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Velocity of a Perturbation in Infinite Lattice Systems

C. Marchioro,[1] A. Pellegrinotti,[1] M. Pulvirenti,[2] and L. Triolo[3]

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The problem of defining and estimating the velocity of disturbances in a crystal is investigated. Some results are given for plane rotors and anharmonic systems.

KEY WORDS: Asymptotic time behavior; group velocity in discrete systems.

1. INTRODUCTION

In the last few years, many papers have been devoted to the investigation of the time evolution of classical systems with infinite degrees of freedom. The earlier papers proved the existence and unicity of the time evolution, which essentially means a weak dependence of the evolution of local observables on distant events.(1–8) Of course, it would be very interesting to know the asymptotic behavior of such systems in time; for instance, how a local perturbation disappears and a nonequilibrium state relaxes to the equilibrium one: this naturally is a very hard problem. Moreover, it is also difficult to obtain interesting estimates for large time because the proofs given in these earlier papers use essentially estimates at fixed time.

In this paper we study a particular asymptotic property of the time evolution of lattice systems; more precisely, we shall try to describe the propagation of the perturbations in a crystal. All our considerations will be made in the framework of classical mechanics and will follow the approach proposed for quantum lattices by Lieb and Robinson in Ref. 9. In that paper the authors consider an infinite quantum lattice and a time evolution derived

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[1] Instituto Matematico dell'Università di Camerino, Camerino (MC), Italy.
[2] Istituto Matematico dell'Università di Roma, Rome, Italy, and Istituto Matematico dell'Università dell'Aquila, L'Aquila, Italy.
[3] Istituto Matematico dell'Università di Camerino, Camerino (MC), Italy, and Istituto di Matematica Applicata, Facoltà di Ingegneria dell'Università di Roma, Rome, Italy.

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by a sufficiently regular Hamiltonian. They proved that if A and B are two observables localized at the origin and B_d is the observable corresponding to B translated a distance d, denoting by α_t the time evolution, then $\|[A, \alpha_t(B_d)]\|_{t\to\infty} \to 0$ exponentially rapidly if $d \geqslant ct$ for some constant c which may be interpreted from a physical point of view as an upper bound for the group velocity.

An analogous approach can be developed, at least in principle, by replacing the commutator with Poisson brackets (this means estimating the derivative of a time-evolved coordinate localized at distance d with respect to the initial data localized at the origin; this quantity gives the dependence of the evolution on a perturbation of the initial data).

We say that we have a finite velocity of perturbation if the following is the case: if A and B are two observables localized at the origin (i.e., two functions in phase space depending only on the spin at the origin), then $\{A, \alpha_t(B_d)\}$ is small if d depends linearly on t, that is, B_d is essentially localized in a sphere of radius ct. One might suspect that $\{A, \alpha_t(B_d)\}$ goes to zero as t increases not because we are displacing the observable B rapidly, but because of the dispersive properties of the matter, according to which, for each pair of local observables, we should have $\{A, \alpha_t(B)\}_{t\to\infty} \to 0$. We remark that at least in the case of the harmonic chain the two properties have different orders of magnitude. In this case, in fact, it is possible to exhibit a cone in spacetime with slope c, such that if A is localized at the origin and B is localized at a distance less than ct, then $\{A, \alpha_t(B)\}$ is of the order of $1/\sqrt{t}$, and is exponentially small in t if B is localized at a distance greater than ct. Unfortunately it is not easy to make a complete analysis of the asymptotic behavior of the Poisson brackets beyond the case of the harmonic chain. In this paper we deal only with the problem of finding a dependence $d = d(t)$ in order to obtain an exponentially small $\{A, \alpha_t(B_d)\}$.

After an analysis of known results on the harmonic chain, we investigate two models. The first one, a lattice of compact spins interacting with bounded forces, is found to exhibit a linear dependence of d on t for all configurations in phase space. The second is a lattice of anharmonic oscillators whose time evolution has been studied in Ref. 3. The dynamical estimates on the displacement and on the momentum of an oscillator are not good enough for our purpose because they take into account all possible energies transferred by the other oscillators; in fact we are interested in thermodynamically relevant configurations and hence with infinite total energy. So one could show that these cooperative effects indicate that $d(t)$ increases more than exponentially with t. Such bad estimates may be greatly improved by the explicit use of the Gibbs state invariance with respect to the time evolution.

So we are induced to investigate the quantity $\int d\omega\, |\{A, \alpha_t(B_d)\}| \equiv \|\{A, \alpha_t(B_d)\}\|$, where ω is a Gibbs state of the system. We prove that the above quantity is exponentially small if $d \approx t^{4/3}$. This result is not satisfactory from the point of view of the group velocity concept, but it remains a nontrivial estimate for asymptotic time behavior. Moreover, we show that disturbances measured in term of local energy do propagate with finite velocity for all configurations typical of the equilibrium state. The $t^{4/3}$ estimate depends on the state ω, while the other result was purely mechanical; this is of course a limitation of the range of our results. On the other hand, as in the simple example which follows, sometimes there simply does not exist a purely mechanical estimate of the velocity of propagation: let us consider a semiinfinite, one-dimensional system of hard rods of length δ placed at the points $\{x_i\}_{i=0}^{\infty}$, $x = x_0 < x_1 \cdots$ and such that $|x_{i+1} - x_i| = \delta + a_i$ and $a_i \leqslant 1/i$. All the hard rods are supposed to have equal masses and to be initially at rest. If we give a velocity v to the first particle directed on the right, according to the exchange of velocity we can follow the perturbation by looking at the moving particle. After the time t the moving particle is at distance $s(t) = vt + h(t)\delta$, where $n(t)$ is the number of collisions that have occurred during the time t; $n(t)$ is the maximal integer for which $\sum_{i=1}^{n(t)} a_i \leqslant vt$. Hence $s(t)$ increases not less than exponentially in time.

Clearly such a configuration is exceptional with respect to a "physical" state, i.e., a state with density less than the close-packing one.

The plan of the paper is the following. In Section 2 we give some definitions. The harmonic chain is treated in Section 3. Sections 4 and 5 are devoted to the plane rotor model (the compact spin model) and to the anharmonic crystal.

2. GENERAL FRAMEWORK

Let us consider the ν-dimensional lattice $\mathbb{Z}^\nu$. At every site $i \in \mathbb{Z}^\nu$ there is associated a space T_i that is $S^1 \times \mathbb{R}^1$ or $\mathbb{R}^1 \times \mathbb{R}^1$. The points of T_i are respectively denoted by $x_i = (\theta_i, \omega_i)$ or $x_i = (q_i, p_i)$.

We denote by $\mathbb{X}$ the phase space $\prod_{i\in\mathbb{Z}^\nu} T_i$. We take $\mathbb{X}$ to be equipped with the product topology and its points are denoted x, y, z, etc.

For any bounded $\Lambda \subset \mathbb{Z}^\nu$ we define the phase space associated with the region Λ by $\mathbb{X}_\Lambda = \prod_{i\in\Lambda} T_i$. Let f_Λ: $\mathbb{X}_\Lambda \to \mathbb{R}$ be a differentiable function with bounded derivative. f_Λ induces a function f: $\mathbb{X} \to \mathbb{R}$ by the following definition:

$$f(x) = f_\Lambda(x_\Lambda), \qquad x \in \mathbb{X} \tag{1}$$

where x_Λ denotes the restriction on $\mathbb{X}_\Lambda$ of x.

We denote by $\mathfrak{A}(\Lambda)$ the algebra of all function of such type and

$$\mathfrak{A} = \bigcup_{\Lambda\subset\mathbb{Z}^\nu, \Lambda\,\text{finite}} \mathfrak{A}(\Lambda)$$

the algebra of all local functions. Since every $f \in \mathfrak{A}$ depends only on a finite number of coordinates, the following definition is reasonable:

$$\{f, g\}(x) = \sum_{i \in \mathbb{Z}^\nu} \left(\frac{\partial f}{\partial P_i} \frac{\partial g}{\partial Q_i} - \frac{\partial g}{\partial P_i} \frac{\partial f}{\partial Q_i} \right)(x) \tag{2}$$

where $(Q_i, P_i) = (\theta_i, \omega_i)$ or (q_i, p_i). It is obvious that (2) still makes sense if we choose $f \in \mathfrak{A}$ and any differentiable function $g: \mathbb{X} \to \mathbb{R}$ because in (2) only a finite number derivatives will appear.

In what follows we shall introduce a subset $\mathbb{X}_0 \subset \mathbb{X}$ in which a one-parameter group of transformations (the solution of the motion equations) $\forall t \in \mathbb{R}$, $S_t: \mathbb{X}_0 \to \mathbb{X}_0$, is defined with the property that $(S_t x)_i = (Q_i(t), P_i(t))$ is a couple of differentiable functions with respect to Q_j or P_j, for all t, where $x = (Q_j, P_j)_{j \in \mathbb{Z}^\nu} \in \mathbb{X}_0$ are the initial coordinates.

Defining by $(V_t g)(x) = g(S_{-t} x)$, $x \in \mathbb{X}_0$ for $g \in \mathfrak{A}$, the main object of our investigation will be $\{f_i, V_t g_j\}(x)$, where $g_0, f_0 \in \mathfrak{A}(0)$ when simultaneously $t \to \infty$ and $|i - j| \to \infty$; here f_i and g_j denote the translates of f and g at the sites i and j, i.e., the same functions f and g thought of as functions respectively from T_i and T_j to $\mathbb{R}$. The following estimates will be useful:

$$\begin{aligned} |\{f_i, V_t g_j\}|(x) &= \left| \frac{\partial f_i}{\partial Q_i} \frac{\partial V_t g_j}{\partial P_i} - \frac{\partial f}{\partial P_i} \frac{\partial V_t g_j}{\partial Q_i} \right|(x) \\ &= \left| \frac{\partial f_i}{\partial Q_i} \left(\frac{\partial g_j}{\partial P_j}\bigg|_{S_{-t}x} \frac{\partial P_j(t)}{\partial P_i} + \frac{\partial g_j}{\partial Q_j}\bigg|_{S_{-t}x} \frac{\partial Q_j(t)}{\partial P_i} \right) \right. \\ &\quad \left. - \frac{\partial f_i}{\partial P_i} \left(\frac{\partial g_j}{\partial P_j}\bigg|_{S_{-t}x} \frac{\partial P_j(t)}{\partial Q_i} + \frac{\partial g}{\partial Q_j}\bigg|_{S_{-t}x} \frac{\partial Q_j(t)}{\partial Q_i} \right) \right|(x) \\ &\leqslant \|\nabla f\|_\infty \|\nabla g\|_\infty u_{j,i}(t, x) \end{aligned} \tag{3}$$

Here $(Q_j(t), P_j(t))$ denotes the j coordinate of $S_{-t}x$, $\|\nabla f\|_\infty$ and $\|\nabla g\|_\infty$ are the maxima of all the derivatives of f and g, and

$$u_{j,i} = 4 \max\left(\left| \frac{\partial P_j(t)}{\partial P_i} \right|, \left| \frac{\partial Q_j(t)}{\partial P_i} \right|, \left| \frac{\partial P_j(t)}{\partial Q_i} \right|, \left| \frac{\partial Q_j(t)}{\partial Q_i} \right| \right) \tag{4}$$

3. THE HARMONIC CHAIN

We consider the one-dimensional lattice $\mathbb{Z}^1 T_i = \mathbb{R}^1 \times \mathbb{R}^1$. The Hamiltonian is

$$H = \sum_1 [(p_i^2/2m) + k(q_i - q_{i-1})^2] \tag{5}$$

and the equations of the motion are

$$\dot{q}_i = p_i/m, \qquad \dot{p}_i = k(q_{i+1} - 2q_i + q_{i-1}) \tag{6}$$

The existence of the solution of Eqs. (6) has been proven for a large class of initial data in $\mathbb{X}$ in Refs. 10 and 11, more precisely for all the configurations $x = (q_i, p_i)$, $i \in \mathbb{Z}$, that have q_i and p_i exponentially increasing as $|i| \to \infty$. The central point in the estimation of the Poisson brackets is that the derivatives of $q_i(t)$ and $p_i(t)$ with respect to initial data satisfy the same equations (6) but with initial data of finite energy. For such initial data the solution can be given explicitly,[11,12] so the Poisson brackets can be evaluated directly. We denote $\chi_j^i = \partial q_j(t)/\partial q_i$; then

$$\begin{aligned} \chi_j^i(t) &= (k/m)[\chi_{j+1}^i(t) - 2\chi_j^i(t) + \chi_{j-1}^i(t)] \\ \chi_j^i(0) &= \delta_{i,j}, \qquad \dot{x}_j^i(0) = 0 \end{aligned} \tag{7}$$

In the case in which $i = 0$, (7) has the solution

$$\chi_j^0(t) = J_{2j}(2ct) \tag{8}$$

where $c = (k/m)^{1/2}$, and the $J_h(z)$ are the Bessel functions of the first type.

Proposition 3.1. Let f and $g \in \mathfrak{A}(0)$. Let us write j for the integer part of $[\beta t]$. Then for all exponentially increasing configuration $x \in \mathbb{X}$ we have:

(i) $\{f, V_t g_j\}(x) \xrightarrow[t\to\infty]{} 0$ as $1/\sqrt{t}$ if $\beta < c$

(ii) $\{f, V_t g_j\}(x) \xrightarrow[t\to\infty]{} 0$ exponentially if $\beta > c$

We remark that the group velocity given in term of Poisson brackets, that is, the slope of the cone separating the asymptotic behavior $1/\sqrt{t}$ from the exponential one, coincides with the natural notion arising by considering the normal modes of the crystal or the analog of Eqs. (6) with the wave equation.

The proof is a simple consequence of the following asymptotic estimates: Let $j > 0$ for large j; then

$$J_{2j}\left(\frac{2j}{\cos\gamma}\right) \sim \left(\frac{1}{j\pi \operatorname{tg}\gamma}\right)^{1/2} \qquad 0 < \gamma < \frac{\pi}{2}$$

$$J_{2j}\left(\frac{2j}{\coth\alpha}\right) \sim \frac{\exp(2j \operatorname{tgh}\alpha - 2j\alpha)}{(4j\pi \operatorname{tgh}\alpha)^{1/2}}, \qquad \alpha > 0$$

4. PLANE ROTATORS

In this section we study the system described by the following formal Hamiltonian:

$$H = \sum_i \left[\tfrac{1}{2}\omega_i^2 + \sum_{j\in V_i} \bar{k}\cos(\theta_i - \theta_j)\right] \tag{9}$$

where $i \in \mathbb{Z}^\nu$, $\theta_i \in S^1$, $\omega_i \in \mathbb{R}^1_i$ $\bar{k}$ is a constant, $V_i = \{j \mid |i-j| = 1\}$, and $|i-j| = \sum_{\alpha=1}^{\nu} |i_\alpha - j_\alpha|$.

The one-particle phase space is $T_i = S^1 \times \mathbb{R}^1$, and the phase space is $\mathbb{X} = \prod_{i \in \mathbb{Z}^\nu} T_i$. The dynamics of the system is described by the following equations:

$$\begin{aligned} \dot{\theta}_i(t) &= \omega_i(t) \\ \dot{\omega}_i(t) &= F_i(x(t)), \qquad (\theta_i(0), \omega_i(0))_{i \in \mathbb{Z}^\nu} = x \in \mathbb{X} \end{aligned} \tag{10}$$

where

$$F_i(x) = \sum_{j \in V_i} \bar{k} \sin(\theta_i - \theta_j), \qquad x = (\theta_i, \omega_i)_{i \in \mathbb{Z}^\nu}$$

Proposition 4.1. For all $x \in \mathbb{X}$, there exists a one-parameter group of transformations $x \to S_t x \equiv x(t) = (\theta_i(t), \omega_i(t))_{i \in \mathbb{Z}^\nu}$ such that $\theta_i(t)$ and $\omega_i(t)$ satisfy (10).

We do not prove Proposition 4.1, which can be easily obtained by the use of the same arguments in Ref. 3.

We note only that the existence of the solution of Eqs. (10) may be obtained by considering that the dynamics would not exist if the kinetic energy of a particle grows too large. But the following estimate prevents this from happening:

$$|d(\omega_i^2/2)/dt| = |\omega_i F_i(x(t))| \leqslant |\omega_i(t)| \bar{k} 2\nu \tag{11}$$

This implies

$$\omega_i^2(t)/2 \leqslant (\omega_i/\sqrt{2} + \bar{k}\sqrt{2}\,\nu t)^2 \tag{12}$$

Furthermore, the derivatives of $\theta_i(t)$ and $\omega_i(t)$ with respective to the initial data exist there and satisfy the following integral equalities:

$$\begin{aligned} \frac{\partial \theta_j(t)}{\partial \theta_i} &= \delta_{i,j} + \int_0^t \frac{\partial \omega_j}{\partial \theta_i}(s)\, ds \\ \frac{\partial \omega_j(t)}{\partial \theta_i} &= \int_0^t ds \left\{ \bar{k} \sum_{k \in V_i} \cos[\theta_j(s) - \theta_k(s)] \left[\frac{\partial \theta_j}{\partial \theta_i}(s) - \frac{\partial \theta_k}{\partial \theta_i}(s) \right] \right\} \\ \frac{\partial \theta_j(t)}{\partial \omega_i} &= \int_0^t \frac{\partial \theta_j(s)}{\partial \omega_i}\, ds \\ \frac{\partial \omega_j(t)}{\partial \omega_i} &= \delta_{i,j} + \int_0^t ds \left\{ \bar{k} \sum_{k \in V_i} \cos[\theta_j(s) - \theta_k(s)] \left(\frac{\partial \theta_j(s)}{\partial \omega_i} - \frac{\partial \theta_k(s)}{\partial \omega_i} \right) \right\} \end{aligned} \tag{13}$$

We put

$$
\begin{aligned}
\Psi_j^i(t) &= \partial\theta_j(t)/\partial\theta_i, \qquad \Psi^i = \{\Psi_j^i\}_{j\in\mathbb{Z}^\nu}, \qquad \delta_i = \{\delta_{i,j}\}_{j\in\mathbb{Z}^\nu} \\
[A(s)\Psi^i]_j &= \sum_k A(s)_{jk}\Psi_k^i \\
A_{jk}(s) &= \cos[\theta_j(s) - \theta_k(s)], \qquad j \neq k, \qquad k \in V_j \\
A_{jj}(s) &= -\sum_{k\in V_j} \cos[\theta_j(s) - \theta_k(s)] \\
A_{jk}(s) &= 0 \qquad \text{if} \quad k \notin V_j
\end{aligned}
\tag{14}
$$

Then

$$
\Psi_j^i(t) = \delta_{i,j} + \int_0^t (t-s)[A(s)\Psi^i(s)]_j\, ds \tag{15}
$$

and hence

$$
\begin{aligned}
\Psi_j^i(t) = \delta_{i,j} + \sum_{n\geq 1}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \\
\times (t - t_1)\cdots(t_{n-1} - t_n)[A(t_1)\cdots A(t_n)\delta_i]_j
\end{aligned}
\tag{16}
$$

The series (16) is easily seen to be absolutely convergent. Furthermore, putting $k = |i - j|$, one has

$$
\begin{aligned}
[A(t_1)\cdots A(t_n)\delta_i]_j &= \sum_{k_1\cdots k_{n-1}} A(t_1)_{j,k_1}\cdots A(t_n)_{k_{n-1},i} \qquad &\text{if} \quad k \leq n \\
&= 0 &\text{otherwise}
\end{aligned}
\tag{17}
$$

Hence

$$
\begin{aligned}
|\Psi_j^i(t)| &\leq \sum_{n\geq k} \frac{|t|^{2n}}{(2n)!}[(2\nu+1)\bar{k}]^{2n} \\
&\leq \frac{|t|^{2k+1}}{(2k+1)!}[(2\nu+1)\bar{k}]^{2k+1}\exp[(2\nu+1)\bar{k}|t|]
\end{aligned}
\tag{18}
$$

Finally, by putting $b = (2\nu + 1)\bar{k}$, and applying Stirling's formula, we obtain

$$
|\Psi_j^i(t)| \leq \frac{1}{[4\pi(2k+1)]^{1/2}}\left(\frac{eb|t|}{2k+1}\right)^{2k+1}\exp(b|t|) \tag{19}
$$

If $|t| \leq k/c$

$$
|\Psi_j^i(t)| \leq \frac{1}{[4\pi(2k+1)]^{1/2}}\frac{eb}{c(2+1/k)}\left[\frac{b}{(2+1/k)c}\exp\left(\frac{b}{2c}+1\right)\right]^{2k} \tag{20}
$$

Choosing c such that

$$
(b/2c)\exp[(b/2c)+1] < 1 \tag{21}
$$

estimating the other derivatives in the same way, and using the inequality (3), we prove the following theorem:

Theorem 4.1. There exists an increasing function μ such that if c satisfies (21), then

$$\lim_{\substack{t\to\infty \\ |i-j|\geq ct}} |\{f_i, V_t f_j\}|(x)e^{\mu(c)t} = 0$$

where $f, g \in \mathfrak{A}(0)$ and $x \in \mathbb{X}$.

5. ANHARMONIC SYSTEMS

In this section we study a system for which $T_i = \mathbb{R}^1 \times \mathbb{R}^1$, $i \in \mathbb{Z}^\nu$, and the Hamiltonian is formally given by

$$\sum_i (\tfrac{1}{2}p_i^2 + kq_i^2 + \lambda q_i^4 - Jq_i \sum_{j\in V_i} q_j) \tag{22}$$

where $k, \lambda, J > 0$.

The Hamiltonian (22) describes a physical model of anharmonic oscillators. If $\lambda = 0$, we have the harmonic case.

We introduce the following functions:

$$\begin{aligned} \mathscr{L}^i(x) &= \tfrac{1}{2}p_i^2 + kq_i^2 + \lambda q_i^4 + 1, \qquad x \in \mathbb{X} \\ \mathscr{L}_\varphi(x) &= \sup_{k\in\mathbb{N}}[1/\varphi(k)] \sup_{i\in\Lambda_k} \mathscr{L}^i(x) \end{aligned} \tag{23}$$

where $\Lambda_k = [-k, k]^\nu$ and $\varphi(k) = \log|k| \vee 1$.

We denote by $\mathbb{X}_0$ the following subset of $\mathbb{X}$:

$$\mathbb{X}_0 = \{x \in \mathbb{X} | \mathscr{L}_\varphi(x) < +\infty\} \tag{23'}$$

The following theorem gives the existence of the time evolution of the infinite system we are considering.

Theorem 5.1. (Lanford *et al.*[3]) There exists a one-parameter group of transformations S_t: $\mathbb{X}_0 \to \mathbb{X}_0$, $t \in \mathbb{R}$, such that:

(i) For all $x \in \mathbb{X}_0$, $S_t x \equiv (q_i(t), p_i(t))_{i\in\mathbb{Z}^\nu}$ represents the solution of the equations of motion

$$dq_i/dt = p_i, \qquad dp_i/dt = F_i(S_t x)$$

Here $F_i(x)$ represents the force induced by the configuration x on the ith oscillator:

$$F_i(x) = -4\lambda q_i^3 - 2kq_i + J\sum_{j\in V_i} q_i, \qquad x \equiv (q_i, p_i)_{i\in\mathbb{Z}^\nu}$$

(ii) The following estimate holds:

$$|d\mathcal{L}^i(S_t x)/dt| \leqslant \sum_{j \in V_i} a_{ij} \mathcal{L}^j(S_t x)$$

and hence

$$\mathcal{L}_\varphi(S_t x) \leqslant e^{a|t|} \mathcal{L}_\varphi(x)$$

where $a_{i,j}$ are independent of x and t and $a = \max_{i,j} a_{ij}$.

Let ω be a state, i.e., a Borel probability measure on $\mathbb{X}$, that satisfies the following superstability estimate:

$$P_\omega(dx_i) \leqslant A \exp(-k_1 p_i^2 - k_2 q_i^4)\, dq_i\, dp_i \tag{24}$$

where P_ω is the relativization of the measure ω on $\mathbb{X}_{(i)}$, $i \in \mathbb{Z}^\nu$, and k_1 and k_2 are constant not depending on i. It is possible to check that $\omega(\mathbb{X}_0) = 1$. We shall suppose that ω is time invariant, i.e., $\omega(V_t f) = \omega(f)$ if $f \in \mathfrak{A}$ and $(V_t f)(x) = f(S_{-t}x)$, $x \in \mathbb{X}_0$. An example of such a state is the Gibbs state obtained as the thermodynamic limit of finite-volume Gibbs states with zero boundary conditions.[13]

Theorem 5.2. Let ω be an invariant state satisfying (24). Then for all $f, g \in \mathfrak{A}(0)$ and $b \in \mathbb{R}^+$

$$\lim_{\substack{t \to \infty \\ |i-j|^{3/4}/t \to \infty}} \|\{f_i, V_t g_j\}\|_1 e^{bt} = 0$$

Here $\|\cdot\|_p$ denotes the norm in $L_p(\omega)$.

Proof. On the basis of (3), we have to estimate the following quantities $[S_t x = (q_j(t), p_j(t))_{j \in \mathbb{Z}^\nu},\ x = (q_j, p_j)_{j \in \mathbb{Z}^\nu} \in \mathbb{X}_0]$:

$$\begin{aligned}
\frac{\partial q_j(t)}{\partial p_i} &= \int_0^t ds \frac{\partial p_j}{\partial p_i}(s) \\
\frac{\partial p_j(t)}{\partial p_i} &= \delta_{i,j} + \int_0^t ds \frac{\partial F_j}{\partial p_i}(x(s)) \\
\frac{\partial q_j(t)}{\partial q_i} &= \delta_{i,j} + \int_0^t ds \frac{\partial p_j}{\partial q_i}(s) \\
\frac{\partial p_j(t)}{\partial q_i} &= \int_0^t ds \frac{\partial F_j}{\partial q_j}(x(s))
\end{aligned} \tag{25}$$

which exist in virtue of Theorem 5.1(ii).

Defining $\varphi_j^i(t) = \partial q_j(t)/\partial q_i$, one has

$$\varphi_j^i(t) = \delta_{i,j} + \int_0^t (t-s)[B(s)\varphi^i(s)]_j\, ds \tag{26}$$

where $\varphi^i = \{\varphi_j^i(s)\}_{j\in\mathbb{Z}^\nu}$, and

$$[B(s)u]_j = [-12\lambda q_j^2(s) - 2k]u_j + J\sum_{l\in V_j} u_l \tag{27}$$

Equality (26) is the integral version of a nonautonomous differential problem whose solution is

$$\varphi_j^i(t) = \delta_{i,j} + \sum_{n=1}^{\infty}\int_0^{t=t_0} dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t} dt_n (t - t_1)\cdots(t_{n-1} - t_n) \times [B(t_1)\cdots B(t_n)\delta_i]_j \tag{28}$$

By the use of the estimate (ii) of Theorem 5.1 the series (28) is easily seen to be absolutely convergent for fixed t and $x \in \mathbb{X}_0$. Unfortunately, for our purpose, this point-by-point estimate is a bad one, so we try an $L_1(\omega)$ estimate with the aim of avoiding high-energy effects by the explicit use of the conservation of the measure.

Let us put $k = |i - j|$; then

$$\begin{aligned}[B(t_1)\cdots B(t_n)\delta_i]_j &= \sum_{k_1\cdots k_{n-1}} B(t_1)_{j,k_1}\cdots B_{k_{n-1},i}(t_n) \qquad &\text{if } k \leqslant n\\ &= 0 &\text{if } k > n\end{aligned} \tag{29}$$

where the $B_{j,k}(s)$ are the components of the operator B defined by

$$B_{jk}(s) = [-12q_j^2(s) - 2k]\delta_{jk} + J\sum_{l\in V_j}\delta_{lk}$$

we can estimate (29) in the following way: every term in (29) represents an n-step walk with some permanences, which arise from the diagonal term $Q_r(t) = [-12\lambda q_r^2(t) - 2k]$, and with the remaining one-step jumps described by the term J.

Then if $k \leqslant n$

$$\begin{gathered}(29) = \sum{}^{*}\sum_{s=0}^{n-k} Q_{r_1}(t_{a_1})\cdots Q_{r_s}(t_{a_s})J^{n-s}\\ 0 < a_i \leqslant n, \qquad a_i \neq a_j, \qquad a_i \in \mathbb{N}\end{gathered} \tag{30}$$

where $\sum^*$ means a sum over all the possible walks with s permanences. By the use of the Hölder inequality, the invariance of the measure, and inequality (24), one finally gets that there exists a constant $R > 1$ such that

$$\begin{aligned}\|(29)\|_1 &\leqslant (2\nu + 1)^n \sum_{s=0}^{n-k} J^{n-s}\int |Q_{r_1}(t_{a_1})\cdots Q_{r_s}(t_{a_s})|\, d\omega\\ &\leqslant (2\nu + 1)^n \sum_{s=0}^{n-k} J^{n-s}\left(\int |Q_{r_1}(0)|^s\, d\omega\right)^{1/s}\cdots\left(\int |Q_{r_s}(0)|^s\, d\omega\right)^{1/s}\\ &\leqslant (2\nu + 1)^n \sum_{s=0}^{n-k} J^{n-s}R^s s^{s/2}\end{aligned} \tag{31}$$

and so

$$\begin{aligned}
\|\varphi_j^i(t)\|_1 &\leqslant \sum_{n=k}^{\infty} \frac{|t|^{2n}}{(2n)!} \sum_{s=0}^{n-k} R^s s^{s/2} J^{n-s} \\
&\leqslant \sum_{n=k}^{\infty} \frac{|t|^{2n} R^{n-k} (J+1)^n [(n-k)/2]^{(n-k)/2}}{(2n)!} (n-k+1) 2^{(n-k)/2} \\
&\leqslant \sum_{n=k}^{\infty} \frac{|t|^{2n} D^n [(n-k+2)/2]!}{[(n-k+2)/2]!\,(\frac{3}{2}n + \frac{1}{2}k - 1)!} \\
&\leqslant \sum_{r=2k-1}^{\infty} \frac{E^r |t|^{\frac{2}{3}(2r-k+2)}}{r!} \\
&\leqslant |t|^{-2k/3+2} \frac{[\exp(Et^{4/3})](Et^{4/3})^{2k}}{(2k)!}
\end{aligned} \tag{32}$$

where D and E are suitable constants. Using the Stirling formula and putting $t^{4/3} = k$, one gets

$$\|\varphi_j^i(t)\|_1 \leqslant k^{-(k/2)+1} (eEe^E)^{2k} \tag{33}$$

The other derivatives may be estimated in the same way. So Theorem 5.2 easily follows from (33) and (3).

Comments on the proof. The estimate (32) may be improved, but not very much, as can be seen by considering only the term $s = n - k$, $n \geqslant k$, in (32). We finally remark that the behavior $s^{s/2}$ in (31) is replaced with $s^{s[(\alpha-1)/\alpha]}$ in the case in which the anharmonic restoring potential is given by $\lambda q^{2\alpha}$. In this case we must take a power smaller than in Theorem 5.2.

If we take the notion of group velocity in terms of Poisson brackets or the commutator as in Ref. 9, Theorem 5.2 is not satisfactory from this point of view. Nevertheless we are able to prove that the perturbations measured in terms of the local energy do propagate with finite velocity.

Theorem 5.3. Let $x, \tilde{x} \in \mathbb{X}_0$ be two configurations differing only in the origin. Then there exists an increasing positive function $\mu = \mu(c)$ such that

$$\lim_{\substack{t\to\infty \\ |k|\geqslant ct}} |\mathscr{L}^k(S_t x) - \mathscr{L}^k(S_t \tilde{x})| e^{\mu(c)t} = 0$$

if c satisfies $(2\nu + 1)a \exp[(a/c) + 1] < c$.

Proof. By Theorem 5.1(ii) we have ($t > 0$)

$$\mathscr{L}^k(S_t x) \leqslant \sum_{j\in V_k} \int_0^t a_{kj} \mathscr{L}^j(S_\tau x)\, d\tau + \mathscr{L}^k(x) \tag{34}$$

and hence ($|k| > 1$)

$$|\mathscr{L}^k(S_t x) - \mathscr{L}^k(S_t \tilde{x})| \leqslant \sum_{j_1 \in V_k} \sum_{j_2 \in V_{j_1}} \cdots \sum_{j_{k-1} \in V_{j_{k-2}}} \frac{|t|^{|k|-1}}{(|k|-1)!} \times a_{kj_1} a_{j_1 j_2} \cdots a_{j_{k-2} j_{k-1}} [\mathscr{L}_\varphi(x) + \mathscr{L}_\varphi(\tilde{x})] e^{a|t|} \log(2|k|)$$

where we have iterated (34) $|k| - 1$ times. Thus

$$|\mathscr{L}^k(S_t x) - \mathscr{L}^k(S_t \tilde{x})| \leqslant (2\nu + 1)^{|k|-1} \log(2|k|) a^{|k|-1} e^{a|t|} [\mathscr{L}_\varphi(x) + \mathscr{L}_\varphi(\tilde{x})] \frac{|t|^{|k|-1}}{(|k|-1)!}$$

$$\leqslant \left[\frac{(2\nu + 1)ate}{|k| - 1}\right]^{|k|-1} [\mathscr{L}_\varphi(x) + \mathscr{L}_\varphi(\tilde{x})] \frac{\log(2|k|)}{(2\pi|k|)^{1/2}} e^{a|t|}$$

and this concludes the proof.

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