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Logarithmic Sobolev Inequalities and Stochastic Ising Models

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We use logarithmic Sobolev inequalities to study the ergodic properties of stochastic Ising models both in terms of large deviations and in terms of convergence in distribution.

KEY WORDS: Logarithmic Sobolev inequalities; stochastic Ising models; large deviations; rates of convergence.

1. INTRODUCTION

The theme of this article is the interplay between logarithmic Sobolev inequalities and ergodic properties of stochastic Ising models.

To be more precise, let g be a Gibbs state for some potential and suppose $\{P_t: t>0\}$ is the semigroup of an associated stochastic Ising model. Then $\{P_t: t>0\}$ determines on $L^2(g)$ a Dirichlet form $\mathscr{E}^g$. A logarithmic Sobolev inequality is a relation of the form

$$\text{(L.S.)} \qquad \int f^2 \log \frac{f^2}{\|f\|^2_{L^2(g)}}\, dg \leqslant \alpha \mathscr{E}^g(f, f), \qquad f \in L^2(g)$$

for some positive α (known as the logarithmic Sobolev constant). In this article we discuss some of the implications that (L.S.) has for the ergodic theory of the stochastic Ising model.

In Section 2 we discuss ergodic properties from the standpoint of large-deviation thoery. In particular, we introduced and compare rate

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functions with which one might hope to measure the large deviations of the normalized occupation time functional. The discussion here is quite general and does not rely on our having (L.S.). Even so, we are able to draw the following qualitative conclusion: given any closed set Γ of nonstationary states, the probability that the normalized occupation time functional up to time T lies in Γ goes to zero exponentially fast as $T \to \infty$. Obviously, this result is more interesting in cases when one knows that the only stationary measures are Gibbs states. Utilizing the ideas developed here, we reprove here the result that in dimensions one and two this is the case.

Section 3 begins our use of (L.S.). In the first place, we show that a complete large-deviation princple follows from (L.S.). Second, (L.S.) provides us with a way to estimate the size of large deviations. Finally, we provide a condition under which one can prove not only that (L.S.) holds, but also that there is precisely one stationary measure.

In Section 4 we begin by showing that (L.S.) plus uniqueness of g implies that shift-invariant initial states converge to g at an exponential rate at least $2/\alpha$. Noting that (L.S.) implies that

$$\left\| P_t f - \int f \, dg \right\|_{L^2(g)} \leqslant \exp(-2t/\alpha) \left\| f - \int f \, dg \right\|_{L^2(g)}$$

we see that this rate is the same as the one we would predict from spectral considerations.

Because we only know a few very special situations in which (L.S.) holds, we study in Section 5 what can be daid if our Gibbs state is very mixing and a logarithmic Sobolev inequality holds for each finite-dimensional conditional with a logarithmic Sobolev constant that tends to ∞ at a certain rate as the size of the system grows. We find that the type of convergence proved in Section 4 (under (L.S.) still occurs, only now at a subexponential rate (depending on the behavior of the logarithmic Sobolev constants for the finite-dimensional conditionals). Section 6 is devoted to the application of Section 5 in the case of one-dimensional Ising models. In this case we find that the above convergence rate is $\exp(-\gamma t/\log t)$ for some $\gamma > 0$.

It should be noted that although we have restricted ourselves here to Ising models with continuous spins, much of what we do applies to any situation in which the appropriate logarithmic Sobolev inequalities are available. Thus, the results of Sections 5 and 6 apply equally well to most Ising models with compact spin states. However, at the present time, the only interesting examples of models for which (L.S.) holds are continuous-spin-state models.

2. RATE FUNCTIONS AND LARGE DEVIATIONS FOR INTERACTING SYSTEMS

Although many of our results are true in more general context, for the sake of definiteness we will restrict our attention to the setting described below.

$(\underline{M}\, \underline{r})$ is a compact, oriented, C^∞-Riemannian manifold of dimension N and λ denotes the associated normalized Riemannian volume element on M.

$\underline{E} = \underline{M}^{Z^\nu}$ is given the product topology and $\mathscr{B}$ denotes the Borel field $\mathscr{B}_{\underline{E}}$ over $\underline{E}$. Given $\varnothing \neq \Lambda \subseteq Z^\nu$, $\underline{E}_\Lambda = \underline{M}^\Lambda$, $\eta \in E \to \eta_\Lambda \in \underline{E}_\Lambda$ is the natural projection of $\underline{E}$ onto $\underline{E}_\Lambda$, and $\mathscr{B}_\Lambda$ is the inverse image under $\eta \to \eta_\Lambda$ of the Borel field $\mathscr{B}_{E_\Lambda}$. Also, if $\mu \in M_1(\underline{E})$ [the space of probability measures on $(\underline{E}\, \mathscr{B})$] and $\varnothing \neq \Lambda \subseteq Z^\nu$, then μ_Λ denotes the marginal distribution of μ on $\underline{E}_\Lambda$ [i.e., $\int_{E_\Lambda} \phi \, d\mu_\Lambda = \int \phi(\eta_\Lambda)\, \mu(d\eta)$ for all $\phi \in \mathscr{B}(\underline{E}_\Lambda)$]. Given $\varnothing \neq \Lambda \subset\subset Z^\nu$ (i.e., Λ is a finite, nonempty subset of Z^ν), $\underline{C}_\Lambda^\infty(\underline{E})$ denotes the inverse image under $\eta \to \eta_\Lambda$ of $C^\infty(\underline{E}_\Lambda)$. Finally,

$$\mathscr{D}(E) = \bigcup \{ C_\Lambda^\infty(\underline{E}) \colon \varnothing \neq \Lambda \subset\subset Z^\nu \}$$

A potential $\mathscr{J}$ is a family $\{J_F \colon \varnothing \neq F \subset\subset Z^\nu\}$ of functions $J_F \in C_F^\infty(E)$. We will always assume that $\mathscr{J}$ has *finite range* R: $J_F = 0$ for $F \subset\subset Z_\nu$ with the property that

$$\max\{|k-l| = \max_{1 \leqslant i \leqslant \nu} |k_i - l_i|, k, l \in F\} > R$$

and we will use Λ_n, $n \geqslant 0$, to denote $\{k \in Z^\nu \colon |k| \leqslant nR\}$ and $\partial\Lambda_n$, $n \geqslant 1$, to stand for $\Lambda_n \backslash \Lambda_{n-1}$. In addition, we will always assume that J is *bounded* in the sense that, for each $m \geqslant 0$, all derivatives of J_F up to order m are bounded independent of $F \subset\subset Z^\nu$. Finally, we will often assume that J is *shift-invariant*, $J_{F+k} = J_F \circ S^k$, $F \subset\subset Z^\nu$ and $k \in Z^\nu$, where $S^k \colon \underline{E} \to \underline{E}$ is the shift map on E induced by the lattice shift on Z^ν.

Given $k \in Z^\nu$, set

$$H_k = \sum_{\{F \subset\subset Z^n : F \ni k\}} J_F$$

and define the linear operator $L \colon \mathscr{D}(E) \to \mathscr{D}(E)$ by

$$L\phi = \sum_{k \in Z^\nu} e^{H_k} \operatorname{div}_k(e^{-H_k} \nabla_k \phi)$$

where div_k and ∇_k refer, respectively, to the divergence and gradient operators on the kth Riemann manifold (M, r).

For a given $\varnothing \neq \Lambda \subseteq Z^\nu$, define

$$(\xi_\Lambda, \eta_{\Lambda^c}) \in E_\Lambda \times E_{\Lambda^c} \to \Phi_\Lambda(\xi_\Lambda \mid \eta_{\Lambda^c}) \in \underline{E}$$

so that $(\Phi_\Lambda(\xi_\Lambda \mid \eta_{\Lambda^c}))_\Lambda = \xi_\Lambda$ and $(\Phi_\Lambda(\xi_\Lambda \mid \eta_{\Lambda^c}))_{\Lambda^c} = \eta_{\Lambda^c}$. In particular, if $\varnothing \neq \Lambda \subset\subset Z^n$, define $g_\Lambda : E_\Lambda \times E_{\Lambda^c} \to R^1$ by

$$g_\Lambda(\xi_\Lambda \mid \eta_{\Lambda^c}) = \exp\left[- \sum_{F: F \cap \Lambda \neq \varnothing} J_F \circ \Phi_\Lambda(\xi_\Lambda \mid \eta_{\Lambda^c}) \right]$$

and set

$$Z_\Lambda(\eta_{\Lambda^c}) = \int_{E_\Lambda} g_\Lambda(\xi_\Lambda \mid \eta_{\Lambda^c})\, \lambda^\Lambda(d\xi_\Lambda)$$

We say that $g \in M_1(\underline{E})$ is a *Gibbs state for the potential* $\mathscr{J}$ and write $g \in \mathscr{G}(\mathscr{J})$ if, for each $\varnothing \neq \Lambda \subset\subset Z^\nu$,

$$\eta_{\Lambda^c} \in E_{\Lambda^c} \to g_\Lambda(\zeta_\Lambda \mid \eta_{\Lambda^c})\, \lambda^\Lambda(d\zeta_\Lambda)/Z_\Lambda(\eta_{\Lambda^c})$$

is a regular conditional probability distribution on E_Λ of g given $\mathscr{B}_{\Lambda^c}$ [i.e., for all $\phi \in \mathscr{B}_E$:

$$\eta \to \int_{E_\Lambda} \phi \circ \Phi_\Lambda(\zeta_\Lambda \mid \eta_{\Lambda^c})\, g_\Lambda(\zeta_\Lambda \mid \eta_{\Lambda^c})\, \lambda^\Lambda(d\zeta_\Lambda(d\zeta_\Lambda/Z_\Lambda(\eta_{\Lambda^c})$$

is the conditional expectation value of ϕ given $\mathscr{B}_{\Lambda^c}$].

$\Omega = C([0, \infty); E)$ with the topology of uniform convergence on finite intervals and $\mathscr{M}$ is the Borel field $\mathscr{B}_\Omega$ over Ω. Given $t \geqslant 0$, $\eta(t): \Omega \to E$ is the evaluation map at time t and $\mathscr{M}_t = \sigma(\eta(s): 0 \leqslant s \leqslant t)$. We say that $P \in M_1(\Omega)$ *solves the martingale problem for* L *at* $\eta \in E$ if

$$\left(\phi(\eta(t)) - \phi(\eta) - \int_0^t L\phi(\eta(s))\, ds, \mathscr{M}_t, P \right)$$

is a mean zero martingale for all $\phi \in \mathscr{D}(E)$.

The following theorem summarizes a few of the basic facts about the situation described above. At least when M is the circle, proofs can be found in Ref. 9. For general (M, r), proofs have been given in the thesis of Clemens.[2]

Theorem 2.1. For each $\eta \in E$ there is precisely one P_η that solves the martingale problem for L at η. Moreover, the family $\{P_\eta : \eta \in E\}$ forms a Feller continuous, strong Markov family. Next, set $P(t, \zeta, \cdot) = P_\zeta \circ \eta(t)^{-1}$,

$(t, \zeta) \in [0, \infty) \times E$, and define $\{P_t : t \geqslant 0\}$ on $\mathscr{B}_E$ by $P_t\phi(\zeta) = \int \phi(\eta)\, P(t, \zeta, d\eta)$. Then for each $\Lambda \subset\subset Z^\nu$ there is a continuous map

$$(t, \zeta) \in (0, \infty) \times E \to p_\Lambda(t, \zeta, \cdot) \in C^\infty(E_\Lambda)^+$$

such that

$$P_\Lambda(t, \zeta, d\eta_\Lambda) = p_\Lambda(t, \zeta, \eta_\Lambda)\, \lambda^\Lambda(d\eta_\Lambda)$$

In fact, $p_\Lambda(t, \zeta, \eta_\Lambda) > 0$ for all $(t, \zeta, \eta_\Lambda) \in (0, \infty) \times E \times E_\Lambda$ and

$$\sup_{\varnothing \neq \Lambda \subset\subset Z^\nu} \max_k \sup_{(t,\zeta) \in [\delta, 1/\delta] \times E} \int \frac{\| [\nabla_k p_\Lambda(t, \zeta, \cdot)](\eta_\Lambda) \|^2}{p_\Lambda(t, \zeta, \eta_\Lambda)}\, \lambda^\Lambda(d\eta_\Lambda) < \infty \tag{2.1}$$

for each $\delta \in (0, 1]$. Also, if $\mu \in M_1(E)$, then μ is $\{P_t : t \geqslant 0\}$-invariant (i.e., $\mu = \mu P_t$, $t \geqslant 0$) if and only if $\int_E L\phi\, d\mu = 0$ for all $\phi \in \mathscr{D}(E)$. Finally, $\mathscr{G}(\mathscr{I})$ is a nonempty, compact, convex subset of $M_1(E)$; $g \in \mathscr{G}(\mathscr{I})$ if and only if, for each $T > 0$, $t \in [0, T] \to \eta(t)$ and $t \in [0, T] \to \eta(T - t)$ have the same distribution under $P_g = \int_E p_\eta\, g(d\eta)$ if and only if $\int_E \phi L\psi\, dg = \int_E \psi L\phi\, dg$ for all $\phi. \psi \in \mathscr{D}(E)$. In particular, for each $g \in \mathscr{G}(\mathscr{I})$: $\{P_t : t \geqslant 0\}$ has a unique extension as a strongly continuous semigroup $\{\bar{P}_t^g : t \geqslant 0\}$ of nonnegativity-preserving self-adjoint contractions on $L^2(g)$;

$$\mathscr{E}^g(\phi, \phi) = \lim_{t \to 0} \frac{1}{t} (\phi - P_t\phi, \phi)_{L^2(g)} = \sup_{t > 0} \frac{1}{t} (\phi - P_t\phi, \phi)_{L^2(g)}, \qquad \phi \in L^2(g)$$

is a Dirichlet form; and g is an extreme element of $\mathscr{G}(\mathscr{I})$ if and only if $\phi = E^g[\phi]$ (a.s.g.) whenever $\phi \in L^2(g)$ and $\mathscr{E}^g(\phi, \phi) = 0$.

One of our aims in this article is to study the long-time asymptotics of the normalized occupation time functional

$$L_t = \frac{1}{t} \int_0^t \delta_{\eta(s)}\, ds$$

under the measures P_η. To begin this program, we introduce Donsker and Varadhan's *rate function* I: $M_1(E) \to [0, \infty]$ given by

$$I(\mu) = \sup \left\{ -\int \frac{Lu}{u}\, d\mu,\ u \in \mathscr{D}(E) \text{ and } u > 0 \right\}$$

Clearly, I is lower semicontinuous [$M_1(E)$ is always given the topology of weak convergence] and convex. In fact, if λ: $C(E) \to R^1$ is defined by

$$\lambda(V) = \lim_{t \to \infty} \frac{1}{t} \log \left\{ \sup_{\eta \in E} E^{P_\eta} \exp\left[\int_0^t V(\eta(s))\, ds \right] \right\}$$

then (cf. Theorem 7.18 and Corollary 7.19 in Ref. 12 and be warned that J is used in place of I throughout that reference) λ and I are duals of one another under the Legendre transform:

$$I(\mu)=\sup\left\{\int V\,d\mu-\lambda(V)\colon V\in C(E)\right\},\qquad \mu\in M_1(E) \tag{2.2}$$

and

$$\lambda(V)=\sup\left\{\int V\,d\mu-I(\mu)\colon \mu\in M_1(E)\right\},\qquad V\in C(E) \tag{2.3}$$

From (1.3) and (1.4) it is quite easy (cf. Corollary 7.26 in Ref. 12) to see that

$$I(\mu)=0 \qquad \text{if and only if} \quad \mu=\mu P_t \quad \text{for all} \quad t\geqslant 0 \tag{2.4}$$

and that (cf. Theorem 8.1 in Ref. 12)

$$\varlimsup_{t\to\infty}\frac{1}{t}\log\sup_{\eta\in E}P_\eta(L_t\in\Gamma)\leqslant-\inf_{\mu\in\bar{\Gamma}}I(\mu) \tag{2.5}$$

for all $\Gamma\in\mathscr{B}_{M_1(E)}$. In particular, if Γ is a closed subset of $M_1(E)$ and Γ contains no $\{P_t\colon t\geqslant 0\}$-invariant measure, then

$$\varlimsup_{t\to\infty}\frac{1}{t}\log\sup_{\eta\in E}P_\eta(L_t\in\Gamma)<0 \tag{2.5$'$}$$

Although (2.5) and (2.5′) are themselves of some interest as they stand, they have two serious drawbacks. First, (2.5) is incomplete in the sense that it lacks an accompanying lower bound. Second, $I(\mu)$ does not lend itself to easy computation or, for that matter, even easy estimation. For these reasons, we now introduce Donsker and Varadhan's other candidate for a rate function. Namely, given a $g\in\mathscr{G}(\mathscr{J})$, define $J_\sigma^g(\mu)$ for $\mu\in M_1(E)$ so that $J_\sigma^g(\mu)=\infty$ if μ is not absolutely continuous with respect to g and

$$J_\sigma^g(\mu)=\mathscr{E}^g(f^{1/2},f^{1/2}) \qquad \text{if} \quad d\mu=f\,dg$$

Using elementary properties of Dirichlet forms, one can check that $f\in L^1(g)^+\to\mathscr{E}^g(f^{1/2},f^{1/2})$ is lower semicontinuous and convex (cf. Lemma 7.40 in Ref. 12); from which it is clear that $\mu\in M_1(E)\to J_\sigma^g(\mu)$ is convex. On the other hand, it does *not* follow that $\mu\in M_1(E)\to J_\sigma^g(\mu)$ is lower semicontinuous; and this circumstance is the source of the major obstruction to a general theory based on J_σ^g. Nevertheless, there are several interesting properties of J_σ^g that do not rely on lower semicontinuity. In particular, let $\overline{L^g}$ denote the generator of $\{\overline{P_t^g}: t \geqslant 0\}$ in $L^2(g)$ and define $\lambda_\sigma^g(V)$ for $V \in C(E)$ by

$$\lambda_\sigma^g(V) = \lim_{t \to \infty} \frac{1}{t} \log E^{P_g} \exp\left[\int_0^t V(\eta(s))\, ds\right]\Bigg\}$$

Then an equivalent expression for $\lambda_\sigma^g(V)$ is

$$\lambda_\sigma^g(V) = \sup\left\{\int V\psi^2\, dg + (\psi, \overline{L^g}\psi)_{L^2(g)} : \psi \in \mathrm{Dom}(\overline{L^g}) \text{ and } \|\psi\|_{L^2(g)} = 1\right\}$$

From this second expression for λ_σ^g it is easy to see that λ_σ^g is the Legendre transform of J_σ^g:

$$\lambda_\sigma^g(V) = \sup\left\{\int V\, d\mu - J_\sigma^g(\mu): \mu \in M_1(E)\right\} \tag{2.6}$$

Unfortunately, unless J_σ^g is lower semicontinuous, one cannot invert (2.6) to conclude that J_σ^g is the Legendre transform of λ_σ^g and hence that there is an upper bound like (2.5) with I replaced by J_σ^g. In order to explain what we can say in this direction, define $S^p(g)$, $p \in [1, \infty]$, to be the set of $\mu \in M_1(E)$ such that there exist $T_p \in [0, \infty)$ and $f_{T_p} \in L^p(g)$ with the property that $d(\mu P_{T_p}) = f_{T_p}\, dg$.

Theorem 2.2. Let $g \in \mathscr{G}(\mathscr{J})$ be given. If g is extreme in $\mathscr{G}(\mathscr{J})$ and $\mu \in S^1(g)$, then

$$\varliminf_{t \to \infty} \tfrac{1}{t} \log P_\mu(L_t \in \Gamma) \geqslant -\inf_{m \in \inf \Gamma} J_\sigma^g(m), \qquad \Gamma \in \mathscr{B}_{M_1(E)} \tag{2.7}$$

On the other hand, if J_σ^g is lower semicontinuous and $\mu \in \bigcap_{p \in [1,\infty)} S^p(g)$, then

$$\varlimsup_{t \to \infty} \frac{1}{t} \log P_\mu(L_t \in \Gamma) \leqslant -\inf_{m \in \Gamma} J_\sigma^g(m), \qquad \Gamma \in \mathscr{B}_{M_1(E)} \tag{2.8}$$

In particular, if $g \in \mathrm{ext}(\mathscr{G}(\mathscr{J}))$ and J_σ^g is lower semicontinuous, then for all $\mu \in \bigcap_{p \in [1,\infty)} S^p(g)$ and $\Phi \in C(M_1(E))$:

$$\begin{aligned} &\lim_{t \to \infty} \frac{1}{t} \log E^{P_\mu}\{\exp[t\Phi(L_t)]\} \\ &\qquad = \sup\{\Phi(m) - J_\sigma^g(m): m \in M_1(E)\} \end{aligned} \tag{2.9}$$

Proof. Suppose $g \in \text{ext}(\mathcal{G}(\mathcal{J}))$. Then, for all $\phi \in L^2(g)$, $\mathcal{E}^g(\phi, \phi) = 0$ if and only if ϕ is m-almost surely constant. Hence, by the same argument as is used to prove Theorem 8.2 in Ref. 12, (2.7) can be shown to hold for all $\mu \in M_1(E)$ with $\mu \ll g$. Thus, if $\mu \in S^1(g)$, then there is a $T \in [0, \infty)$ such that (2.7) holds when μ is replaced by $\mu_T = \mu P_T$. But if $\theta_T \colon \Omega \to \Omega$ denotes the time shift map, then $P_{\mu_T}(L_t \in \Gamma) = P_\nu(L_t \circ \theta_T \in \Gamma)$ and clearly $\|L_t - L_t \circ \theta_T\|_{\text{var}} \leqslant 2T/t$. Hence, if $m \in \text{int}\, \Gamma$ and B is an open neighborhood of m such that $\bar{B}$ is a positive variation norm distance from Γ^c, then

$$\begin{aligned}\varliminf_{t \to \infty} \frac{1}{t} \log P_\mu(L_t \in \Gamma) &\geqslant \varliminf_{t \to \infty} \frac{1}{t} \log P_\mu(L_t \colon \theta_T \in B)\\ &= \varliminf_{t \to \infty} \frac{1}{t} \log P_{\mu_T}(L_t \in B)\\ &\geqslant -\inf_{\beta \in B} J_\sigma^g(\beta) \geqslant -J_\sigma^g(m)\end{aligned}$$

Next, assume that J_σ^g is lower semicontinuous. Then, by Lemma 8.18 in Ref. 12

$$\varlimsup_{t \to \infty} \frac{1}{t} \log P_g(L_t \in \Gamma) \leqslant -\inf_{\mu \in \Gamma} J_\sigma^m(\mu)$$

Hence, if $d\mu = f\, dg$, where $f \in L^p(g)$, then, by Hölder's inequality,

$$\varlimsup_{t \to \infty} \frac{1}{t} \log P_\mu(L_t \in \Gamma) \leqslant -\inf_{m \in \bar{\Gamma}} \frac{1}{p'} J_\sigma^g(m)$$

where p' is the Hölder conjugate of p. Now suppose that $\mu \in \bigcap_{p \in [1,\infty)} S^p(g)$. Then, for each $p \in [1, \infty)$ there is a $T_p \in [0, \infty)$ such that

$$\varlimsup_{t \to \infty} \frac{1}{t} \log P_{\mu_{T_p}}(L_t \in \Gamma) \leqslant -\frac{1}{p'} \inf_{m \in \bar{\Gamma}} J_\sigma^g(m)$$

By the same reasoning as was used in the preceding paragraph, we can now conclude that for any $\varepsilon > 0$

$$\begin{aligned}\varlimsup_{t \to \infty} \frac{1}{t} \log P_\mu(L_t \in \Gamma) &\leqslant \varlimsup_{t \to \infty} \frac{1}{t} p_{\mu_{T_p}}(L_t \in \Gamma^\varepsilon)\\ &\leqslant -\frac{1}{p'} \inf_{m \in \Gamma^\varepsilon} J_\sigma^g(m)\end{aligned} \tag{2.10}$$

where

$$\Gamma^\varepsilon := \{\mu' \colon \|\mu - \mu'\|_{\text{var}} < \varepsilon \text{ for some } \mu \in \Gamma\}$$

Since (2.10) holds for all $p \in [1, \infty)$,

$$\overline{\lim_{t \to \infty}} \frac{1}{t} \log P_\mu(L_t \in \Gamma) \leqslant -\inf_{m \in \Gamma^\varepsilon} J_\sigma^g(m)$$

for all $\varepsilon > 0$, and clearly (2.8) results from this and the lower semicontinuity of J_σ^g. ∎

Comparing (2.8) and (2.5), one is inclined to ask whether I and J_σ^g are not closely related. A partial answer is provided in the work of Donsker and Varadhan. Namely, one has (cf. Theorem 7.44 in Ref. 12) that

$$I(\mu) \leqslant J_\sigma^g(\mu), \qquad \mu \in M_1(E) \tag{2.11}$$

and that

$$I(\mu) = J_\sigma^g(\mu), \qquad \mu \in M_1(E) \quad \text{with } \mu P_t \ll g \quad \text{for all } t > 0 \tag{2.12}$$

Obviously, if (as can be the case when $\nu \geqslant 3$) $\mathscr{G}(\mathscr{J})$ has more than one element, then $I(\mu) = J_\sigma^g(\mu)$ must fail for some $\mu \in M_1(E)$. Indeed, if $\mathscr{G}(\mathscr{J})$ contains more than one element, then so does $\text{ext}(\mathscr{G}(\mathscr{J}))$. Let g and g' be distinct elements of $\text{ext}(\mathscr{G}(\mathscr{J})$. Then $g \perp g'$, and so $J_\sigma^g(g') = \infty$, whereas $I(g') = 0$.

The difference between I and J_σ^g is, of course, a manifestation of the weak ergodicity of the processes under consideration. In particular, we do not even know, in general, that every $\{P_t : t \geqslant 0\}$-invariant measure is a Gibbs state. As we will now show, one can make effective use of the function I to study such problems; namely, we use I to prove that, when $\nu \in \{1, 2\}$, every $\{P_t : t \geqslant 0\}$-invariant measure is a Gibbs state. This result was obtained by us in Ref. 9 using the full force of Theorem (2.1; the present proof is much more elementary [in particular we do not use relation 2.1)]. In Section 3 we will use similar ideas to show that, when $\nu = 1$, there are nontrivial choices of $\mathscr{J}$ for which one can show that $I = J_\sigma^g$ [when $\nu = 1$, $\mathscr{G}(\mathscr{J})$ contains only one element and so the choice of g is unambiguous].

In the following, $H'(E_{\Lambda_n})$ denotes the Hilbert space obtained by completing $C^\infty(E_{\Lambda_n})$ with respect to $\|\cdot\|_{H'(E_{\Lambda_n})}$ given by

$$\|\psi\|_{H'(E_{\Lambda_n})}^2 = \|\psi\|_{L^2(E_{\Lambda_n})}^2 + \sum_{k \in \Lambda_n} \|\nabla_k \psi\|_{L^2(E(\Lambda_n))}^2$$

Lemma 2.3. If $I(\mu) < \infty$, then, for each $n \geqslant 0$, $d\mu_{\Lambda_n} = f_n \, d\lambda^{\Lambda_n}$, where $f_n^{1/2} \in H'(E_{\Lambda_n})$. In fact, there is a $B \in (0, \infty)$ such that

$$\sum_{k \in \Lambda_n} \int_{E_{\Lambda_n}} \|\nabla_k[(\exp H_k^n/2) f_n^{1/2}]\|^2 (\exp H_k^n) \, d\lambda^{\Lambda_n} \leqslant 2I(\mu) + B\,|\partial \Lambda_n|$$

for $n \geqslant 1$, where

$$H_k^n \equiv \sum_{\{F \subseteq \Lambda_n:\, F \ni k\}} J_F$$

Proof. Set $E_n = E_{\Lambda_n}$, $\mu_n = \mu_{\Lambda_n}$, and $\lambda_n = \lambda^{\Lambda_n}$.

Noting that

$$I(\mu) \geqslant -\int_{E_{n+1}} \frac{Lu}{u}\, d\mu_{n+1}$$

for all $u \in C^\infty(E_n)$ that are strictly positive and taking $\psi = \log u$, we see that

$$I(\mu) \geqslant -\sum_{k \in \Lambda_n} \int_{E_n} \|\nabla_k \psi\|^2\, d\mu_n - \int_{E_{n+1}} L\psi\, d\mu_{n+1}$$

for all $\psi \in C^\infty(E_n)$. Next define $L_n \colon C^\infty(E_n) \to C^\infty(E_n)$ by

$$L_n \psi = \sum_{k \in \Lambda_n} (\exp H_k^n) \operatorname{div}_k[(\exp H_k^n) \nabla_k \psi]$$

Then, by the preceding,

$$\begin{aligned} I(\mu) \geqslant{}& -2 \sum_{k \in \Lambda_n} \int_{E_n} \|\nabla_k \psi\|^2\, d\mu_n - \int_{E_n} L_n \psi\, d\mu_n \\ &+ \sum_{k \in \Lambda_n} \int_{E_{n+1}} (\operatorname{grad}_k \psi \mid \nabla_k \psi - \nabla_k \hat{H}_k^n)\, d\mu_{n+1} \\ \geqslant{}& -2 \sum_{k \in \Lambda_n} \int_{E_n} \|\nabla_k \psi\|^2\, d\mu_n - \int_{E_n} L_n \psi\, d\mu_n \\ &- \frac{1}{4} \sum_{k \in \partial \Lambda_n} \int_{E_{n+1}} \|\nabla_k \hat{H}_k^n\|^2\, d\mu_{n+1} \end{aligned}$$

where $\hat{H}_k^n = H_k - H_k^n$. Hence, if

$$\begin{aligned} I^n(\mu) &= \sup \left\{ -\int_{E_n} \frac{L_n u}{u}\, d\mu \colon u \in C^\infty(E_n) \text{ and } u > 0 \right\} \\ &= \sup \left\{ -\sum_{k \in \Lambda_n} \int_{E_n} \|\nabla_k \psi\|^2\, d\mu - \int_{E_n} L_n \psi\, d\mu \colon \psi \in C^\infty(E_n) \right\} \\ &= 2 \sup \left\{ -2 \sum_{k \in \Lambda_n} \int_{E_n} \|\nabla_k \psi\|^2\, d\mu - \int_{E_n} L_n \psi\, d\mu \colon \psi \in C^\infty(E_n) \right\} \end{aligned}$$

for $\mu \in M_1(E_n)$, then

$$I^n(\mu_n) \leqslant 2I(\mu) + B\,|\partial\Lambda_n| \tag{2.13}$$

where

$$B = \frac{1}{2} \sup_{\substack{n \geqslant 1 \\ k \in \Lambda_n}} \sup_{\eta_{\Lambda_n} \in E_n} \|\nabla_k \hat{H}^n_k(\eta_{\Lambda_n})\|^2$$

To complete the proof, let $\{P^n_t\colon t \geqslant 0\}$ be the diffusion semigroup on $C(E_n)$ determined by L_n [i.e., $P^n_t\psi - \psi = \int_0^t P^n_t L_n \psi\, ds$ for $t > 0$ and $\psi \in C^\infty(E_n)$] and set

$$g_n(d\eta_{\Lambda_n}) = \exp\left[-\sum_{F \subset \Lambda_n} J_F(\eta_{\Lambda_n})\right] \lambda_n(d\eta_{\Lambda_n})/Z_n$$

where $Z_n = \int_{E_n} \exp(-\sum_{F \subset \Lambda_n} J_F(\eta_{\Lambda_n}))\, \lambda_n(d\eta_{\Lambda_n})$. Then, since

$$\int_{E_n} \phi L_n \psi\, dg_n = -\sum_{k \in \Lambda_n} \int_{E_n} (\nabla_k \phi \,|\, \nabla_k \psi)\, dg_n$$

for all $\phi, \psi \in C^\infty(E_n)$, $\{P^n_t\colon t \geqslant 0\}$ is the diffusion semigroup associated with the Dirichlet form $\mathscr{E}_n$ given by

$$\mathscr{E}_n(\psi, \psi) = \sum_{k \in \Lambda_n} \int \|\nabla_k \psi\|^2\, dg_n$$

for $\psi \in H'(E_n)$. Moreover, since L_n is elliptic, P^n_t is given by a smooth kernel. Hence, for all $\mu \in M_1(E_n)$ and $t > 0$, $\mu P^n_t \ll g_n$; and so (cf. Theorem 7.44 in Ref. 12) $J^n(\mu) < \infty$ if and only if $d\mu = f\, dg_n$, where $f^{1/2} \in H'(E_n)$, in which case $J^n(\mu) = \mathscr{E}_n(f^{1/2}, f^{1/2})$. Applying this with $\mu = \mu_n$, our result follows now from Lemma 2.3. ∎

Theorem 2.4. If $I(\mu) = 0$, then, for each $n \geqslant 1$, $d\mu_{\Lambda_n} = f_n\, d\lambda^{\Lambda_n}$, where $f_n^{1/2} \in H'(E_{\Lambda_n})$ and

$$\begin{aligned}&\sum_{k \in \Lambda_{n-1}} \int_E \|\nabla_k(e^{H_k/2} f_n^{1/2})\|^2\, e^{H_k}\, d\lambda^{\Lambda_n} \\ &\qquad \leqslant B^{1/2}\, |\partial\Lambda_n|^{1/2} \left[\sum_{k \in \partial\Lambda_n} \int_{E_{n+1}} \|\nabla_k(e^{H_n/2} f_n^{1/2})\|^2\, e^{H_k}\, d\lambda^{\Lambda_{n+1}}\right]^{1/2}\end{aligned} \tag{2.14}$$

In particular, if $\nu \in \{1, 2\}$, then every $\{P_t : t \geqslant 0\}$-invariant $\mu \in M_1(E)$ is a Gibbs state for $\mathscr{J}$, and for all ν, every translation-invariant, $\{P_t : t \geqslant 0\}$-invariant $\mu \in M_1(E)$ is a Gibbs state for $\mathscr{J}$.

Proof. We continue with the notation used in the proof of Lemma 2.3.

Observe that (Ref. 9) once (2.14) has been proved, the identification of $\{P_t : t \geqslant 0\}$-invariant measures as Gibbs states is quite easy. Thus, we will concentrate on the proof of (2.14). As a first step, note that (cf. Remark at the end of this section), as a consequence of Theorem 2.1, $\mu = \mu P$ implies that $d\mu_n = f_n \, d\lambda_n$, where f_n is a strictly positive element of $C^\infty(E_n)$ for each $n \geqslant 1$. Second, as in the proof of Lemma 2.3, $I(\mu) = 0$ implies that

$$0 \geqslant -\sum_{k \in \Lambda_n} \left[\int_{E_n} \|\nabla_k \psi\|^2 \, d\mu_n + \int_{E_{n+1}} e^{H_k} \operatorname{div}_k(e^{-H_k} \nabla_k \psi) \, d\mu_{n+1} \right]$$

for all $\psi \in C^\infty(E_n)$. Noting that for $k \in \Lambda_{n-1}$

$$\begin{aligned} &-\int_{E_n} \|\nabla_k \psi\|^2 \, d\mu_n - \int_{E_{n+1}} e^{H_k} \operatorname{div}_k(e^{-H_k} \nabla_k \psi) \, d\mu_{n+1} \\ &\quad = -\int_{E_n} (f_n^{1/2} \nabla_k \psi \mid f_n^{1/2} \nabla_k \psi - 2e^{-H_k/2} \nabla_k(e^{H_k/2} f_n^{1/2})) \, d\lambda_n \end{aligned}$$

and that for $k \in \partial \Lambda_n$

$$\begin{aligned} &-\int_{E_n} \|\nabla_k \psi\|^2 \, d\mu_n - \int_{E_{n+1}} e^{H_k} \operatorname{div}_k(e^{-H_k} \nabla_k \psi) \, d\mu_{n+1} \\ &\quad \equiv -\int_{E_n} \|f_n^{1/2} \nabla_k \psi\|^2 \, d\lambda_n \\ &\qquad + 2 \int_{E_{n+1}} (f_{n+1}^{1/2} \nabla_k \psi \mid e^{-H_k/2} \nabla_k(e^{H_k/2} f_{n+1}^{1/2}) \, d\lambda_{n+1} \\ &\quad \geqslant -\int_{E_n} \|f_n^{1/2} \nabla_k \psi\|^2 \, d\lambda_n \\ &\qquad - 2 \left(\int_{E_n} \|f_n^{1/2} \nabla_k \psi\|^2 \, d\lambda_n \right)^{1/2} \\ &\qquad \times \left[\int_{E_{n+1}} \|\nabla_k(e^{H_k/2} f_{n+1}^{1/2})\|^2 \, e^{-H_k} \, d\lambda_{n+1} \right]^{1/2} \end{aligned}$$

we arrive at

$$\sum_{k \in \partial \Lambda_n} \int_{E_n} \|f_n^{1/2} \nabla_k \psi\|^2 \, d\lambda_n$$
$$+ 2 \left(\sum_{k \in \partial \Lambda^n} \int_{E_n} \|f_n^{1/2} \nabla_k \psi\|^2 \, d\lambda_n \right)^{1/2}$$
$$\times \left[\sum_{k \in \partial \Lambda_n} \int_{E_{n+1}} \|\nabla_k (e^{H_k/2} f_{n+1}^{1/2})\|^2 e^{-H_k} \, d\lambda_{n+1} \right]^{1/2}$$
$$\geqslant \sum_{k \in \Lambda_{n-1}} \int_{E_n} (f_n^{1/2} \nabla_k \psi \mid -f_n^{1/2} \nabla_k \psi + 2e^{-H_k/2} \nabla_k (e^{H_k/2} f_n^{1/2})) \, d\lambda_n$$

for all $\psi \in C^\infty(E_n)$. In particular, taking $\psi = \psi_\varepsilon = \frac{1}{2}\varepsilon(\sum_{F \subseteq \Lambda_n} J_F + \log f_n)$ and noting that

$$f_n^{1/2} \nabla_k \psi_\varepsilon = \frac{\varepsilon}{2} f_k^{1/2} \left(\nabla_k H_k^n + \frac{1}{f_n} \nabla_k f_n \right)$$
$$= \varepsilon [\exp(-H_k^n/2)] \, \nabla_k \{ [\exp(H_k^n/2)] \, f_n^{1/2} \}$$

for $k \in \Lambda_n$ and that $H_k^n = H_k$ for $k \in \Lambda_{n+1}$, the preceding together with (2.13) yields

$$\varepsilon^2 \sum_{k \in \partial \Lambda_n} \int_{E_n} \|\nabla_k \{ [\exp(H_k^n/2)] \, f_n^{1/2} \}\|^2 \exp(-H_k^n) \, d\lambda_n + 2\varepsilon B^{1/2} \, |\partial \Lambda_n|^{1/2}$$
$$\times \left(\sum_{k \in \partial \Lambda_n} \int_{E_{n+1}} \|\nabla_k \{ [\exp(H_k/2)] \, f_{n+1}^{1/2} \}\|^2 \exp(-H_k) \, d\lambda_{n+1} \right)^{1/2}$$
$$\geqslant (2\varepsilon - \varepsilon^2) \sum_{k \in \Lambda_{n-1}} \int_{E_n} \|\nabla_k \{ [\exp(H_k/2)] \, f_n^{1/2} \}\|^2 \exp(-H_k) \, d\lambda_n \tag{2.15}$$

After dividing by ε and letting $\varepsilon \to 0$, we obtain (2.14). ∎

Remark. As mentioned before, Theorem 2.4 was proved in Ref. 9 using the estimates in Theorem 2.1, especially (2.1). In the proof given here, we have used the much simpler fact that $P_{\Lambda_n}(1, \eta, \cdot)$ admits a smooth, positive density with respect to λ^{Λ_n}. Actually, we could have avoided using even this relatively elementary fact. Indeed, the existence of f_n, $n \geqslant 1$, with $f_n^{1/2} \in H'(E_{\Lambda_n})$ comes from Lemma 2.3. In addition, a mollification procedure (cf. Ref. 13) allows one to find, for a given $n \geqslant 1$, a sequence

$\{\mu^l\}_{l=1}^{\infty} \subset M_1(E)$ such that $\mu^l \to \mu$, $I(\mu^l) \to I(\mu)$, $d(\mu^l)_{\Lambda_{n+1}} = f^l_{n+1}\, d\lambda^{\Lambda_{n+1}}$, where f^l_{n+1} is a strictly positive element of $C^\infty(E_{\Lambda_n})$, and

$$\|(f^l_{n+1})^{1/2} - f^{1/2}_{n+1}\|_{H'(E_{n+1})} \to 0$$

Hence, we could have arrived at (2.14) via a limit procedure in which μ is replaced by μ^l and l is allowed to become infinite.

3. LOGARITHMIC SOBOLEV INEQUALITIES AND GIBBS STATES

In this section we give conditions that imply the existence of a logarithmic Sobolev inequality for some Gibbs states. We then show how a logarithmic Sobolev inequality allows us to prove that $I = J^g_\sigma$ when $v = 1$ and to obtain an upper bound on $-\inf_{\mu \in \Gamma} J^g_\sigma(\mu)$ [and therefore on $\overline{\lim}_{t\to\infty}(1/t)\log P_g(L_t \in \Gamma)$] for any v when

$$\Gamma = \left\{\mu \in M_1(E) \colon \int \phi\, d\mu - \int \phi\, dg \geqslant \varepsilon\right\}$$

for some $\phi \in C(E)$ and $\varepsilon > 0$.

The theorem that gives us a logarithmic Sobolev inequality is the following.

Theorem 3.1. Let Ric denote the Ricci curvature tensor for (M, r) and assume that $\text{Ric} \geqslant \beta r$ (in the sense of quadratic forms) on $T(M) \times T(M)$ for some $\beta \in (0, \infty)$. In addition assume that there is a $\gamma\colon Z^v \to [0, \infty)$ and an $0 < \varepsilon < 1$ such that $\sum_{k \in Z^v} \gamma(k) \leqslant (1-\varepsilon)\beta$ and

$$\begin{aligned} \sum_{F \supseteq \{k,l\}} &|\text{Hess}(J_F)(\nabla_k f, \nabla_l f)| \\ &\leqslant \sum_{k,l \in Z^v} \gamma(k-l)\, \|\nabla_k f\|\ \|\nabla_l f\| \end{aligned} \tag{3.1}$$

for all k, $l \in Z^v$ and $f \in \mathscr{D}$. Then $\mathscr{G}(\mathscr{J})$ contains precisely one element g. Moreover, if

$$G_{\Lambda_n,\eta}(d\zeta_{\Lambda_n}) = g_{\Lambda_n}(\zeta_{\Lambda_n} \mid \eta_{\Lambda_n^c})\, \lambda^{\Lambda_n}(d\zeta_{\Lambda_n})/Z_{\Lambda_n}(\eta_{\Lambda_n^c})$$

for $n \geqslant 0$ and $\eta \in E$, then

$$\begin{aligned} \int_{E_n} &\phi(\zeta_{\Lambda_n})^2 \log[|\phi(\zeta_{\Lambda_n})|/\|\phi\|_{L_2(G_{\Lambda_n,\eta})}] \\ &\leqslant \frac{4}{\varepsilon\beta} \sum_{k \in \Lambda_n} \int \|\nabla_k \phi(\zeta_{\Lambda_n})\|^2\, G_n(d\zeta_{\Lambda_n} \mid \eta_{\Lambda_n^c}) \end{aligned} \tag{3.2}$$

for all $\phi \in C^\infty(E_n)$. In particular,

$$\int_E \phi(\zeta)^2 \log(|\phi(\zeta)|/\|\phi\|_{L^2(g)}) \leqslant \frac{4}{\varepsilon\beta} \mathscr{E}^g(\phi, \phi), \qquad \phi \in L^2(g) \tag{3.3}$$

Proof. When $M = S^d$ and $g \in \operatorname{ext}(\mathscr{G}(\mathscr{J}))$, (3.2) and (3.3) are proved in Ref. 1. Since the general manifold case is exactly the same as when $M = S^d$, we will restrict our attention here to the proof that $\mathscr{G}(\mathscr{J})$ contains only one element.

To prove that there is only one element in $\mathscr{G}(\mathscr{J})$, we will produce a Markov semigroup $\{\hat{P}_t : t \geqslant 0\}$ with the properties that every $g \in \mathscr{G}(\mathscr{J})$ is $\{\hat{P}_t : t \geqslant 0\}$-invariant and

$$\lim_{T \to \infty} \sup_{\zeta, \eta \in E} |\hat{P}_T\phi(\zeta) - \hat{P}_T\phi(\eta)| = 0 \tag{3.4}$$

for each $\phi \in \mathscr{D}(E)$. To this end, define $\hat{L}_n : C^\infty(E_n) \to C^\infty(E_n)$ by

$$\hat{L}_n\phi = \sum_{k \in \Lambda_n} 2^{|k|} \operatorname{div}_k\{[\exp(-H_k^n)] \nabla_k \phi\}$$

where

$$H_k^n = \sum_{(F \subseteq \Lambda_n : F \ni k)} J_F$$

and denote by $\{\hat{P}_t^n : t \geqslant 0\}$ the associated Markov semigroup on $C(E_n)$. Then $C^\infty(E_n)$ is $\{\hat{P}_t^n : t \geqslant 0\}$-invariant. Moreover, by the same reasoning as was used in Ref. 1, if

$$\Gamma_1^n(\phi, \phi) \equiv \frac{1}{2}[\hat{L}_n\phi^2 - 2\phi\hat{L}_n\phi] = \sum_{k \in \Lambda_n} 2^{|k|} \|\nabla_k\phi\|^2$$

and

$$\Gamma_2^n(\phi, \phi) \equiv \frac{1}{2}[\hat{L}_n\Gamma_1^n(\phi, \phi) - 2\Gamma_1^n(\phi, \hat{L}_n\phi)]$$

then

$$\Gamma_2^n(\phi, \phi) \geqslant \varepsilon\beta\Gamma_1^n(\phi, \phi), \qquad \phi \in C^\infty(E_n)$$

Next, note that for each $T > 0$ and $\phi \in C^\infty(E_n)$

$$\frac{d}{dt}\hat{P}_t^n\Gamma_1^n(\hat{P}_{T-t}^n\phi, \hat{P}_{T-t}^n\phi) = \hat{P}_t^n\Gamma_2^n(\hat{P}_{T-t}^n\phi, \hat{P}_{T-t}^n\phi), \qquad t \in [0, T]$$

Thus,

$$\|\Gamma_1^n(\hat{P}_T^n\phi < \hat{P}_T^n\phi)\|_{C(E_n)} \leqslant e^{-\varepsilon\beta T}\,\|\Gamma_1^n(\phi, \phi)\|_{C(E_n)}$$

At the same time, by the mean-value theorem, there is a $K \in (0, \infty)$, which is independent of n, such that

$$\sup_{\zeta,\eta \in E_n} |\psi(\zeta) - \psi(\eta)| \leqslant K\,\|\Gamma_1^n(\psi, \psi)\|_{C(E_n)}^{1/2}, \qquad \psi \in C^\infty(E_n)$$

Thus, we conclude that

$$\sup_{\zeta,\eta \in E_n} |\hat{P}_T^n\phi(\zeta) - \hat{P}_t^n\phi(\eta)| \leqslant K e^{-2\varepsilon\beta T}\,\|\Gamma_1^n(\phi, \phi)\|_{C(E_n)} \tag{3.5}$$

for all $n \geqslant 0$, $T > 0$, and $\phi \in C^\infty(E_n)$.

Finally, let $\{\hat{P}_t : t \geqslant 0\}$ be the Markov semigroup on $C(E)$ associated with $\hat{L}: \mathscr{D}(E) \to \mathscr{D}(E)$ given by

$$\hat{L}\phi = \sum_{k \in Z^\nu} 2^{|k|} e^{H_k}\,\mathrm{div}_k(e^{-H_k}\nabla_k\phi)$$

Then every $g \in \mathscr{G}(\mathscr{J})$ is $\{\hat{P}_t : t \geqslant 0\}$-invariant (in fact, reversible). Also, for each $T > 0$ and $\phi \in C(E)$, $[\hat{P}_T^n\phi \circ \pi_{A_n}] \circ \pi_{A_n} \to \hat{P}_T\phi$ uniformly on E. Hence, by (3.5), (3.4) holds for each $\phi \in \mathscr{D}(E)$. ∎

Note that Theorem 3.1 applies only to manifolds with a nonzero Ricci curvature. For example, it applies to S^2, where the Ricci curvature equals the usual metric. Thus, in this case, if the interaction is

$$J_F(x) = \begin{cases} \beta(x_i \cdot x_j) & \text{if} \quad F = \{i, j\} \quad \text{with} \quad |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

then for $\beta < 1/(4\nu)$ this process (the stochastic Heisenberg model) has a unique stationary measure, and that stationary measure, which is necessarily a Gibbs state, satisfies a logarithmic Sobolev inequality.

Our next goal is to show that if $g \in \mathscr{G}(\mathscr{J})$ satisfies (L.S.), then J_σ^g can sometimes be used in place of I to estimate $\overline{\lim}_{t \to \infty}(1/t)\log P(L_t \in \Gamma)$. We begin by showing that, when $\nu = 1$, (L.S.) implies that I actually coincides with J_σ^g [recall that, when $\nu = 1$, there is only one $g \in \mathscr{G}(\mathscr{J})$]. We know of no nontrivial examples in which $I = J_\sigma^g$ when $\nu \geqslant 2$; and we cannot rule out the possibility that $I = J_\sigma^g$ whenever $|\mathscr{G}(\mathscr{J})| = 1$, or at least whenever $|\mathscr{G}(\mathscr{J})| = 1$ and the unique $g \in \mathscr{G}(\mathscr{J})$ satisfies (L.S.).

We begin with the following lemma.

Lemma 3.2. Assume that g is the only element of $\mathscr{G}(\mathscr{J})$ and that g satisfies (3.2). Let $\mu^{(n)} = \mu_{\Lambda_n}$, $n \geqslant 1$, where $\mu \in M_1(\Omega)$, and assume that $d\mu^{(n)} = f_n \, d\lambda^n$, where

$$\sup_n \sum_{k \in \Lambda_{n-1}} \int e^{-H_k} \|\nabla_k(e^{H_k/2} f_n^{1/2})\|^2 \, d\lambda < \infty$$

Then $\mu \ll g$.

Proof. Let $g_n(\cdot \mid \eta)$ be the conditional density of g on Λ_{n+1} given $\eta \in E_{\Lambda_n^c}$. Then denoting $\varepsilon\beta/4$ by α and applying (3.2), we have

$$\begin{aligned}
&\sum_{k \in \Lambda_{n-1}} \int e^{-H_k} \|\nabla_k(e^{H_k/2} f_n^{1/2})\|^2 \, d\lambda \\
&\quad = \sum_{k \in \Lambda_{n-1}} \int g_n(\zeta \mid \eta) \left\| \nabla_k \left(\frac{f_n(\zeta, \eta)}{g_n(\zeta \mid \eta)} \right)^{1/2} \right\|^2 d\zeta \, d\eta \\
&\quad \geqslant \alpha \iint \frac{f_n(\zeta \mid \eta)}{g_n(\zeta \mid \eta)} \log \left(\frac{f_n(\zeta \mid \eta)}{g_n(\zeta \mid \eta)} \right) f_{\partial \Lambda_n}(\eta) \, g_n(\zeta \mid \eta) \, d\zeta \, d\eta \qquad (3.6)
\end{aligned}$$

Let $h_n(\zeta) = \int f_{\partial \Lambda_n}(\eta) \, g_n(\zeta \mid \eta) \, d\eta$. Then by Jensen's inequality applied to $x \log x$ and the $d\eta$ integral, we bound the right side of (2.8) below by

$$\begin{aligned}
&\alpha \int_{\Lambda_{n-1}} f_{n-1}(\zeta) \log \frac{f_{n-1}(\zeta)}{h_n(\zeta)} \, d\zeta_{\Lambda_{n-1}} \\
&\qquad \geqslant \alpha \int f_m(\zeta_{\Lambda_m}) \log \frac{f_m(\zeta_{\Lambda_m})}{(h_n)_{\Lambda_m}(\zeta_{\Lambda_m})} \, d\zeta_{\Lambda_m} \qquad (3.7)
\end{aligned}$$

for $m \leqslant n-1$. Here we have applied Jensen's inequality again, this time to the variables $\zeta_{\Lambda_n \setminus \Lambda_m}$. Note that $(h_n)_{\Lambda_m} \to g_{\Lambda_m}$ as $n \to \infty$ by the uniqueness of the Gibbs state. Thus,

$$\sup_m \int \frac{f_m}{g_{\Lambda_m}} \log \left(\frac{f_m}{g_{\Lambda_m}} \right) g_{\Lambda_m} \, d\zeta_{\Lambda_m} < \infty$$

Therefore $\{f_m / g_{\Lambda_m} : n \geqslant 1\}$ is uniformly integrable with respect to g, and hence $\mu \ll g$. ∎

Since $|\partial \Lambda_n|$ does not depend on n if $\nu = 1$, from Lemma 2.3, (2.11), (2.12), Lemma 3.2, and (2.5) we obtain the following theorem.

Theorem 3.3. If $\nu = 1$ and (3.2) holds, then there is precisely one $g \in \mathscr{G}(\mathscr{J})$ and $I = J_\sigma^g$. In particular, in this case we have that

$$\overline{\lim_{t \to \infty}} \frac{1}{t} \log[\sup_{\eta \in E} P\eta(L_t \in \Gamma)] \leqslant -\inf_{\Gamma} J_\sigma^g \qquad (3.8)$$

for all closed $\Gamma \subseteq M_1(\Omega)$ and that

$$\varliminf_{t\to\infty} \frac{1}{t} \log[P_\mu(L_t \in \Gamma)] \geqslant -\inf_\Gamma J_\sigma^g \tag{3.9}$$

for all open $\Gamma \subseteq M_1(\Omega)$ and all $\mu \in S^1(g)$.

When $v \geqslant 2$ and (L.S.) holds, we can still give an upper bound in terms of J_σ^g.

Theorem 3.4. Let $g \in \mathscr{G}(\mathscr{J})$ and assume that g satisfies (L.S.). Then J_σ^g is lower semicontinuous and $M_1(\Omega)$ and $\bigcup_{p\in(1,\infty]} S^p(g) \subseteq \bigcap_{p\in(1,\infty)} S^p(g)$. In particular,

$$\varlimsup_{t\to\infty} \frac{1}{t} \log[P_\mu(L_t \in \Gamma)] = -\inf_\Gamma J_\sigma^g \tag{3.10}$$

for all $\mu \in \bigcup_{p\in(1,\infty]} S^p(g)$.

Proof. To prove that J_σ^g is lower semicontinuous, suppose that $\mu_n \to \mu$ in $M_1(\Omega)$ and that $\sup_n J_\sigma^g(\mu_n) < \infty$. Then, $d\mu_n = f_n\, dg$, where $\mathscr{E}^g(f_n^{1/2}, f_n^{1/2}) = J_\sigma^g(\mu_n)$ is bounded. Hence, by (L.S.), $\int f_n \log(f_n)\, dg$ is bounded and so $\{f_n\}$ is uniformly g-integrable. But this means that $d\mu = f\, dg$ and that $f_n \to f$ in $L^1(g)$. In particular,

$$J_\sigma^g(\mu) = \mathscr{E}^g(f1/2, f^{1/2}) \leqslant \varliminf_{n\to\infty} \mathscr{E}^g(f_n^{1/2}, f_n^{1/2}) = \varliminf_{n\to\infty} J_\sigma^g(\mu_n)$$

To see that $S^p(g) \subseteq \bigcap_{q\in(1,\infty)} S^q(g)$ for all $p \in (1,\infty)$, it suffices to check that $L^p(g) \subseteq S^q(g)$ for all $1 < p < q < \infty$. But, by Gross's theorem (cf. Theorem 9.10 in Ref. 12)

$$\|P_t\|_{p\to q} = 1 \qquad \text{for } \frac{q-1}{p-1} \leqslant e^{2/\alpha t} \quad ∎$$

Given $g \in \mathscr{G}(\mathscr{J})$, set

$$\Gamma_\varepsilon^g(\phi) = \left\{\mu \in M_1(\Omega) \colon \int \phi\, d\mu - \int \phi\, dg \geqslant \varepsilon\right\}$$

for $\phi \in C(E)$ and $\varepsilon > 0$. We conclude this section by showing that when g satisfies (L.S.), then

$$-\inf_{\Gamma_\varepsilon^g(\phi)} J_\sigma^g \leqslant -\varepsilon^2/[\alpha B(\phi)], \qquad \varepsilon > 0 \tag{3.11}$$

where $B(\phi) \in (0, \infty)$ is a certain number, which depends on ϕ alone.

The first step in the derivation of (3.11) is the simple observation that (L.S.) implies that

$$-\inf_{\Gamma} J_\sigma^g \leqslant -\frac{1}{\alpha} \inf\left\{\int f \log(f)\, dg\colon f\, dg \in \Gamma\right\} \tag{3.12}$$

The second step is taken in the following lemma.

Lemma 3.5. Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and let ϕ be a bounded, continuous, real-valued function on Ω such that $\int_\Omega \phi(x)\, \mu(dx) = 0$. Define

$$\Phi(a) = \int e^{a\phi(x)}\, \mu(dx)$$

Then for all $\varepsilon > 0$,

$$\begin{aligned}\inf\bigg\{&\int f(x) \log f(x)\, \mu(dx)\colon f \geqslant 0, \int f(x)\, \mu(dx) = 1, \text{ and}\\ &\int \phi(x)\, f(x)\, \mu(dx) \geqslant \varepsilon\bigg\}\\ &\geqslant \sup_a\, [a\varepsilon - \log \Phi(a)]\end{aligned}$$

Proof. By a theorem of Sanov (see Lemma 3.38 in Ref. 12), for each $f \geqslant 0$ such that $\int f(x)\, \mu(dx) = 1$, we have

$$\int f(x) \log f(x)\, \mu(dx) = \sup_\psi \left\{\int \psi(x)\, f(x)\, \mu(dx) - \log\left[\int e^{\psi(x)}\, \mu(dx)\right]\right\} \tag{3.13}$$

where the supremum over ψ is over all bounded, measurable functions ψ. Letting ψ be of the form $\psi(x) = a\phi(x)$, we see that

$$\begin{aligned}&\int f(x) \log f(x)\, \mu(dx)\\ &\qquad\geqslant \sup_a \left\{\int a\phi(x)\, f(x)\, \mu(dx) - \log\left[\int e^{a\phi(x)}\, \mu(dx)\right]\right\}\end{aligned} \tag{3.14}$$

Note that $\int \phi(x)\, \mu(dx) = 0$ implies that $\log[\int e^{a\phi(x)}\, \mu(dx)] \geqslant 0$ for all a. Thus, if in addition $\int f(x)\, \phi(x)\, \mu(dx) \geqslant \varepsilon$, then we have

$$\int f(x) \log f(x)\, \mu(dx) \geqslant \sup_a \left\{a\varepsilon - \log\left[\int e^{a\phi(x)}\, \mu(dx)\right]\right\} \quad \blacksquare \tag{3.15}$$

Let ϕ be a bounded, continuous function with $\int \phi(x)\, g(dx) = 0$. We denote $\log[\int e^{a\phi(x)}\, g(dx)]$ by $F(a)$.

Corollary 3.6. If (L.S.) holds and if $\Gamma = \{\mu : \int \phi(x)\, \mu(dx) \geqslant \varepsilon\}$, $\varepsilon > 0$, then

$$- \inf_{\mu \in \Gamma} J_\sigma^g(\mu) \leqslant -(1/\alpha) \sup_a [a\varepsilon - F(a)] \tag{3.16}$$

Proof. This follows immediately from (3.12) and Lemma 3.5. ∎

We now let $K(\varepsilon) = \sup_a [a\varepsilon - F(a)]$. Since $F(0) = 0$ and $F'(0) = 0$ and $F(a) \geqslant 0$ for all a, we have $K(0) = 0$ and $K(\varepsilon) > 0$ for all $\varepsilon > 0$. Note that if $G(x) \geqslant F(x)$ for all $x \geqslant 0$, then

$$K(\varepsilon) = \sup_{a \geqslant 0} [\varepsilon a - F(a)] \geqslant \sup_{a \geqslant 0} [\varepsilon a - G(a)] \tag{3.17}$$

Since $F(0) = F'(0) = 0$ and $F(a) \leqslant a\, \|\phi\|_\infty$ for all a, there is a constant, $B_\phi < \infty$, such that $F(a) \leqslant B_\phi a^2$ for all $a \geqslant 0$. Thus, by (3.17), $K(\varepsilon) \geqslant \varepsilon^2/4B_\phi$ for all $\varepsilon > 0$ and thus

$$- \inf_{\mu \in \Gamma} J_\sigma^g(\mu) \leqslant -\varepsilon^2/4\alpha B_\phi \tag{3.18}$$

The constant $4\alpha B_\phi$ in (3.18) is probably not optimal, but in the case where the $J_F = 0$ for all F (i.e., there is no interaction) one sees that $\inf_{\mu \in \Gamma} J_\sigma^g(\mu)$ is asymptotically a constant times ε^2 as ε goes to zero. Thus, (3.18) is qualitatively correct.

We collect a few of the above observations together for easy reference in the next two sections.

Lemma 3.7. Let ϕ be a bounded, continuous function such that $\int \phi(x)\, g(dx) = 0$. Then for all $f \geqslant 0$ such that $\int f(x)\, g(dx) = 1$,

$$\int \phi(x)\, f(x)\, g(dx) \leqslant 2B \left[\int f(x) \log f(x)\, g(dx) \right]^{1/2} \tag{3.19}$$

for any B such that $\log[\int e^{a\phi(x)}\, g(dx)] \leqslant B^2 a^2$ for all a.

Proof. Let $\varepsilon = \int \phi(x)\, f(x)\, g(dx)$. If $\varepsilon \leqslant 0$, then (3.19) is immediate. Otherwise, from Lemma 3.5 we have

$$\int f(x) \log f(x)\, g(dx) \geqslant K(\varepsilon) \geqslant \varepsilon^2/4B^2 \quad ∎$$

4. FREE ENERGY

In this section the potential $\mathscr{J}$ and all probability measures on Z^ν that occur are assumed to be translation-invariant.

The point of this section is to show that if (3.1) holds (and hence the unique Gibbs state admits a logarithmic Sobolev inequality), then, starting from translation-invariant initial states, the corresponding stochastic Ising model converges exponentially fast to equilibrium.

Our main tool in this and the following section is the Helmholtz free energy. In order to take advantage of the translation invariance of the initial distribution, we work with the specific Helmholtz free energy (i.e., the energy per lattice site) in this section. In the next section we will be concerned with one large but finite cube at a time, and hence in that section we will not need to divide the free energy by the volume of the cube in order to keep the quantities with which we are dealing finite.

The free energy in a cube Λ at time t is defined as follows. Let μ_0 be any initial distribution and let $\mu_t^{(\Lambda)}$ denote the marginal distribution on M^Λ of $\mu_0 P_t$. If $G^{(\Lambda)}(d\xi)$ is the marginal of the [unique if (3.1) holds] Gibbs state, then by Theorem 2.1, $\mu_t^{(\Lambda)} \ll G^{(\Lambda)}$ for all $t>0$. We denote $d\mu_t^{(\Lambda)}/d(G^{(\Lambda)})$ by $f_t^{(\Lambda)}$. The *free energy of* μ_t *on* Λ is defined to be

$$\int_{M^\Lambda} f_t^{(\Lambda)}(\xi) \log[f_t^{(\Lambda)}(\xi)]\, G^{(\Lambda)}(d\xi) \tag{4.1}$$

and the *specific free energy of* μ_t is given by

$$\lim_{\Lambda \to Z^\nu} |\Lambda|^{-1} \int_{M^\Lambda} f_t^{(\Lambda)}(\xi) \log[f_t^{(\Lambda)}(\xi)]\, G^{(\Lambda)}(d\xi) \tag{4.2}$$

If μ_0 is translation-invariant, then μ_t is also translation-invariant and hence the limit in (4.2) exists (possibly $+\infty$) by Theorem 7.2.7 in Ref. 11.

We need the following two facts.

I. There is a constant $C<\infty$ such that for all finite cubes Λ and all initial distributions μ_0

$$\int_{M^\Lambda} f_1^{(\Lambda)}(\xi) \log[f_1^{(\Lambda)}(\xi)]\, G^{(\Lambda)}(d\xi) \leqslant C\,|\Lambda| \tag{4.3}$$

II. For all $\delta>0$ and all $t\in[\delta, \delta^{-1}]$ there is a constant, $C(\delta)<\infty$, such that for all cubes Λ, $f_t^{(\Lambda)}$ and $\log f_t^{(\Lambda)}$ are in the domain of L and

$$\begin{aligned} &\frac{d}{dt}\int f_t^{(\Lambda)}(\xi) \log[f_t^{\Lambda}(\xi)]\, G^{(\Lambda)}(d\xi) \\ &\quad\leqslant \int f_t^{(\Lambda)}(\xi)\, L[\log f_t^{\Lambda}(\xi)]\, G^{(\bar\Lambda)}(d\xi) + |\partial\bar\Lambda|\, C(\delta) \end{aligned} \tag{4.4}$$

where $\bar{\Lambda} = \{k \in Z^\nu: \text{dist}(k, \Lambda) \leqslant R\}$ and $\partial\bar{\Lambda} = \bar{\Lambda} \backslash \Lambda$. Fact I follows from (2.1) just as Theorem 4.14 follows from Theorem 3.9 in Ref. 9. For Fact II see (4.21) and Lemma 4.22 of Ref. 9.

Lemma 4.1. If (L.S.) holds, then for any initial distribution μ_0 and any cube Λ and all $t > 0$

$$\int f_t^{(\Lambda)}(\xi)\, L[\log f_t^{(\Lambda)}(\xi)]\, g(d\xi)$$

$$\leqslant -\frac{4}{\alpha} \int f_t^{(\Lambda)}(\xi) \log f_t^{(\Lambda)}(\xi)\, g(d\xi) \tag{4.5}$$

Proof. Let $\bar{L}^g$ be the generator of the semigroup $\{\bar{P}^g_t: t > 0\}$ in Theorem 2.1. Then, for $\phi, \psi \in \text{Dom}(\bar{L}^g)$

$$-\int \phi \bar{L}^g \psi\, dg = \mathscr{E}^g(\phi, \psi)$$

where

$$\mathscr{E}^g(\phi, \psi) = \frac{1}{4}\left[\mathscr{E}^g(\phi+\psi, \phi+\psi) - \mathscr{E}^g(\phi-\psi, \phi-\psi)\right]$$

and $\mathscr{E}^g$ is described in Theorem 2.1. Next, set $m_t(d\xi \times d\eta) = P(t, \xi, d\eta)\, g(d\xi)$, where $P(t, \xi, \cdot)$ is the transition probability function in Theorem 2.1. Then (cf. Lemma 7.38 in Ref. 12)

$$\mathscr{E}^g(\phi, \psi) = \lim_{t \to 0} \frac{1}{t} \int [\phi(\eta) - \phi(\xi)][\psi(\eta) - \psi(\xi)]\, m_t(d\xi, d\eta)$$

Hence, applying (L.S.) to $(f_t^{(\Lambda)})^{1/2}$, we will prove (4.5) once we show that

$$(a-b)(\log a - \log b) \geqslant 4(a^{1/2} - b^{1/2})^2$$

for all $a, b > 0$. Equivalently, we must show that

$$(x-1)\log x \geqslant 4(x^{1/2} - 1)^2$$

for all $x > 0$. But $x \in (0, \infty) \to (x-1)\log x - 4(x^{1/2}-1)^2$ is a convex function whose minimum occurs at $x = 1$. ∎

Lemma 4.2. If (L.S.) holds, then for all $\delta > 0$ and all $t \in [\delta, \delta^{-1}]$,

$$\frac{d}{dt} \int f_t^{(\Lambda)}(\xi) \log[f_t^{(\Lambda)}(\xi)]\, G^{(\Lambda)}(d\xi)$$

$$\leqslant -\frac{4}{\alpha} \int f_t^{(\Lambda)}(\xi) \log[f_t^{(\Lambda)}(\xi)]\, g^{(\Lambda)}(d\xi) + C(\delta)\, |\partial\bar{\Lambda}| \tag{4.6}$$

Proof. This follows immediately from (4.4) and Lemma 4.1. ∎

Note that by (4.3) and Lemma 4.2 for all $t \in [1, \delta^{-1}]$,

$$\int f_t^{(\Lambda)}(\xi) \log[f_t^{(\Lambda)}(\xi)]\, G^{(\Lambda)}(d\xi) \leqslant e^{-(4/\alpha)(t-1)} C\, |\Lambda| + \tfrac{1}{4}\alpha C(\delta)\, |\partial\bar{\Lambda}| \tag{4.7}$$

Lemma 4.3. If $g \in \mathscr{G}(\mathscr{J})$, and (L.S.) holds for g, then for all $\phi \in \mathscr{D}(E)$ there is a constant $A = A(\phi, \mathscr{J}, \alpha)$ and an $\varepsilon = \varepsilon(\mathscr{J}, \alpha)$ such that

$$\left| \int (\phi \circ S^k)(\phi \circ S^j)\, dg - \int \phi \circ S^k\, dg \int \phi \circ S^j\, dg \right| \leqslant A e^{-\varepsilon|k-j|}$$

Proof. (L.S.) implies that there is a gap of length at least $2/\alpha$ between 0 and the rest of the spectrum of L on $L^2(g)$ (see Ref. 10). The rest follows just as in the proof of Theorem 2.18 in Ref. 8. ∎

Lemma 4.4. Assume that $\mathscr{J}$ satisfies (3.1). Let g be the unique element of $\mathscr{G}(\mathscr{J})$ and $\phi \in \mathscr{D}(E)$ with $\int \phi\, dm = 0$. Define

$$F_\Lambda(a) = \log\left[\int \exp\left(a \sum_k \phi \circ S^k\right) dg\right]$$

where the summation is over all k such that $\phi \circ S^k \in \mathscr{D}(\Lambda)$. Then there is a constant $A < \infty$ and a $\delta > 0$ such that for all $|a| < \delta$ and all cubes Λ

$$\frac{d^2}{da^2} F_\Lambda(a) \leqslant A\, |\Lambda| \tag{4.8}$$

Proof. Let Λ be fixed and suppress it from the notation. Differentiating F twice, we have

$$\begin{aligned} F''(a) = {} & \left\{ \int \left(\sum_k \phi \circ S^k\right)^2 \exp\left(a \sum_j \phi \circ S^j\right) dg \int \exp\left(a \sum_j \phi \circ S^j\right) dg \right. \\ & \left. - \left[\int \sum_k \phi \circ S^k \exp\left(a \sum_j \phi \circ S^j\right) dg\right]^2 \right\} \\ & \times \left[\int \exp\left(a \sum_j \phi \circ S^j\right) dg\right]^{-2} \end{aligned} \tag{4.9}$$

Now let $\mathscr{J}(a, \Lambda) = \mathscr{J} \cup \{a\phi \circ S^j\colon j \text{ such that } \phi \circ S^j \in \mathscr{D}(\Lambda)\}$. That is, $\mathscr{J}(a, \Lambda)$ consists of the elements of $\mathscr{J}$ together with all translates of $a\phi$ that are measurable inside Λ. If $\mathscr{J}$ satisfies (3.1), then there is a $\delta > 0$ such that for all $|a| < \delta$, $\mathscr{J}(a, \Lambda)$ also satisfies (3.1) with ε replaced by $\varepsilon/2$. Assume that $|a|$ is less than this δ and let the unique element in $\mathscr{J}(a, \Lambda)$ be denoted by g_a. Then note that (4.9) is equivalent to

$$\begin{aligned} F''(a) &= \int \Big(\sum_k \phi \circ S^k\Big)^2 dg_a - \Big(\int \sum_k \phi \circ S^k \, dg_a\Big)^2 \\ &= \sum_k \sum_j \Big[\int (\phi \circ S^k)(\phi \circ S^j)\, dg_a - \Big(\int \phi \circ S^k \, dg_a\Big)\Big(\int \phi \circ S^j \, dg_a\Big)\Big] \end{aligned} \tag{4.10}$$

Thus, by Theorem 3.3, (L.S.) holds with an α that may be taken independently of a for $|a| < \delta$. The lemma now follows from the mixing property of Lemma 4.3. ∎

Theorem 4.5. Let $\mathscr{J}$ satisfy (3.1) and denote $4/\varepsilon\beta$ [see (3.2)] by α. Let $\mathscr{G}(\mathscr{J}) = \{g\}$. Then, for all $\phi \in \mathscr{D}(E)$ with $\int \phi \, dg = 0$, there is a constant B_ϕ such that for all translation-invariant initial states μ_0

$$\int \phi(\xi)\, \mu_t(d\xi) \leqslant B_\phi e^{-(2/\alpha)t} \tag{4.11}$$

Proof. Fix a finite cube Λ and note that by translation invariance

$$\int \phi(\xi)\, \mu_t(d\xi) = |\Lambda|^{-1} \sum_{k \in \Lambda} \int \phi \circ S^k(\xi)\, f_t^{(\Lambda + \Lambda_0)}(\xi)\, g(d\xi)$$

where Λ_0 is such that $\phi \in C^\infty_{\Lambda_0}(E)$ and $f_t^{(\Lambda + \Lambda_0)}$ is as in the first part of this section. Then, by (3.19) and (4.7), for any $\delta > 0$ and all $t \in [1, \delta^{-1}]$, we have

$$\int \phi(\xi)\, \mu_t(d\xi) \leqslant 2B_\Lambda \{e^{-(4/\alpha)(t-1)} C\, |\Lambda_0 + \Lambda| + \tfrac{1}{4}\alpha C(\delta)\, |\partial(\overline{\Lambda_0 + \Lambda})|\}^{1/2} \tag{4.12}$$

where B_Λ satisfies $F_{\Lambda + \Lambda_0}(a/|\Lambda|) \leqslant B_\Lambda^2 a^2$ for all $a \geqslant 0$, and $F_{\Lambda + \Lambda_0}$ is as in Lemma 4.4. Note that since $F_{\Lambda + \Lambda_0}(0) = 0$ and $F'_{\Lambda + \Lambda_0}(0) = \int \sum_{k \in \Lambda} \phi \circ S^k \, dg = 0$ and $F_{\Lambda + \Lambda_0}(a) \leqslant a\, |\Lambda|\, \|\phi\|_\infty$ for all a, the existence of such a B_Λ is guaranteed by Lemma 4.8. Moreover, again by Lemma 4.8, we see that there is a constant $B_\phi < \infty$ such that $B_\Lambda^2 \leqslant B_\phi^2/|\Lambda|$ for all cubes Λ. Substituting this into (4.12), we have

$$\int \phi(\xi)\, \mu_t(d\xi) \leqslant 2B_\phi \{e^{-(4/\alpha)(t-1)} C\, |\Lambda + \Lambda_0|/|\Lambda| + \tfrac{1}{4}\alpha C(\delta)\, |\partial(\overline{\Lambda + \Lambda_0})|/|\Lambda|\}^{1/2} \tag{4.13}$$

for all finite cubes Λ. Letting $\Lambda \to Z^\nu$ and noting that $|\Lambda + \Lambda_0|/|\Lambda| \to 1$ and that $|\partial(\overline{\Lambda + \Lambda_0})/|\Lambda| \to 0$, we have the desired result. ∎

Remark. Notice that $2/\alpha$ is the estimate for the gap in the spectrum of L predicted by (L.S.). What we have shown is that, at least when μ_0 is shift-invariant, $2/\alpha$ is a lower bound on the exponential rate at which $\int \phi\, d\mu_t$ approaches $\int \phi\, dg$ when $\phi \in \mathscr{D}(E)$.

5. MORE FREE ENERGY

In this section we weaken the logarithmic Sobolev hypothesis and replace it with a strong mixing condition on the Gibbs state. We then derive a rate of convergence that is slower than exponential. How much slower depends on how much the logarithmic Sobolev hypothesis has been weakened. The method used here has the advantage that it works for any initial distributions, not only translation-invariant ones.

For $\Lambda \subset\subset Z^\nu$, recall the functions $\Phi_\Lambda: E_\Lambda \times E_{\Lambda^c}$ and $g_\Lambda: E_{\Lambda^c} \to (0, \infty)$ introduced in Section 2 and define $G_{\Lambda,\eta} \in M_1(E)$ by

$$\int f(\xi)\, G_{\Lambda,\eta}(d\xi) = \int f \circ \Phi(\xi_\Lambda \mid \eta_{\Lambda^c})\, g_\Lambda \mid \eta_{\Lambda^c})\, \lambda^\Lambda(d\xi_\Lambda)/Z_\Lambda(\eta_{\Lambda^c})$$

for $\eta \in E$ and $f \in C(E)$. Also define $\gamma(\Lambda)$ to be the smallest number γ such that

$$\int f^2(\xi) \log \frac{f^2(\xi)}{\|f\|^2_{L^c(G_{\Lambda,\eta})}}\, G_{\Lambda,\eta}(d\xi)$$
$$\leqslant -\gamma \int f(\xi)\, Lf(\xi)\, G_{\Lambda,\eta}(d\xi), \qquad f \in C_\Lambda^\infty(E) \tag{5.1}$$

for all $\eta \in E$.

Lemma 5.1. For each $\Lambda \subset\subset Z^\nu$, $\gamma(\Lambda) < \infty$.

Proof. Observe that (5.1) is equivalent to

$$\int f^2(\xi) \log \frac{f^2(\xi)}{\|f\|^2_{L^2(G_{\Lambda,\eta})}}\, G_{\Lambda,\eta}(d\xi) \leqslant \gamma \int \sum_{k \in \Lambda} \|\nabla_k f(\xi)\|^2\, G_{\Lambda,\eta}(d\xi)$$

Also, for any probability measure m and any $f \in L^2(m)$

$$\int f^2(\xi) \log \frac{f^2(\xi)}{\|f\|^2_{L^2(m)}}\, m(d\xi)$$
$$= \inf_{x>0} \int [f^2(\xi) \log f^2(\xi) - f^2(\xi) \log x - f^2(\xi) + x]\, m(d\xi)$$

and for each $x>0$ the integrand on the right side of the above equation is nonnegative. Also, if the left side in the above equality is finite, then the infimum on the right side is achieved when $x=\int f^2(\xi)\,m(d\xi)$. Hence, one easily checks that for any probability measures m and μ with $m \ll \mu$,

$$\int f^2(\xi)\log\frac{f^2(\xi)}{\|f\|^2_{L^2(m)}}\,m(d\xi)\leqslant\left\|\frac{dm}{d\mu}\right\|_\infty\int f^2(\xi)\log\frac{f^2(\xi)}{\|f\|^2_{L^2(\mu)}}\,\mu(d\xi)$$

Thus, since g_Λ is bounded above and below by positive constants, we need only check that

$$\int_{E_\Lambda} f^2(\xi)\log\frac{f^2(\xi)}{\|f\|^2_{L^2(\lambda^\Lambda)}}\,\lambda^\Lambda(d\xi)\leqslant\gamma\int_{E_\Lambda}\sum_{k\in\Lambda}\|\nabla_k f(\xi)\|^2\,\lambda^\Lambda(d\xi)$$

for some $\gamma<\infty$. But, because logarithmic Sobolev inequalities are preserved under tensor products (cf. Ref. 4 or Lemma 9.13 in Ref. 12), the preceding will follow once we show that

$$\int_M f^2(\xi)\log\frac{f^2(\xi)}{\|f\|^2_{L^2(\lambda)}}\,\lambda(d\xi)\leqslant\gamma\int_M\|\nabla f(\xi)\|^2\,\lambda(d\xi),\qquad f\in C^\infty(M)\quad(5.1')$$

That a logarithmic Sobolev inequality holds for the Brownian motion on a connected compact manifold was first observed by Rothaus.[10] For the sake of completeness, we sketch a proof here. By standard elliptic theory, the heat flow semigroup $e^{t\Delta}$ admits a smooth density $q(t,x,y)$ that, for each $t>0$, is uniformly positive. In particular, e^Δ is a Hilbert–Schmit operator on $L^2(M)$, and therefore 0 is the only possible accumulation point of its spectrum. In addition, 1 is its largest eigenvalue and, because $q(1,x,y)$ is uniformly positive, it is clear that 1 is a simple eigenvalue. From these considerations, we see that

$$\left\|e^{t\Delta}f-\int f\,d\lambda\right\|_{L^2(\lambda)}\leqslant\left\|f-\int f\,d\lambda\right\|_{L^2(\lambda)}e^{-\varepsilon t},\qquad t\geqslant 0$$

for some $\varepsilon>0$ and all $f\in L^2(\lambda)$. At the same time, because $q(1,x,y)$ is bounded, it is clear that $\|e^\Delta f\|_{L^4(\lambda)}\leqslant C\,\|f\|_{L^2(\lambda)}$ for some $C<\infty$. Hence, by a simple argument, due to Glemm,[3] there is a $T\geqslant 1$ such that $\|e^{T\Delta}f\|_{L^4(\lambda)}\leqslant\|f\|_{L^2(\lambda)}$. But (cf. p. 181 in Ref. 12) $\|e^{T\Delta}\|_{L^2(\lambda)\to L^4(\lambda)}=1$ implies (5.1′) with $\gamma=4T$. ∎

The point of this section is that we will not require that $\{\gamma(\Lambda): \Lambda\subset\subset Z^\nu\}$ be bounded as we did in the previous section, but only that $\gamma(\Lambda)$ not grow too rapidly as $\Lambda\to Z^\nu$. To compensate for this relaxation of the logarithmic Sobolev hypothesis, we need the following mixing conditions.

Mixing condition. There is a $\delta > 0$ such that for all finite Λ_0 and all f that are bounded and $\mathscr{B}_{E_{\Lambda_0}}$ measurable, there is a constant $A_{1,f}$ such that for all $\eta \in E$ and all $\Lambda \supset \Lambda_0$

$$\left| \int f(\xi)\, G_{\Lambda,\eta}(d\xi) - \int f(\xi)\, g(d\xi) \right| \leqslant A_{1,f} \exp[-\delta \operatorname{dist}(\Lambda_0, \Lambda^c)] \quad (5.2)$$

where g is the unique [because of (5.2)] element in $\mathscr{G}(\mathscr{J})$.

Given $\Lambda \subset\subset Z^\nu$ and $\eta \in E$, let $\{P^{\Lambda,\eta}: t > 0\}$ denote the Markov semigroup on $C(E)$ such that

$$P_t^{\Lambda,\eta} f - f = \int_0^t P_s^{\Lambda,\eta} L^{\Lambda,\eta} f \, ds, \qquad t \geqslant 0$$

where

$$L^{\Lambda,\eta} f(\xi) = \frac{1}{g_\Lambda(\xi_\Lambda \mid \eta_{\Lambda^c})} \sum_{k \in \Lambda} \operatorname{div}_k [g_\Lambda(\xi_\Lambda \mid \eta_{\Lambda^c}) \nabla_k f] \circ \Phi(\xi_\Lambda \mid \eta_{\Lambda^c})$$

for $f \in \mathscr{D}(E)$. It is an easy matter to check that $G_{\Lambda,\eta}$ is $\{P_t^{\Lambda,\eta}: t > 0\}$-reversible.

If M were a finite set, the proof of the next lemma could be found in Ref. 7. The changes needed in that proof to cover the present situation are purely notational. In particular, if one replaces Δ_K there by ∇_k, the proof goes through nearly word for word.

Lemma 5.2. There is a constant $c < \infty$ such that for all finite Λ_0 and all $f \in C_{\Lambda_0}^\infty(E_{\Lambda_0})$ there is a constant $A_{2,f}$ such that for all $\eta \in E$

$$|P_t f(\eta) - P_t^{\Lambda,\eta} f(\eta)| \leqslant A_{2,f} e^{ct} \frac{(ct)^{N+2}}{(N+2)!}$$

where $N = [\operatorname{dist}(\Lambda_c, \Lambda^c)/R]$.

Theorem 5.3. Assume that the above mixing condition holds for some $\delta > 0$. In addition, assume that there are $\gamma \in (0, \infty)$, $\sigma \in [0, 1/\nu)$, and $\tau \in [0, \infty)$ such that

$$\gamma(\Lambda) \leqslant \gamma\, |\Lambda|^\sigma (\log |\Lambda|)^\tau \quad (5.3)$$

for all cubes $\Lambda \subset\subset Z^\nu$. Then there is an $\varepsilon > 0$ such that for all initial distributions μ_0 and $\phi \in \mathscr{D}(E)$

$$\left| \int \phi(\xi)\, g(d\xi) - \int \phi(\xi)\, \mu_t(d\xi) \right| \leqslant B(\phi) \exp\left[-\varepsilon \frac{t^{1-\sigma\nu}}{(\log t)^\tau} \right], \qquad t \geqslant 2 \tag{5.4}$$

where $B(\phi) \in (0, \infty)$.

Proof. Let $\phi \in C^\infty_{\Lambda_0}(E)$. If Λ_0 has side length l, let $\Lambda(t)$ be the cube with side length $l + 8cRt$ and having the same center as Λ_0. Here c is as in Lemma 5.2. Then

$$\begin{aligned}\left| P_t\phi(\eta) - \int \phi(\xi)\, g(d\xi) \right| &\leqslant |P_t\phi(\eta) - P_t^{\Lambda(t),\eta}\phi(\eta)| \\ &\quad + \left| P_t^{\Lambda(t),\eta}\phi(\eta) - \int \phi(\xi)\, G_{\Lambda,\eta}(d\xi) \right| \\ &\quad + \left| \int \phi(\xi)\, G_{\Lambda,\eta}(d\xi) - \int \phi(\xi)\, g(d\xi) \right| \end{aligned} \tag{5.5}$$

The first term on the right side of (4.9) is bounded by

$$A_{2,\phi} e^{ct} \frac{(ct)^{4ct+2}}{(4ct+2)!} \leqslant A_{2,\phi} \left[e\left(\frac{e}{4}\right)^4 \right]^{ct} \leqslant A_{2,\phi} e^{-ct/2}$$

By the above mixing condition the third term on the right side of (5.5) is bounded by $A_{1,\phi} e^{-4\delta Rct}$. Thus, we need only bound the second term. To do that we return to the free energy considerations of the previous section. First, note that if

$$F_t(a) = \log\left\{ \int \exp\left[a\left(\phi(\xi) - \int \phi(\sigma)\, G_{\Lambda(t),\eta}\,(d\sigma)\right) \right] G_{\Lambda(t),\eta}(d\xi) \right\}$$

then $F_t(a) = 0 = F_t'(0)$ and $F_t''(a) \leqslant 4\, \|\phi\|_\infty^2$ for all a. Thus, for all $a \geqslant 0$, $F_t(a) \leqslant 2\, \|\phi\|_\infty^2\, a^2$, and by (3.19)

$$\begin{aligned}&\left| P_t^{\Lambda(t),\eta}\phi(\eta) - \int \phi(\xi)\, G_{\Lambda(t),\eta}(d\xi) \right| \\ &\qquad \leqslant 2^{5/2}\, \|\phi\|_\infty \left[\int f_t^{\Lambda(t)}(\xi) \log f_t^{\Lambda(t)}(\xi)\, G_{\Lambda(t),\eta}(d\xi) \right]^{1/2}\end{aligned} \tag{5.6}$$

where $f_s^{\Lambda(t)}(\cdot) = d\mu_s^{\Lambda(t)}(\cdot)/dG_{\Lambda(t),\eta}(\cdot)$ and $\mu_s^{\Lambda(t)} = (P_s^{\Lambda(t),\eta})^*\, \delta_\eta(\cdot)$. Now, by (4.3) we have

$$\int f_1^{\Lambda(t)}(\xi) \log f_1^{\Lambda(t)}(\xi)\, G_{\Lambda(t),\eta}(d\xi) \leqslant C\, |\Lambda(t)| \tag{5.7}$$

Also, by a straightforward computation (see Ref. 9) and Lemma 4.1

$$\frac{d}{ds}\int f_s^{\Lambda(t)}(\xi)\log f_s^{\Lambda(t)}(\xi)\, G_{\Lambda(t),\eta}(d\xi)$$

$$= \int f_s^{\Lambda(t)}(\xi)\, L^{\Lambda(t),\eta} \log f_s^{\Lambda(t)}(\xi)\, G_{\Lambda(t),\eta}(d\xi)$$

$$\leqslant -\frac{4}{\gamma(\Lambda(t))}\int f_s^{\Lambda(t)}(\xi)\log f_s^{\Lambda(t)}(\xi)\, G_{\Lambda(t),\eta}(d\xi) \tag{5.8}$$

Thus,

$$\begin{aligned}
&\int f_t^{\Lambda(t)}(\xi)\log f_t^{\Lambda(t)}(\xi)\, G_{\Lambda(t),\eta}(d\xi)\\
&\quad\leqslant C\,|\Lambda(t)|\exp[-4(t-1)/\gamma(\Lambda(t))]\\
&\quad\leqslant C(l+8cRt)^\nu \exp\{-4(t-1)/\gamma(l+8cRt)^{\sigma\nu}\,[\log(l+8cRt)^\nu]^\tau\}\\
&\quad\leqslant B_0\exp[-\varepsilon t^{1-\sigma\nu}/(\log t)^\tau]
\end{aligned} \tag{5.9}$$

for some $B_0<\infty$ that depends on ϕ only through l, and some $\varepsilon>0$ that does not depend on l, and all $t\geqslant 2$. ▮

6. ONE DIMENSION

In this section we show that, in one dimension, the hypotheses of Theorem 5.3, with $\sigma=0$ and $\tau=1$, are satisfied for all finite-range, translation-invariant potentials $\mathscr{J}$.

The first hypothesis is (5.2). That this holds for Gibbs states with finite-range interaction in one dimension is well known. It can be proved by considering intervals whose length is the length of the interaction and noting that the conditional Gibbs state $G_{\Lambda,\eta}(\cdot)$ is just a Markov chain conditioned to have specific values at both ends of an interval of length $|\Lambda|/l$. Moreover, the state space of this Markov chain is compact and the translation function is uniformly positive. (See the discussion of one-dimensional systems in Ref. 11 for the basic ideas.)

It is considerably more work to check that $\gamma(\Lambda)\leqslant\gamma\log|\Lambda|$ for some $\gamma<\infty$. We begin with the following lemma.

Lemma 6.1. Let $\Lambda_0=[-R/2, R/2]$. There is a constant γ_1 such that if Λ is any interval containing Λ_0 and $\eta\in E$, then for all $f\in C^\infty_{\Lambda_0}(E)$

$$\int f^2(\xi)\log f^2(\xi)\, G_{\Lambda,\eta}(d\xi)\leqslant\gamma_1\sum_{k\in\Lambda_0}\int \|\nabla_k f(\xi)\|^2\, G_{\Lambda,\eta}(d\xi)$$

$$+\int f^2(\xi)\, G_{\Lambda,\eta}(d\xi)\log\int f^2(\xi)\, G_{\Lambda,\eta}(d\xi) \tag{6.1}$$

Proof. Note that for any $\Lambda \supset \Lambda_0$ and any $\eta \in E$ the marginal distribution of $G_{\Lambda,\eta}$ on M^{Λ_0} has a density with respect to λ^{Λ_0} that is bounded away from infinity and zero uniformly in Λ and η. The rest of the proof is just a in Lemma 5.1. ▌

Our next step is to prove that there is some number $\varepsilon > 0$ such that for all Λ and all η, $L^{\Lambda,\eta}$ acting on $L^2(G_{\Lambda,\eta}(\cdot))$ has a gap of length at least ε between 0 and the rest of its spectrum. We do this by first introducing a jump process for which this result has already been proved.

For $f \in \mathscr{D}(E)$ let

$$\Omega f(\eta) = \sum_k \int_M [f \circ \Phi_{\{k\}}(\sigma \mid \eta_{\{k\}^c}) - f(\eta)]\, G_{\{k\},\eta}(d\sigma)$$

Ω generates a positive contraction semigroup $(S_t\colon t \geqslant 0)$ on $C(E)$ and Ω is self-adjoint on $L^2(g)$.[5] Moreover,[5,8]

$$\int f(\eta)\, \Omega f(\eta)\, g(d\eta)$$
$$= -\frac{1}{2} \sum_k \int \left\{ \int_M [f \circ \Phi_{\{k\}}(\sigma \mid \eta_{\{k\}^c}) - f(\eta)]\, G_{\{k\},\eta}(d\sigma) \right\}^2 g(d\eta) \qquad (6.2)$$

The following lemma can be proved by merely changing the notation in the proof of Theorem 0.4 of Ref. 6.

Lemma 6.2. There is an $\varepsilon_0 > 0$ such that for all $f \in L^2(g)$,

$$-\int f(\xi)\, \Omega f(\xi)\, g(d\xi) \geqslant \varepsilon_0 \int \left[f(\xi) - \int f(\eta)\, g(d\eta) \right]^2 g(d\xi) \qquad (6.3)$$

Lemma 6.3. There is an $\varepsilon_1 > 0$ such that if $f \in C_\Lambda^\infty(E)$ for some finite Λ, then

$$\sum_k \int \|\nabla_k f(\xi)\|^2\, g(d\xi) \geqslant \varepsilon_1 \int \left[f(\xi) - \int f(\eta)\, g(d\eta) \right]^2 g(d\xi) \qquad (6.4)$$

Proof. To simplify the notation, we make the following convention. For $k \in Z^\nu$, $\eta \in E$, and $\omega \in M$ we write $\eta_k \omega$ for the element of E that is equal to η at all sites except k and is equal to ω at k. Thus, instead of writing $f \circ \Phi(\omega \mid \eta_{\{k\}^c})$ we write simply $f(\eta_k \omega)$.

Now, by (6.2) and (6.3)

$$\sum_k \int \left\{ \int_M [f(\eta_k \sigma) - f(\eta)]\, G_{\{k\},\eta}(d\sigma) \right\}^2 g(d\eta)$$
$$\geqslant \varepsilon_0 \int \left[f(\xi) - \int f(\eta)\, g(d\eta) \right]^2 g(d\xi) \qquad (6.5)$$

But

$$\int \left\{ \int_M [f(\eta_k \omega) - f(\eta)]\, G_{\{k\},\eta}(d\omega) \right\}^2 g(d\eta)$$

$$= \iint_M \left[f(\eta_k \omega) - \int_M f(\eta_k \sigma)\, G_{\{k\},\eta}(d\sigma) \right]^2 G_{\{k\},\eta}(d\omega)\, g(d\eta)$$

$$\leqslant \iint_M \left[f(\eta_k \omega) - \int_M f(\eta_k \sigma)\, \lambda(d\sigma) \right]^2 \lambda(d\omega)\, g(d\eta)$$

$$\times \max_{\omega,\eta} g_{\{k\}}(\omega \mid \eta_{\{k\}^c}) / Z_{\{k\}}(\eta_{\{k\}^c}) \tag{6.6}$$

Now the Laplace–Beltrami operator on the compact manifold M has a gap at 0 in its spectrum (cf. the proof of Lemma 5.1). Thus, there is an $\varepsilon_2 > 0$ such that

$$\int_M \left[f(\eta_k \omega) - \int_M f(\eta_k \sigma)\, \lambda(d\sigma) \right]^2 \lambda(d\omega)$$

$$\leqslant -\frac{1}{\varepsilon_2} \int_M f(\eta_k \sigma)\, \mathrm{div}_k \nabla_k f(\eta_k \sigma)\, \lambda(d\sigma)$$

$$= \frac{1}{\varepsilon_2} \int_M \|\nabla_k f(\eta_k \sigma)\|^2\, \lambda(d\sigma)$$

Substituting this into the right side of (6.6) and using translation invariance, we have

$$\int \left\{ \int_M [f(\eta_k \sigma) - f(\eta)]\, G_{\{k\},\eta}(d\sigma) \right\}^2 g(d\eta)$$

$$\leqslant \frac{1}{\varepsilon_2} \max_{\omega,\xi} \frac{g_{\{k\}}(\omega \mid \xi_{\{k\}^c})}{Z_{\{k\}}(\xi_{\{k\}^c})} \max_{\omega,\xi} \frac{Z_{\{k\}}(\xi_{\{k\}^c})}{g_{\{k\}}(\omega \mid \xi_{\{k\}^c})}$$

$$\times \iint_M \|\nabla_k f(\eta_k \sigma)\|^2\, G_{\{k\},\eta}(d\sigma)\, g(d_\eta) \tag{6.7}$$

The lemma follows from (6.5) and (6.7). ∎

Lemma 6.4. There is an $\varepsilon > 0$ such that for all intervals Λ, all $\eta \in E$, and all $f \in C_\Lambda^\infty(E)$,

$$\sum_{k \in \Lambda} \int \|\nabla_k f(\sigma)\|^2\, G_{\Lambda,\eta}(d\sigma)$$

$$\geqslant \varepsilon \int \left[f(\sigma) - \int f(\omega)\, G_{\Lambda,\eta}(d\omega) \right]^2 G_{\Lambda,\eta}(d\sigma) \tag{6.8}$$

Proof. Note that since $|\partial\Lambda|$ is independent of Λ in one dimension, there is a constant $\alpha>0$ such that for all η and all $A\in\mathscr{B}_\Lambda$, $1/\alpha\geqslant G_{\Lambda,\eta}(A)/g(A)\geqslant\alpha$. Thus, the left side of (6.8) is bounded below by

$$\begin{aligned}
\alpha \sum_{k\in\Lambda} & \int \|\nabla_k f(\sigma)\|^2\, g(d\sigma)\\
&\geqslant \alpha\varepsilon_1 \int \left[f(\sigma)-\int f(\omega)\, g(d\omega)\right]^2 g(d\sigma)\\
&\geqslant \alpha^2\varepsilon_1 \int \left[f(\sigma)-\int f(\omega)\, g(d\omega)\right]^2 G_{\Lambda,\eta}(d\sigma)\\
&\geqslant \varepsilon \int \left[f(\sigma)-\int f(\omega)\, G_{\Lambda,\eta}(d\omega)\right]^2 G_{\Lambda,\eta}(d\sigma) \qquad (6.9)
\end{aligned}$$

where $\varepsilon=\alpha^2\varepsilon_1$. ∎

Lemma 6.5. Let g be a one-dimensional Gibbs state whose range of interaction is R and let $\gamma(\Lambda)$ be as in Section 5. Then there is a constant $k_0<\infty$ such that for all $l_1,\ l_2\geqslant 1$

$$\begin{aligned}
\gamma&\left(\left[-l_1-\frac{1}{2}R,\, l_2+\frac{1}{2}R\right]\right)\\
&\leqslant \left\{\gamma\left(\left[-l_1-\frac{1}{2}R,\, -\frac{1}{2}R-1\right]\right)\vee\gamma\left(\left[\frac{1}{2}R+1,\, \frac{1}{2}R+l_2\right]\right)\right\}+k_0 \qquad (6.10)
\end{aligned}$$

Proof. First note that if Λ is an interval, then $\gamma(\Lambda)$ depends only on $|\Lambda|$. Therefore we write $\gamma(l)$ instead of $\gamma(\Lambda)$ when Λ is an interval containing l integers.

Now let $\Lambda_1=[-l_1-\frac{1}{2}R,\ -\frac{1}{2}R-1]$, $\Lambda_2=[-\frac{1}{2}R,\frac{1}{2}R]$, and $\Lambda_3=[\frac{1}{2}R+1,\ \frac{1}{2}R+l_2]$ and set $\Lambda=\Lambda_1\cup\Lambda_2\cup\Lambda_3$. If $\sigma\in M^\Lambda$ and $\omega_i\in M^{\Lambda_i}$, we write $\sigma=\omega_1\omega_2\omega_3$ to mean $\sigma(k)=\omega_i(k)$ if $k\in\Lambda_i$. If $\omega_2\in M^{\Lambda_2}$ and $\eta\in E$, we will let $\eta\omega_2$ denote the configuration that is equal to η off of Λ_2 and equal to ω_2 on Λ_2. We denote the conditional distribution of g given $\mathscr{B}_{\Lambda^c\cup\Lambda_2}$ by $\bar{G}_{\eta\omega_2}(\cdot)$ and note that since $|\Lambda_2|=R$, $\bar{G}_{\eta\omega_2}=G_{\Lambda_1,\eta\omega_2}\times G_{\Lambda_3,\eta\omega_2}$. If $A\in\mathscr{B}_{\Lambda_2}$, we denote $G_{\Lambda,\eta}(A)$ by $g_\Lambda^{(\Lambda_2)}(A\,|\,\eta)$.

Let $f\in C_\Lambda^\infty$. By conditioning on $\mathscr{B}_{\Lambda_2}$, we have

$$\begin{aligned}
&\int f^2(\sigma)\log f^2(\sigma)\, G_{\Lambda,\eta}(d\sigma)\\
&\quad=\iint f^2(\omega_1\omega_2\omega_3)\log f^2(\omega_1\omega_2\omega_3)\, \bar{G}_{\eta\omega_2}(d\omega_1\, d\omega_3)\, g_\Lambda^{(\Lambda_2)}(d\omega_2\,|\,\eta) \qquad (6.11)
\end{aligned}$$

Thus, by first factoring $\bar{G}_{\eta\omega_2}(\cdot)$ and then applying Lemma 9.13 in Ref. 12, we bound the right side of (6.11) above by

$$\int \{[\gamma(l_1) \vee \gamma(l_2)] \sum_{k \in \Lambda_1 \cup \Lambda_3} \iint \|\nabla_k f(\omega_1 \omega_2 \omega_3)\|^2 G_{\Lambda_1,\eta\omega_2}(d\omega_1)\, G_{\Lambda_3,\eta\omega_2}(d\omega_3)$$
$$+ F^2(\eta\omega_2) \log F^2(\eta\omega_2)\} \, g_\Lambda^{(\Lambda_2)}(d\omega_2 \mid \eta) \tag{6.12}$$

where

$$F^2(\eta\omega_2) = \iint f^2(\omega_1 \omega_2 \omega_3)\, G_{\Lambda_1,\eta\omega_2}(d\omega_1)\, G_{\Lambda_3,\eta\omega_2}(d\omega_3)$$

By applying Lemma 6.1 to the part of (6.12) that involves F^2, we may bound (6.12) above by

$$[\gamma(l_1) \vee \gamma(l_2)] \sum_{k \in \Lambda_1 \cup \Lambda_3} \int \|\nabla_k f(\sigma)\|^2 G_{\Lambda,\eta}(d\sigma)$$
$$+ \gamma_1 \sum_{k \in \Lambda_2} \int \|\nabla_k F(\eta\omega_2)\|^2 g_\Lambda^{(\Lambda_2)}(d\omega_2)$$
$$+ \int f^2(\sigma)\, G_{\Lambda,\eta}(d\sigma) \log\left[\int f^2(\sigma)\, G_{\Lambda,\eta}(d\sigma)\right] \tag{6.13}$$

Denote $d\bar{G}_{\eta\omega_2}/d\lambda^{\Lambda_1 \cup \Lambda_3}$ by $\bar{g}_{\eta\omega_2}$ and concentrate on the second term in (6.13). For any $k \in \Lambda_2$

$$\begin{aligned}
&\|\nabla_k F(\eta\omega_2)\|^2 \\
&= \left\| \frac{\iint 2f(\omega_1\omega_2\omega_3)\, \nabla_k f(\omega_1\omega_2\omega_3)\, \bar{G}_{\eta\omega_2}(d\omega_1\, d\omega_3)}{2F(\eta\omega_2)} \right. \\
&\qquad \left. + \frac{\iint f^2(\omega_1\omega_2\omega_3)\, \nabla_k \bar{g}_{\eta\omega_2}(\omega_1\omega_3)\, \lambda^{\Lambda_1 \cup \Lambda_3}(d\omega_1\, d\omega_3)}{2F(\eta\omega_2)} \right\|^2 \\
&\leqslant 2 \iint \|\nabla_k f(\omega_1\omega_2\omega_3)\|^2\, \bar{G}_{\eta\omega_2}(d\omega_1\, d\omega_3) \\
&\qquad + \frac{1}{2} \left\| \frac{\iint f^2(\omega_1\omega_2\omega_3)\, \nabla_k \bar{g}_{\eta\omega_2}(\omega_1\omega_3)\, \lambda^{\Lambda_1 \cup \Lambda_3}(d\omega_1\, d\omega_3)}{F(\eta\omega_2)} \right\|^2
\end{aligned} \tag{6.14}$$

Now

$$\iint \nabla_k\, \bar{g}_{\eta\omega_2}(d\omega_1\, d\omega_3)\, \lambda^{\Lambda_1 \cup \Lambda_3}(d\omega_1\, d\omega_3) = \nabla_k 1 = 0$$

Thus, for any number W,

$$\left\| \iint f^2(\omega_1\omega_2\omega_3)\,\nabla_k\,\bar{g}_{\eta\omega_2}(\omega_1\omega_3)\,\lambda^{A_1\cup A_3}(d\omega_1\,d\omega_3)\right\|^2$$

$$= \left\| \iint [f(\omega_1\omega_2\omega_3)-W]^2\,\frac{\nabla_k\,\bar{g}_{\eta\omega_2}(\omega_1\omega_3)}{\bar{g}_{\eta\omega_2}(\omega_1\omega_3)}\,\bar{g}_{\eta\omega_2}(\omega_1\omega_3)\,\lambda^{A_1\cup A_3}(d\omega_1\,d\omega_3)\right.$$

$$\left. + 2W \iint [f(\omega_1\omega_2\omega_3)-W]\,\frac{\nabla_k\,\bar{g}_{\eta\omega_2}(\omega_1\omega_3)}{\bar{g}_{\eta\omega_2}(\omega_1\omega_3)}\,\bar{g}_{\eta\omega_2}(\omega_1\omega_3)\,\lambda^{A_1\cup A_3}(d\omega_1\,d\omega_3)\right\|^2$$

$$\leqslant 2\,\|\nabla_k \log \bar{g}_{\eta\omega_2}(\omega_1\omega_3)\|_\infty^2 \iint [f(\omega_1\omega_2\omega_3)-W]^2\,\bar{G}_{\eta\omega_2}(d\omega_1\,d\omega_3)$$

$$+ 4W^2 \iint \|\nabla_k \log \bar{g}_{\eta\omega_2}(\omega_1\omega_3)\|^2\,\bar{G}_{\eta\omega_2}(d\omega_1\,d\omega_3)$$

$$\times \iint [f(\omega_1\omega_2\omega_3)-W]^2\,\bar{G}_{\eta\omega_2}(d\omega_1\,d\omega_3) \tag{6.15}$$

Setting

$$W = \iint f(\omega_1\omega_2\omega_3)\,\bar{G}_{\eta\omega_2}(d\omega_1\,d\omega_3)$$

and noting that $\|\nabla_k \log \bar{g}_{\eta\omega_2}(\omega_1\omega_3)\|$ is bounded uniformly in all of its variables, we see that the second term on the right side of (6.14) is bounded by

$$K_1 W^2 \iint [f(\omega_1\omega_2\omega_3)-W]^2\,\bar{G}_{\eta\omega_2}(d\omega_1\,d\omega_3)$$

for some finite constant K_1, which is independent of l_1, l_2, η, and k. Since $W^2 \leqslant F^2(\eta\omega_2)$, upon substituting this into (6.14) and then substituting the resulting inequality into (6.13), we have

$$\int f^2(\sigma)\log f^2(\sigma)\,G_{A,\eta}(d\sigma)$$

$$\leqslant [\gamma(l_1)\vee\gamma(l_2)] \sum_{k\in A_1\cup A_3} \int \|\nabla_k f(\sigma)\|^2\,G_{A,\eta}(d\sigma)$$

$$+ \gamma_1 \sum_{k\in A_2} \int \|\nabla_k f(\sigma)\|^2\,G_{A,\eta}(d\sigma)$$

$$+ k_1 \sum_{k\in A_2} \iiint [f(\omega_1\omega_2\omega_3)-W]^2\,\bar{G}_{\eta\omega_2}(d\omega_1\,d\omega_3)\,g_A^{(A_2)}(d\omega^2\,|\,\eta) \tag{6.16}$$

Since $\tilde{G}_{\eta\omega_2} = G_{\Lambda_1,\eta\omega_2} \times G_{\Lambda_3,\eta\omega_2}$, we apply Lemma 6.3 to the tensor product $L^2(G_{\Lambda_1,\eta\omega_2}) \otimes L^2(G_{\Lambda_3,\eta\omega_2})$ to conclude that the last term on the right side of (6.16) is bounded by

$$K_1 \frac{1}{\varepsilon_1} \sum_{k \in \Lambda_2} \sum_{j \in \Lambda_1 \cup \Lambda_3} \iiint \|\nabla_j f(\omega_1 \omega_2 \omega_3)\|^2 \, \tilde{G}_{\eta\omega_2}(d\omega_1 \, d\omega_3) \, g_\Lambda^{\Lambda_2}(d\omega_2)$$

$$= K_1 \, |\Lambda_2| \, \varepsilon^{-1} \sum_{j \in \Lambda_1 \cup \Lambda_3} \int \|\nabla_j f(\sigma)\|^2 \, G_{\Lambda,\eta}(d\sigma)$$

Thus the lemma is proved with $k_0 = \gamma_1 \vee [K_1 R \varepsilon_1^{-1}]$. ▌

Theorem 6.6. Let g be a one-dimensional Gibbs state with finite-range potential, and let $\gamma(|\Lambda|)$ be as in Section 5. Then there is a constant γ such that $\gamma(\Lambda) \leqslant \gamma \log |\Lambda|$ for all $|\Lambda| \geqslant 2$.

Proof. By induction on i it is easily seen from Lemma 6.5 that if $(2^i - 1) R < m \leqslant (2^{i+1} - 1) R$, then

$$\gamma(m) \leqslant \bar{\gamma} + ik_0 \tag{6.17}$$

where $\bar{\gamma} = \max_{1 \leqslant i \leqslant R} \gamma(i)$. Also, if $(2^i - 1) R < m \leqslant (2^{i+1} - 1) R$, then $\log R + (i-1) \log 2 \leqslant \log m$. Thus,

$$\overline{\lim_{m \to \infty}} \frac{\gamma(m)}{\log m} \leqslant \overline{\lim_{m \to \infty}} \frac{\bar{\gamma} + ik_0}{\log R + (i-1) \log 2} = \frac{k_0}{\log 2}$$

and hence there is a constant $\gamma < \infty$ such that

$$\gamma(m) \leqslant \gamma \log m \qquad \text{for all} \quad m \geqslant 2 \quad ▌$$

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