\infty} y_c^4\, dt = \tfrac{1}{3} \tag{27}$$

It has been named "instanton" or "pseudoparticle" in the literature. We thus obtain

$$F_K \underset{K\to\infty}{\sim} K!\,(-3)^K \tag{28}$$

Fluctuations around the saddle point (22)–(23) may be systematically calculated. The parameter of this saddle-point expansion is $1/K$. Equation (22) is invariant under time translation, and the origin t_0 on the periodic trajectory is arbitrary. One may integrate properly over t_0; this may be done by the method of collective coordinates. The result is

$$F_K \underset{K\to\infty}{\sim} \beta\Gamma(K + \tfrac{1}{2})(-3)^K(6/\pi^{3/2})[1 + O(1/K)] \tag{29}$$

We now have to take the logarithm of $F(g)$ and divide by $-1/\beta$ to obtain $E(g)$. At large order, this is very easy. Indeed

$$\begin{aligned}&\log(1 + F_1 g + \cdots + F_K g^K + \cdots)\\ &\qquad = F_1 g + \cdots + g^K[F_K - F_1 F_{K-1} + F_{K-2}(F_2 - F_1^2) + \cdots]\end{aligned}$$

and since F_K grows like $K!$, F_{K-1}/F_K is of order $1/K$, F_{K-2}/F_K of order $1/K^2$, etc. Therefore we can neglect all these terms. A diagrammatic way of visualizing this property consists in noticing that the diagrams that contribute to the free energy are only the connected ones and at large orders there is only a fraction $1/K$ of disconnected diagrams. This completes the asymptotic calculation of E_K:

$$E_K \underset{K}{\sim} -\Gamma(K + \tfrac{1}{2})(-3)^K(6/\pi^{3/2})[1 + O(1/K)] \tag{30}$$

The same result can be obtained also by using the traditional WKB method.[12] However, it is possible to show that in the one-dimensional quantum mechanical case, the approach presented here and the WKB method are deeply connected and they give identically the same answer. This makes manifest the divergence of the perturbation series of $E(g)$.

Let us apply the same procedure to the Landau–Ginzburg–Wilson system. The M-point functions are given by the functional integral

$$G_M(x_1, \alpha_1; \ldots; x_M, \alpha_M) = \frac{\int \mathscr{D}\varphi\varphi_{\alpha_1}(x_1) \cdots \varphi_{\alpha_M}(x_M) \exp(-S\{\varphi\})}{\int \mathscr{D}\varphi \exp(-S\{\varphi\})} \tag{31}$$

$$S(\varphi) = \int d^dx \{\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \tfrac{1}{2}m^2\boldsymbol{\varphi}^2 + \tfrac{1}{4}gm^{4-d}(\boldsymbol{\varphi}^2)^2 + \tfrac{1}{2}(r_0 - m^2)\boldsymbol{\varphi}^2\} \tag{32}$$

in which m is the inverse of the magnetic susceptibility χ. The Kth order of the expansion of G_M in powers of g is obtained by a contour integral as in Eq. (17), and we apply the saddle-point method to the integrations over g and $\boldsymbol{\varphi}(x)$,

$$\begin{pmatrix} \delta/\delta g \\ \delta/\delta\varphi^\alpha(x) \end{pmatrix} \left\{ K \log g + \int d^dx \, [\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2 + \tfrac{1}{2}m^2(\boldsymbol{\varphi}^2) + \tfrac{1}{4}gm^{4-d}(\boldsymbol{\varphi}^2)^2] \right\} = 0 \tag{33}$$

The mass counter-term of Eq. (32) is only relevant at the subleading order, which is discussed below. The variational equations are then

$$\frac{K}{g_{0c}} + \tfrac{1}{4}m^{4-d} \int d^dx \, (\boldsymbol{\varphi}_c{}^2)^2 = 0, \qquad \Delta\varphi_c{}^\alpha = m^2\varphi_c{}^\alpha + g_c m^{4-d}(\boldsymbol{\varphi}_c{}^2)\varphi_c{}^\alpha$$

or with the rescaling

$$\varphi_c{}^\alpha(x) = \frac{m^{(d-1)/2}}{(-g_c)^{1/2}} \Phi_c{}^\alpha(mx) \tag{34}$$

we obtain

$$g_c = -\frac{1}{4K}\int (\Phi_c{}^2)^2 \, d^dx \tag{35}$$

$$\Delta\Phi_c{}^\alpha = \Phi_c{}^\alpha - 4\Phi_c{}^2\Phi_c{}^\alpha \tag{36}$$

Here again g_c is negative and it goes to zero when K goes to infinity. At the saddle point, an easy calculation gives

$$\left(\frac{1}{g_c}\right)^K \exp[-S(\varphi_c)] = \left[-\frac{4}{\int (\boldsymbol{\Phi}_c{}^2)^2 \, d^dx}\right]^K \exp(K \log K - K) \tag{37}$$

showing a behavior very similar to that of the anharmonic oscillator. The leading contribution is thus given by the instanton, the solution of (36) that minimizes the integral of $(\Phi^2)^2$. This solution is spherically symmetric,[13] nodeless, and corresponds to a fixed direction in internal space

$$\Phi_c^{\alpha} = u^{\alpha} f(x), \qquad \mathbf{u}^2 = 1 \tag{38}$$

$$\left(\frac{d^2}{dx^2} + \frac{d-1}{x}\frac{d}{dx}\right) f(x) = f(x) - f^3(x) \tag{39}$$

At large distance $f(x)$ decreases as $e^{-x}x^{-(d-1)/2}$. The solution has been determined by numerical computation in two and three dimensions.[14]

This shows that any Green's function of the theory has an expansion in powers of g which at large order behaves as $g^K \Gamma(K + b) a^K c$. The number a is the same for all Green's functions independent of the number of external legs of the values of the external momenta:

$$a = -\left[\tfrac{1}{4}\int (\Phi_c^2)^2\, d^d x\right]^{-1} \tag{40}$$

The number b requires a more detailed calculation; for an M-point function

$$b = \tfrac{1}{2}(M + N + d - 1) \tag{41}$$

The constant c is a function of all the variables, the external momenta, etc. It is calculated in the second part of this article for the functions that appear in the Callan–Symanzik equations.

2.6. Large-Order Behavior and Summation Techniques

We have seen in the previous section that perturbation theory is divergent for any value of the coupling constant. However, with the use of the large-order information, it is possible in various ways to obtain convergent algorithms. Let us study as an example the problem of the ground-state energy of the anharmonic oscillator of Eqs. (14)–(15) for a coupling constant equal to unity. If we use the truncated series for $g = 1$ the result is absurd. However, since we know that E_K grows as $K!$, it is indicated that we perform a Borel transformation

$$E(g) = \int_0^{\infty} dt\, e^{-t} \sum_0^{\infty} (tg)^K \frac{E_K}{K!} \tag{42}$$

(in which we recover the ordinary perturbation series if the summation and integration are interchanged). There is actually a proof[15] that $E(g)$ is indeed given by (41). The Borel transformation of $E(g)$

$$f(b) = \sum_0^{\infty} b^K \frac{E_K}{K!} \tag{43}$$

is analytic in a circle of radius 1/3 as indicated by (28) and the singularity closest to the origin in the b plane is at $b = -1/3$. The function $f(b)$ is furthermore analytic in the vicinity of the real, positive axis. If we replace $f(b)$ by

$$f_L(b) = \sum_0^L b^K \frac{E_K}{K!} \tag{44}$$

we recover the perturbation series truncated at order L. But we can replace the function $f_L(b)$ by a rational function, a Padé approximant, before performing the integration (42). In Ref. 1 the authors have taken as an example $L = 6$; the error between the approximate value of $E(1)$ and the exact one (which can be obtained by variational methods) is 10^{-3}.

However, we can use more than just the $K!$ of E_K and notice that Eq. (30) implies that $f(b)$ has a square root branch point at $b = -1/3$. Assuming analyticity in the whole b plane cut along the negative axis from $-1/3$ to $-\infty$, we can map the whole b plane in the interior of a circle by the conformal mapping

$$z = [(1 + 3b)^{1/2} - 1]/[(1 + 3b)^{1/2} + 1] \tag{45}$$

The natural representation of the corresponding function of z is a Taylor expansion and the knowledge of L coefficients E_K determines the L first coefficients a_K of the representation

$$\tilde{f}_L(b) = \sum_0^L a_K z^K(b) \tag{46}$$

which is then transformed by (42). With the same value $L = 6$ the error drops to 3×10^{-4}. It is thus manifest that the large-order behavior is a useful systematic guide to transforming the knowledge of the low-order coefficients into a modified convergent scheme.

These ideas have been developed and applied with success in Ref. 1 to the calculation of the critical exponents. The problem is exactly of the same nature. The fixed point is of order unity. The series diverges in the same fashion. However, the critical exponents have been computed with an accuracy which is higher than that achieved by all previous methods.

3. NORMALIZATION OF THE LARGE-ORDER ESTIMATES

Previously we have shown that the coefficients of all the Green's functions at large order behave as

$$g^K \Gamma(K + b) a^K c[1 + O(1/K)] \tag{47}$$

The calculation of a relies on the value of the classical action for the instanton solution. We present here the calculation of b and c for the functions that appear as coefficients in the Callan–Symanzik equations, from which the critical exponents are computed. The technique involves the calculation of the contribution to the functional integral of the Gaussian fluctuations near the instanton solution. The main body of the calculation is to evaluate the determinant of the quadratic form of the fluctuations around the instanton, namely

$$\Delta_L = \det\left\{1 + \frac{1}{-\Delta + 1}\left[-3\Phi_c^2(x)\right]\right\} \tag{48}$$

$$\Delta_T = \det\left\{1 + \frac{1}{-\Delta + 1}\left[-\Phi_c^2(x)\right]\right\} \tag{49}$$

in which we have distinguished between longitudinal fluctuations in the direction of a given classical solution and transverse fluctuations along perpendicular directions in internal space; Δ_T is needed only if $N \neq 1$.

Some caution is, however, needed in order to take into account properly the zero modes. Indeed the classical equation (36) has an infinite set of lowest action solutions which differ from one another by a translation, or an $O(N)$ rotation in internal space. It follows from these invariances that (i) the operator $[-\Delta + 1 - 3\Phi_c^2(x)]$ has a zero eigenvalue d-fold degenerate, the corresponding eigenvectors being $\partial\Phi_c(x)/\partial x_\alpha$, $\alpha = 1, 2,\ldots, d$; (ii) the operator $[-\Delta + 1 - \Phi_c^2(x)]$ also has a zero mode corresponding to the eigenfunction Φ_c itself.

These modes should be properly quantized, by using the collective coordinates method,[16] which performs the integral exactly (without Gaussian approximation) in the direction in which the action $S\{\Phi\}$ remains constant. The result is then the product of a simple Jacobian corresponding to the collective coordinate change of variable and of the determinants of the operators restricted to the subspace orthogonal to the zero eigenmodes $\Delta_L^\perp, \Delta_T^\perp$.

This gives for the $2M$-point function at zero external momenta

$$[G_{2M}(\mathbf{0}, n;\ldots; \mathbf{0}, n)]_K = [\exp(-S_{\text{cl}})]J\left[\frac{4KI_1^2}{I_4}\right]^M \frac{\Gamma(M + 1/2)\Gamma(N/2)}{\sqrt{\pi}\,\Gamma(M + N/2)}$$
$$\times (\Delta_L^\perp)^{-1/2}(\Delta_T^\perp)^{-(N-1)/2}\{\exp[-\tfrac{1}{2}(N + 2)I_2G_2^F(0)]\} \tag{50}$$

in which

$$\exp(-S_{\text{cl}}) = \frac{K!}{(2\pi K)^{1/2}}\left(\frac{4}{I_4}\right)^K\left[1 + O\left(\frac{1}{K}\right)\right] \tag{51}$$

$$I_p = \int d^dx\,[\Phi_c(x)]^p \tag{52}$$

and the Jacobian

$$J = \frac{\sqrt{2}\,\pi^{(N-1)/2}}{\Gamma(N/2)} K^{[(N+d)/2]-1}(2\pi)^{-d/2-(N-1)/2} \tag{53}$$

The mass counterterm in the action is responsible for the last factor in (50) and it involves the divergent integral

$$G_2^F(0) = \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2+1} \tag{54}$$

which will be discussed below.

The calculation of the integrals I_p is done by numerical integration and the problem is reduced to that of computing the $\Delta^\perp$. We found it convenient to introduce the function

$$D(z) = \det\left[1 - \frac{3z}{-\Delta+1}\,\Phi_c^2(x)\right] \tag{55}$$

from which we recover $\Delta_L^\perp$ and $\Delta_T^\perp$ as

$$\bar{D}(1) = \lim_{z\to 1} \frac{D(z)}{(1-z)^d} = \Delta_L^\perp \lim_{z\to 1} \frac{\det\{1-[3z/(-\Delta+1)]\Phi_c^2\}}{(1-z)^d} \tag{56}$$

in which the last determinant is d over d in the subspace of the longitudinal zero modes $\partial_\alpha \Phi_c$. This leads to

$$\Delta_L^\perp = \bar{D}(1)\left[\frac{\frac{1}{4} d I_4}{I_6 - I_4}\right]^d \tag{57}$$

Similarly we obtain

$$\Delta_T^\perp = \bar{D}(1/3)/4 \tag{58}$$

$$\bar{D}(1/3) = \lim_{z\to 1/3} [D(z)/(1-3z)] \tag{59}$$

This determinant $D(z)$ may be expressed in terms of the eigenvalues[17] λ_n defined by the equation

$$(-\Delta\psi_n + \psi_n) = \lambda_n 3\Phi_c^2(x)\psi_n \tag{60}$$

as

$$D(z) = \prod_n \left(1 - \frac{z}{\lambda_n}\right) \tag{61}$$

in which it is understood that an m-fold degenerate eigenvalue appears m times in the product. In two or more dimensions this infinite product is divergent. The mass counterterm of Eq. (50) is also infinite, but since

$$\sum_n (1/\lambda_n) = G_2^F(0) 3 I_2 \tag{62}$$

as may be checked by expanding $D(z)$ at first order in z, there is an exact cancellation in less than four dimensions when we combine the mass counterterm and the divergent part of $D(z)$ at $z = 1$. We can thus eliminate the counterterm contribution from (50), absorb it into $D(z)$, and define the renormalized convergent determinant

$$D_R(z) = \prod_n \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n} \tag{63}$$

The convergence of the infinite product holds provided that

$$\sum_n 1/\lambda_n{}^2 < \infty$$

which is true below four dimensions since the sum on the left-hand side is related to the one-loop diagram with four external legs, which is finite.

In order to perform the numerical computation of $D_R(z)$ we can calculate explicitly the lowest λ_n's and perform the product (63). However, it is important to improve this procedure by evaluating the asymptotic behavior of the large λ_n's. Let us discuss in some detail the three-dimensional case.

All the eigenvalues are real and positive, since the operator $(-\Delta + 1)^{-1/2}\Phi_c{}^2(x)(-\Delta + 1)^{-1/2}$ is Hermitian and positive. Let $N(\lambda)$ be the number of eigenvalues smaller than λ. Equivalently $N(\lambda)$ may be defined as the number of eigenvalues of the Schrödinger problem

$$H = p^2 - 3\lambda\Phi_c{}^2(x) \tag{64}$$

with an energy smaller than minus one. The asymptotic behavior of $N(\lambda)$ for λ large may thus be obtained from this last remark, since it is known that the asymptotic number $B(V)$ of bound states in a large attractive potential $-V$ is correctly given by the Thomas–Fermi approximation as[18]

$$B(V) \sim (1/6\pi^2) \int d^3x\, V^{3/2}(x) + O(V^{1/2}) \tag{65}$$

In the present problem since $V(x) = 3\lambda\Phi_c{}^2(x)$ this gives

$$N(\lambda) \underset{\lambda\to\infty}{\sim} (3^{1/2}/2\pi^2) I_3 \lambda^{3/2} \equiv C\lambda^{3/2} \tag{66}$$

The method will thus consist in using this asymptotic information together with the explicitly computed lowest eigenvalues. This procedure turns out to be quite useful in order to improve the convergence, as will be shown below. Specifically, this is done in the following way. Let $\rho(\lambda) = dN/d\lambda$ be the exact density of eigenvalues and $\rho_{as}(\lambda) = \frac{3}{2}C\lambda^{1/2}$. Then from (63) we may write

$$D_R(z) = \exp \int_0^\infty d\lambda\, \rho(\lambda)\left[\ln\left(1 - \frac{z}{\lambda}\right) + \frac{z}{\lambda}\right]$$
$$\simeq \exp \int_0^{\bar\lambda} d\lambda\, \rho(\lambda)\left[\ln\left(1 - \frac{z}{\lambda}\right) + \frac{z}{\lambda}\right]$$
$$\times \exp\left\{\frac{3}{2}\, C \int_{\bar\lambda}^\infty d\lambda\, \lambda^{1/2}\left[\ln\left(1 - \frac{z}{\lambda}\right) + \frac{z}{\lambda}\right]\right\} \tag{67}$$

which, for large $\bar\lambda$, reduces to

$$D_R(z) = \left\{\prod_{\lambda_n < \bar\lambda}\left(1 - \frac{z}{\lambda_n}\right)\exp\frac{z}{\lambda_n}\right\}\exp\left(-\frac{3}{2}\, Cz^2\bar\lambda^{-1/2}\right) \tag{68}$$

Given the numerical value

$$C = (3^{1/2}/2\pi^2)I_3 = 2.78083 \tag{69}$$

this gives a correction factor to the truncated product which is about 50% for $\bar\lambda \sim 50$ at $z = 1$. Furthermore, in order to avoid numerical oscillations when $\bar\lambda$ passes through an eigenvalue, the correction factor may be replaced by an asymptotically equivalent one, namely

$$D_R(z) = \left\{\prod_{n=1}^{N}\left(1 - \frac{z}{\lambda_n}\right)\exp\frac{z}{\lambda_n}\right\}\exp\left(-\frac{3}{2}\, C^{4/3}N^{-1/3}z^2\right) \tag{70}$$

in which it is understood that the λ_n are ordered increasingly with n. The lowest 900 λ_n counted with their multiplicity have been computed by solving the Schrödinger equation (64) expanded in partial waves up to $l_{\max} = 14$, which corresponds to a $\bar\lambda$ of about 48. In Table I we reproduce the results for the lowest λ_n up to $\bar\lambda = 20$.

Table I. Eigencouplings of $(-\Delta + 1 - 3\lambda\Phi_c^2)$ up to $\bar\lambda = 20$

$l = 0$	1	2	3	4	5	6	7	8
0.33333	1.00000	2.11863	3.69159	5.71487	8.18630	11.1049	14.4703	18.2822
1.38758	2.46066	4.02308	6.06067	8.55922	11.5114	14.9137	18.7645	
3.11042	4.57446	6.56661	9.05841	12.0251	15.4529	19.3353		
5.49860	7.34474	9.75643	12.6936	16.1213				
8.55053	10.7730	13.5965	16.9715					
12.2653	14.8599	18.0894						
16.6423	19.6059							

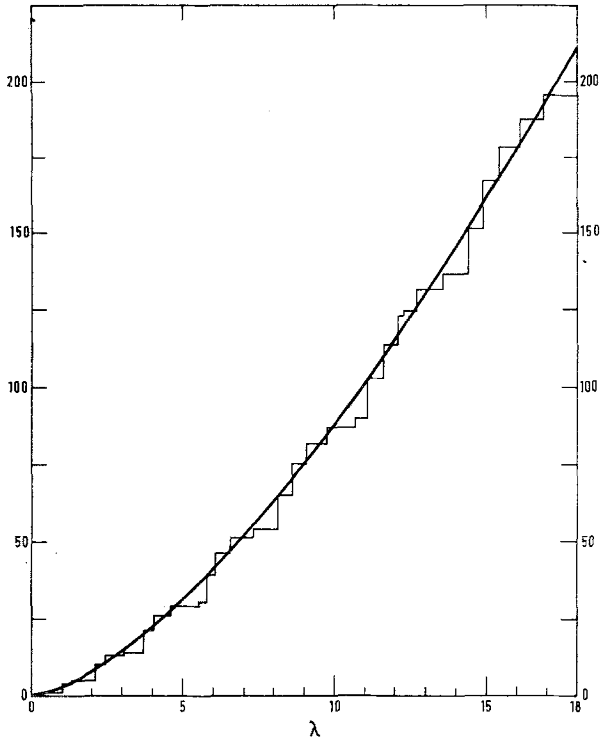

Fig. 2

In order to see how the asymptotic regime (66) is reached, we have represented the true $N(\lambda)$ in Fig. 2 and compared it with $N_{as}(\lambda)$. The results are

$$\bar{D}_R(1) = 10.544 \pm 0.004 \tag{71a}$$

$$\bar{D}_R(1/3) = 1.4571 \pm 0.004 \tag{71b}$$

in which the errors come from the extrapolation from $N = 900$ to infinity.

Putting the results into Eq. (50) for the four-point function at zero momentum, we obtain

$$g = g_0 - g_0{}^2 \frac{N + 8}{8\pi} + \cdots + \omega_K g_0{}^K + \cdots \tag{72}$$

$$\omega_K \underset{K \to \infty}{\sim} (-)^{K+1} a^K \Gamma(K + b) c \left[1 + O\left(\frac{1}{K}\right)\right] \tag{73}$$

with

$$a = 4/I_4 \tag{74}$$

$$b = 3 + \tfrac{1}{2}N \tag{75}$$

$$c = \frac{2^{N/2+2}3^{-3/2}\pi^{-2}}{\Gamma(\frac{1}{2}N + 2)} \left(\frac{I_1{}^2}{I_4}\right)^2 \left(\frac{I_6}{I_4} - 1\right)^{3/2}$$
$$\times\ [\bar{D}_R(1)]^{-1/2}[\bar{D}_R(1/3)]^{-(N-1)/2} \tag{76}$$

Inverting (72) for g_0 in terms of g, we obtain

$$g_0 = g + g^2 \frac{N + 8}{8\pi} + \cdots + \delta_K g^K + \cdots \tag{77}$$

$$\delta_K \underset{K\to\infty}{\sim} -\omega_K\left(\exp - \frac{N + 8}{8\pi a}\right)\left[1 + O\left(\frac{1}{K}\right)\right] \tag{78}$$

The field strength renormalization is asymptotically (i.e., for large K) negligible, and thus we can use the formula

$$\tilde{\beta}(g) = -g\left(1 + g\frac{d}{dg}\ln\frac{g_0}{g}\right)$$

This leads to

$$\tilde{\beta}(g) = -g + \frac{N + 8}{8\pi} g^2 + \cdots + g^K\tilde{\beta}_K + \cdots \tag{79}$$

with

$$\tilde{\beta}_K = (-)^K a^K \Gamma(K + b + 1)c \exp - \frac{N + 8}{8\pi a} \tag{80}$$

Using, as in Ref. 9, the normalization

$$g = \frac{8\pi}{N + 8} V \tag{81}$$

$$\tilde{\beta}(g) = \frac{N + 8}{8\pi}\beta(V) \tag{82}$$

we find

$$\beta(V) = -V + V^2 + \cdots + \beta_K V^K + \cdots \tag{83}$$

$$\beta_K \underset{K\to\infty}{\sim} (-A)^K\Gamma(K + B)C \tag{84}$$

Table II. Fredholm Determinants and Integrals

d	$\bar{D}_R(1)$	$\bar{D}_R(1/3)$	I_1	I_4	I_6	H_3
3	10.544 ± 0.004	1.4571 ± 0.0001	31.691522	75.589005	659.868352	13.563312
2	135.3 ± 0.1	1.465 ± 0.001	15.10965	23.40179	71.08023	9.99118

with

$$A = 32\pi/(N + 8)I_4 \tag{85}$$

$$B = 4 + \tfrac{1}{2}N \tag{86}$$

$$C = \frac{(N + 8)2^{N/2-1}3^{-3/2}}{\pi^3\Gamma(2 + \frac{1}{2}N)} \left(\frac{I_1^2}{I_4}\right)^2 \left(\frac{I_6}{I_4} - 1\right)^{3/2} \times [\bar{D}_R(1)]^{-1/2}\left[\bar{D}_R\left(\frac{1}{3}\right)\right]^{-(N-1)/2} \exp\left(-\frac{N+8}{32} I_4\right) \tag{87}$$

A similar calculation in two dimensions gives, if

$$g = \frac{4\pi}{N+8} V \tag{88}$$

and

$$\tfrac{1}{2}\beta(V) = -V + V^2 + \cdots + \beta_K V^K + \cdots \tag{89}$$

Table III. Parameters Characterizing the Asymptotic Behavior

	$d = 3, N = 0$	$d = 3, N = 1$	$d = 3, N = 2$
A	0.1662460	0.14777422	0.1329968
B	4	4.5	5
C	$(8.5489 \pm 16\cdot 10^{-4}) \times 10^{-2}$	$(3.9962 \pm 6\cdot 10^{-4}) \times 10^{-2}$	$(1.6302 \pm 3\cdot 10^{-4}) \times 10^{-2}$
	$d = 3, N = 3$	$d = 2, N = 1$	
A	0.12090618	0.238659	
B	5.5	4	
C	$(5.9609 \pm 10^{-3}) \times 10^{-3}$	$(4.886 \pm 5\cdot 10^{-4}) \times 10^{-2}$	

Table IV. Comparison of the Low-Order Calculations and of the Asymptotic Behavior for the Function $\beta(g)$

	$d = 3, N = 0$		$d = 3, N = 1$		$d = 3, N = 2$		$d = 3, N = 3$		$d = 2, N = 1$	
K	$(-)^K\beta_K$	$\bar{\beta}_K$	$(-)^K\beta_K$	$\bar{\beta}_K$	$(-)^K\beta_K$	$\bar{\beta}_K$	$(-)^K\beta_K$	$\bar{\beta}_K$	$(-)^K\beta_K$	$\bar{\beta}_K$
2	1	30.15×10^{-2}	1	15.91×10^{-2}	1	7.85×10^{-2}	1	3.66×10^{-2}	1	14.6×10^{-2}
3	0.4398	13.29×10^{-2}	0.4225	7.00×10^{-2}	0.4029	3.40×10^{-2}	0.3832	1.54×10^{-2}	0.7162	7.32×10^{-2}
4	0.3899	10.13×10^{-2}	0.3511	5.26×10^{-2}	0.3149	2.50×10^{-2}	0.2829	1.11×10^{-2}	0.9308	5.69×10^{-2}
5	0.4473	8.74×10^{-2}	0.3765	4.48×10^{-2}	0.3179	2.11×10^{-2}	0.2703	9.23×10^{-3}	1.5824	5.07×10^{-2}
6	0.6339	8.27×10^{-2}	0.4955	4.20×10^{-2}	0.3911	1.95×10^{-2}	0.3126	8.41×10^{-3}	—	—
7	1.0349	8.13×10^{-2}	0.7497	4.09×10^{-2}	0.5524	1.88×10^{-2}	0.4149	8.03×10^{-3}	—	—
C		8.55×10^{-2}		4.00×10^{-2}		1.63×10^{-2}		5.96×10^{-3}		4.89×10^{-2}

a formula for β_K analogous to (84) with

$$A = 16\pi/(N+8)I_4 \tag{90}$$

$$B = 7/2 + N/2 \tag{91}$$

$$C = \frac{(N+8)2^{(N-3)/2}\pi^{-5/2}}{\Gamma(2+\frac{1}{2}N)}\left(\frac{I_1^{\,2}}{I_4}\right)^2\left(\frac{I_6}{I_4}-1\right) \times [\bar{D}_R(1)]^{-1/2}\left[\bar{D}_R\left(\frac{1}{3}\right)\right]^{-(N-1)/2}\exp\left(-\frac{N+8}{16\pi}I_4\right) \tag{92}$$

In order to complete this calculation we give the numerical results of Table II, where

$$I_2 = (1 - \tfrac{1}{4}d)I_4 \tag{93}$$

$$I_3 = I_1 \tag{94}$$

$$H_3 = \int d^dx\, x^2\Phi_c^{\,3}(x) \tag{95}$$

This leads to the results of Table III.

These asymptotic formulas for K large are now compared with the explicit calculations of Ref. 9. Table IV gives the β_K provided by perturbation theory, and also these numbers divided by their leading asymptotic estimates

$$\bar{\beta}_K = |\beta_K|/\Gamma(K+B)A^K \tag{96}$$

For $K = 7$, $d = 3$, $\bar{\beta}_K$ is within 5% of its asymptotic limit for $N = 0$ or 1. However, the agreement gets worse for larger values of N, indicating a slower approach to the asymptotic limit. This is to be connected with the crossover to the large-N regime, in which the coefficients of the powers of $1/N$ are analytic at $g = 0$.

The critical exponents are calculated from two additional series

$$\eta_4(V) = -V\frac{(N+2)}{(N+8)} + \cdots + V^K\gamma_K + \cdots \tag{97}$$

$$\eta(V) = \frac{V^2(N+2)}{2(N+8)^2} + \cdots + V^K\delta_K + \cdots \tag{98}$$

Table V. The Normalization of the Large-Order Behavior of $\eta_4(g)$ and $\eta(g)$

	$d = 3, N = 0$	$d = 3, N = 1$	$d = 3, N = 2$	$d = 3, N = 3$	$d = 2, N = 1$
C'	1.0107×10^{-2}	6.2991×10^{-3}	3.0836×10^{-3}	1.2813×10^{-3}	1.049×10^{-2}
C''	2.8836×10^{-3}	1.7972×10^{-3}	0.8798×10^{-3}	0.3656×10^{-3}	3.468×10^{-3}

Table VI. Comparison of the Asymptotic Behavior and of the Low-Order Calculations for the Function $\eta_4(g)$

	$d = 3, N = 0$		$d = 3, N = 1$		$d = 3, N = 2$		$d = 3, N = 3$		$d = 2, N = 1$	
K	$(-)^K\gamma_K$	$\bar{\gamma}_K$	$(-)^K\gamma_K$	$\bar{\gamma}_K$	$(-)^K\gamma_K$	$\bar{\gamma}_K$	$(-)^K\gamma_K$	$\bar{\gamma}_K$	$(-)^K\gamma_K$	$\bar{\gamma}_K$
1	25×10^{-2}	2.506×10^{-2}	33.33×10^{-2}	43.11×10^{-3}	40×10^{-2}	25.1×10^{-3}	45.45×10^{-2}	13.06×10^{-3}	0.667	11.64×10^{-2}
2	6.25×10^{-2}	1.881×10^{-2}	7.41×10^{-2}	11.79×10^{-3}	8×10^{-2}	6.28×10^{-3}	8.26×10^{-2}	3.02×10^{-3}	0.250	3.65×10^{-2}
3	3.58×10^{-2}	1.081×10^{-2}	4.43×10^{-2}	7.34×10^{-3}	4.95×10^{-2}	4.18×10^{-3}	5.26×10^{-2}	2.12×10^{-3}	0.233	2.38×10^{-2}
4	3.44×10^{-2}	0.893×10^{-2}	3.95×10^{-2}	5.91×10^{-3}	4.08×10^{-2}	3.23×10^{-3}	4.00×10^{-2}	1.57×10^{-3}	0.323	1.97×10^{-2}
5	4.09×10^{-2}	0.799×10^{-2}	4.44×10^{-2}	5.28×10^{-3}	4.38×10^{-2}	2.90×10^{-3}	4.13×10^{-2}	1.41×10^{-3}	—	—
6	5.97×10^{-2}	0.779×10^{-2}	6.03×10^{-2}	5.11×10^{-3}	5.56×10^{-2}	2.77×10^{-3}	4.91×10^{-2}	1.32×10^{-3}	—	—
C'		1.01×10^{-2}		6.26×10^{-3}		3.08×10^{-3}		1.28×10^{-3}		1.05×10^{-2}

Table VII. Comparison of the Asymptotic Behavior and the Low-Order Calculations for the Function $\eta(g)$

	$d=3, N=0$		$d=3, N=1$		$d=3, N=2$		$d=3, N=3$		$d=2, N=0$	
K	$(-)^K\delta_K$	$\tilde{\delta}_K$	$(-)^K\delta_K$	$\tilde{\delta}_K$	$(-)^K\delta_K$	$\tilde{\delta}_K$	$(-)^K\delta_K$	$\tilde{\delta}_K$	$(-)^K\delta_K$	$\tilde{\delta}_K$
2	9.25×10^{-3}	13.95×10^{-3}	10.97×10^{-3}	9.60×10^{-3}	11.85×10^{-3}	5.58×10^{-3}	12.24×10^{-3}	29.09×10^{-4}	3.40×10^{-2}	24.8×10^{-3}
3	-0.77×10^{-3}	-1.40×10^{-3}	-0.91×10^{-3}	-0.98×10^{-3}	-0.99×10^{-3}	-0.58×10^{-3}	-1.02×10^{-3}	-3.08×10^{-4}	0.20×10^{-2}	1.24×10^{-3}
4	1.59×10^{-3}	2.89×10^{-3}	1.80×10^{-3}	2.01×10^{-3}	1.84×10^{-3}	1.17×10^{-3}	1.79×10^{-3}	5.97×10^{-4}	1.14×10^{-2}	4.88×10^{-3}
5	0.66×10^{-3}	1.03×10^{-3}	0.65×10^{-3}	0.66×10^{-3}	0.59×10^{-3}	0.35×10^{-3}	0.504×10^{-3}	1.64×10^{-4}	—	—
6	1.41×10^{-3}	1.66×10^{-3}	1.39×10^{-3}	1.12×10^{-3}	1.25×10^{-3}	0.80×10^{-3}	1.090×10^{-3}	3.44×10^{-4}	—	—
C''		2.88×10^{-3}		1.80×10^{-3}		0.88×10^{-3}		3.66×10^{-4}		3.47×10^{-3}

from the zero of $\beta(V)$:

$$\beta(V^*) = 0, \qquad \beta'(V^*) > 0 \tag{99}$$

$$\eta_4(V^*) = \eta + (1/\nu) - 2 \tag{100}$$

$$\eta(V^*) = \eta \tag{101}$$

Similar calculations lead to the asymptotic formulas

$$\gamma_K \underset{K\to\infty}{\sim} (-)^K A^K \Gamma(K + B')C' \tag{102}$$

$$\delta_K \underset{K\to\infty}{\sim} (-)^K A^K \Gamma(K + B'')C'' \tag{103}$$

with

$$B' = \tfrac{1}{2}(d + N + 5) \tag{104}$$

$$B'' = \tfrac{1}{2}(d + N + 3) \tag{105}$$

$$C' = \begin{cases} C\dfrac{N+2}{N+8}\, 8\pi \dfrac{I_2}{I_1{}^2} & \text{in three dimensions} \\[2ex] C\dfrac{N+2}{N+8}\, 4\pi \dfrac{I_2}{I_1{}^2} & \text{in two dimensions} \end{cases}$$

and

$$C'' = C' \frac{2H_3}{I_1 d(4-d)}$$

in which the I_p and H_3 have been defined in Eqs. (52) and (95) and tabulated in Table II. The results are given in Table V and compared with the perturbation series in Tables VI and VII.

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