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Anderson Localization for One-Dimensional Difference Schrödinger Operator with Quasiperiodic Potential

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The Schrödinger difference operator considered here has the form

$$(H_\varepsilon(\alpha)\,\psi)(n) = -\varepsilon(\psi(n+1)+\psi(n-1)) + V(n\omega+\alpha)\,\psi(n)$$

where V is a C^2-periodic Morse function taking each value at not more than two points. It is shown that for sufficiently small ε the operator $H_\varepsilon(\alpha)$ has for a.e. α a pure point spectrum. The corresponding eigenfunctions decay exponentially outside a finite set. The integrated density of states is an incomplete devil's staircase with infinitely many flat pieces.

KEY WORDS: Schrödinger operator; eigenfunction; eigenvalue; Green's function; continued fraction.

1. INTRODUCTION

The main subject of this paper concerns the properties of localization of eigenfunctions (e.f.) of the self-adjoint operator $H_\varepsilon(\alpha)$ acting in $l^2(-\infty, \infty)$ by the formula

$$(H_\varepsilon(\alpha)\,\psi)(n) = -\varepsilon(\psi(n+1)+\psi(n-1)) + V(n\omega+\alpha)\,\psi(n) \qquad (1.1)$$

where V is a C^2-smooth periodic function of period 1, having one nondegenerate maximum and minimum and strictly monotone with nonzero derivatives between them. A typical example is $V(\alpha) = \cos 2\pi\alpha$. The rotation number ω is a typical irrational number. More precise assumptions concerning ω will be formulated below.

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Equation (1.1) is a particular example of the one-dimensional difference Schrödinger operator with a random potential. In a more general setting one considers a measure space $(M, \mathscr{M}, \mu)$ with a probability measure μ, its measure-preserving ergodic automorphism T, and a measurable function $V(x)$. Each random realization of the potential is a sample of values of V along a random trajectory, i.e., $V_n = V(T^n x)$. Thus, the whole randomness stems from the randomness of an initial point x distributed according to the measure μ. Thus, the general form of (1.1) is

$$(H_\varepsilon(x)\,\psi)(n) = -\varepsilon(\psi(n+1) + \psi(n-1)) + V(T^n x)\,\psi(n) \qquad (1.2)$$

Equation (1.1) corresponds to the case of $M = S^1$, with μ the Lebesgue measure and T the rotation of M to the angle ω.

We shall say that for (1.2) the complete Anderson localization holds if, with μ-probability 1, the operator $H_\varepsilon(x)$ has a complete system of eigenfunctions (e.f.) belonging to $l^2(-\infty, \infty)$. Naturally, the sets of e.f. and corresponding eigenvalues (e.v.) are functions of x, i.e., are random variables in an appropriate sense.

The property of localization has been investigated mostly in cases where $\{V(T^n x)\}$ was a sequence of identically distributed independent random variables (iirv). The first explanations of localization were given by Mott and Twose[1] and Borland.[2] Namely, consider the equation

$$(H_\varepsilon \psi)(n) = -\varepsilon(\psi(n+1) + \psi(n-1)) + (V_n - E)\,\psi(n) \qquad (1.3)$$

where V_n is a sequence of iirv. Then it follows from Furstenberg's theorem that for each E with μ-probability 1 there exist $\psi_E^+(0)$, $\psi_E^+(1)$ such that the corresponding solutions of (1.3) for $n > 0$ with these initial data decay exponentially as $n \to \infty$. In the same manner there exist $\psi_E^-(0)$, $\psi_E^-(1)$ that have this property as $n \to -\infty$. According to Refs. 1 and 2, the e.v. correspond to those E for which $\psi_E^+(0) = \psi_E^-(0)$, $\psi_E^+(1) = \psi_E^-(1)$. The main mathematical difficulty in this approach is due to the fact that the appearing sets of measure zero depend on E and for a typical sequence V_n the function $\psi_E^+(1)/\psi_E^+(0)$, which is defined only almost everywhere in x, is not continuous in E. Nevertheless, the final conclusion concerning the localization is true. The first mathematical proofs for a slightly different situation were given by Goldsheid *et al.*[4,5] Kunz and Souillard[5] considered the case (1.2) where the iirv have a probability distribution with a bounded density. The idea of Ref. 5 is slightly different from that of Refs. 3 and 4. One of the main ingredients of all these and subsequent proofs is the statement that all Liapunov exponents of the corresponding monodromy matrices are different from zero, which in fact implies the exponential decay of solutions $\psi^\pm$. The exact results about the exponents defining the decay of e.f. were obtained by Molchanov[6] and Carmona.[7] The recent survey article by Souillard[8] contains a rich mine of information about proofs and results on localization, based upon the technique of Liapunov exponents.

The next landmark in developing the mathematical theory of localization was the paper by Fröhlich and Spencer.[9] They considered the multidimensional version of (1.2) where again the random potential consisted of a sequence of iirv whose distribution has a bounded density. The main results of Ref. 9 gave, under appropriate assumptions, some estimations of Green's functions and in fact a construction of infinitely many localized e.f. provided that ε is sufficiently small. Later Fröhlich *et al.*[10] showed that for small enough ε the complete Anderson localization takes place. In this connection an earlier paper by Jona-Lasinio *et al.*[11] should also be mentioned. The localization in this situation was proven by Delyon *et al.*[12] and Simon *et al.*[13]

The main idea of Ref. 9 was based upon the notion of quantum tunneling and resonant e.f. The authors invented a very interesting approach to the construction of exact e.f., which resembles in a sense the methods of KAM theory. Namely, assuming that e.f. of the operator in a bounded domain with the Dirichlet boundary conditions are constructed, the authors write down the series containing Green's functions for extended e.f. in larger domains. The series is rapidly converging if a nonresonant condition holds. This condition is formulated in terms of differences of e.v. in different domains. Using Wegner's lemma,[14] the authors estimate the probability distribution for these differences. From the estimation the statement of localization follows with the help of the Borel–Cantelli lemma. In the one-dimensional situation Fröhlich *et al.*,[10] using their methods, reproduced the results of Kunz and Souillard.[5] Recently Carmona *et al.*[15] extended the technique and proved the localization for cases where the random variables V_n take a finite number of values. Let us emphasize that for iirv Anderson localization takes place for all values of ε.

Now return to the operator (1.1). The class of quasiperiodic potentials is the simplest one after the class of periodic potentials. In the latter case the spectrum has a zone structure and e.f. have the form of Bloch functions $\psi(n)=e^{ipn}\varphi(n)$, where $\varphi(n)$ is a periodic function whose period is equal to the period of the potential, and p is a quasimomentum. The general interest in problems concerning spectral properties of (1.1) has increased recently in connection with the discovery of quasicrystals. One-dimensional problems are also of more than mathematical interest. For example, an optimistic interpretation of the results of experimental work[16] sees them as evidence of the existence of compounds that are periodic along two directions and quasiperiodic in the third direction. In Ref. 17, concerning the motion of forced, damped pendulums, the problems of its dynamics are directly connected with the spectral properties of (1.1). One should also mention the interesting papers by Kohmoto *et al.*[18] and Kalugin *et al.*[19] where some results for spectra of (1.1) were obtained for quasiperiodic potentials taking a finite number of values. The results by Delyon and Petritis[20] show that in this case the spectrum might be only singular.

The first rigorous results for one-dimensional continuous Schrödinger operators with quasiperiodic potentials were obtained in Refs. 21–23. Later more exact estimations were given by Rüssmann,[24] and Bellissard *et al.*[25] extended the technique of Refs. 21, 22, and 24 to (1.1). The main result of these studies shows that for sufficiently large ε or for sufficiently small potentials one can construct a set $S_\varepsilon \subset [-2\varepsilon + \min V, 2\varepsilon + \max V]$ not depending on α such that for each $\lambda \in S_\varepsilon$ there exist two Bloch e.f. of the form $e^{ipn}\Phi(n\omega + \alpha)$ and $e^{-ipn}\overline{\Phi(n\omega + \alpha)}$ having the same e.v. Moreover, $l(S_\varepsilon)/\max V - \min V \to 1$ as $\varepsilon \to \infty$, where l in this section means the Lebesgue measure. The set S_ε constructed in Refs. 21–25 is a Cantor-type set of positive measure. It gives a contribution to the limit density of states which turns out to have a devil's staircase component. The components of the complement or gaps are analogs of forbidden zones, the union of which is, as expected, an everywhere dense set. Strictly speaking, the method, which is based on KAM estimation, does not give any information about proper ties of the spectrum for points of this set. However, some results by Avron and Simon[26] show that generically the limit density of states of (1.1) is a Cantor devil's staircase, while Johnson and Moser[27] give a beautiful description of the forbidden zones (f.z.). Moser and Pöschel[28] show that in typical situations the f.z. have positive length.

The main result of this paper is a theorem that states that for sufficiently small ε the spectrum of $H_\varepsilon(\alpha)$ is pure point and each e.f. decays exponentially outside a finite set. The set of e.v. $\{\lambda_i(\alpha)\}$ really depends on α, or, using the terminology of the theory of dynamical systems, the spectrum of $H_\varepsilon(\alpha)$ depends sensitively on α. Thus, we encounter in (1.1) two different types of spectra: a Bloch spectrum not depending on α and a pure point spectrum sensitively depending on α. The transition from one type to another under the change of ε is apparently complicated and the notion of mobility edge as a strict boundary between two types of spectra needs some clarification here. Probably this transition has something in common with the transition occurring in the bifurcation of invariant KAM circles into cantori (see, e.g., Ref. 30).

The method of this paper was inspired by the paper of Fröhlich and Spencer.[9] Our main idea consists in a detailed analysis of the process of tunneling. Namely, suppose that we have a method that gives a possibility for any e.f. $\psi = \{\psi(n)\}$ of $H_\varepsilon(\alpha)$ [see (1.1)] to construct an essential support (e.s.) $Z(\psi)$, which is a finite subset of the lattice $\mathbb{Z}^1$ having the following two natural properties:

(a) Outside of $Z(\psi)$ the values of $\psi(n)$ decay exponentially with the distance of n to $Z(\psi)$.

(b) If ψ is an e.f. of $H_\varepsilon(\alpha)$, then $T\psi$ is an e.f. of $H_\varepsilon(\alpha+\omega)$ and $Z(T\psi) = TZ(\psi)$; here, by the same letter T we denote the translation of the lattice $\mathbb{Z}^1$ to the left and the induced transformation of sequences $(T\psi)(n) = \psi(n+1)$.

The properties (a) and (b) do not determine $Z(\psi)$ uniquely, but in the following sections we shall elaborate a concrete procedure for their construction. If we have already defined $Z(\psi)$ for all ψ, we may introduce the following two new objects:

$\Phi(\alpha)$ is the set of all e.f. ψ of the operator $H_\varepsilon(\alpha)$ for which $0 \in Z(\psi) \subset [0, \infty)$.

$\Lambda(\alpha)$ is the set of all e.v. $\lambda(\alpha)$ of e.f. belonging to $\Phi(\alpha)$.

In general $\Phi(\alpha)$ and $\Lambda(\alpha)$ are multivalued functions of α taking finitely many values. In our situation they are "measurable" in α. In the case of complete Anderson localization the set of all e.v. of $H_\varepsilon(\alpha)$ is equal to $\bigcup_{m=-\infty}^{\infty} \Lambda(\alpha + m\omega)$, i.e., the whole spectrum consists of values of a measurable function along a trajectory of rotation. It was always clear that in the domain of Anderson localization the whole set of e.v. of $H_\varepsilon(x)$ in (1.2) is a nonmeasurable function of x. The last expression shows explicitly the nature of this nonmeasurability. The basis of all e.f. can be written as $\bigcup_{m=-\infty}^{\infty} T^m \Phi(\alpha + m\omega)$. It is easy to show also that the limit density of states is the distribution function of $\Lambda(\alpha)$. Certainly objects like $\Phi(\alpha)$ and $\Lambda(\alpha)$ can be defined not only in our situation, but also in the general setting of (1.2). The investigation of their properties might be useful for many problems, for example, a limiting distribution of spacings between the nearest e.v. and others.

The first idea of the possibility of Anderson localization in (1.1) with $V(\alpha) = \cos 2\pi\alpha$ appeared in Ref. 29, where the Aubry duality was discovered and it was proven that for small enough ε Liapunov exponents for all values of the spectral parameter E are positive.[2] Later Simon[(31)] and Pastur and Figotin[(32)] gave a more rigorous derivation of these results. It follows from the proof of our theorem that the integrated density of states of $H_\varepsilon(\alpha)$ is, for small enough ε, a Cantor devil's staircase concentrated on a Cantor set of positive measure and for all E the Liapunov exponents are positive for general potentials V. Anderson localization for the potential $V(\alpha) = \mathrm{tg}(\alpha\pi)$ was proved by Pastur and Figotin[(33)] and Simon.[(34)] Other examples of potentials where one can establish Anderson localization for small ε appeared in a paper by Pöschel.[(35)] The fact that the integrated density of states is a Cantor staircase is connected with the fact that we are dealing with quasiperiodic potentials with one incommensurate density.

[2] In fact, the results are sharper.

As is clear from what was said above, our method consists in constructing functions $\Phi(\alpha)$ and $\Lambda(\alpha)$ for the operators $H_\varepsilon(\alpha)$. It is based upon a renormalization group analysis currently popular in the theory of dynamical systems. Namely, let us consider the continued fraction expansion of ω, i.e., $\omega = [k_1, k_2, \ldots, k_s, \ldots]$. We assume that (1) $k_s \leqslant \text{const} \cdot s^2$; (2) if $\omega_s = p_s/q_s$ is the sth approximant of ω, i.e., $\omega_s = [k_1, k_2, \ldots, k_s]$, then $\lim_{s \to \infty}(1/s) \ln q_s$ exists. It is well known that almost all ω have both properties. Our analysis goes by induction in s. On the sth step we consider the operator

$$(H_\varepsilon^{(s)}(\alpha)\,\psi)(n) = -\varepsilon(\psi(n+1) + \psi(n-1)) + V(n\omega_s + \alpha)\,\psi(n) \qquad (1.4)$$

in the finite-dimensional space of periodic sequences $\{\psi(n)\}$ with the period q_s, for which we define the corresponding objects $\Phi^{(s)}(\alpha)$ and $\Lambda^{(s)}(\alpha)$. Then we develop a perturbation theory that makes it possible to pass from $s \to s+1$. In the next section we discuss in detail the initial step of the construction.

In our proof several orders of smallness appear. The smallness q_s^{-1} is a smallness of distances between the e.v. of $H_\varepsilon^{(s)}(\alpha)$. Next is the smallness q_s^{-2}, which is a smallness of perturbation of the potential under the transition $s \to s+1$. Then we introduce a smallness that is intermediate between these and is connected with the cutoff of e.f. (see later). We choose this to be equal to $q_s^{-3/2}$; however, it is possible to have $q_s^{-\gamma}$ for arbitrary γ, $1 < \gamma < 2$. We also consider slightly perturbed smallness, such as $q_s^{-1 \pm c(\ln 1/\varepsilon)^{-1}}$, $q_s^{-2 \pm c(\ln 1/\varepsilon)^{-1}}$, or $q_s^{-3/2 \pm c(\ln 1/\varepsilon)^{-1}}$. The constants c and C are absolute constants not depending on ε and s. In one part of the construction we also need $q_s^{-2+\delta_1}$, where $c(\ln 1/\varepsilon)^{-1} \leqslant \delta_1 \leqslant 1/2$. The rotation of the circle to the angle ω is denoted by R_ω.

Remark. J. Fröhlich has informed me that he and T. Spencer, using their methods,[9,10] have also proved the complete Anderson localization for (1.1).

2. BEGINNING OF THE INDUCTIVE PROCEDURE

We consider the operator $H_\varepsilon^{(s)}(\alpha)$ acting in the space of periodic sequences $\psi = \{\psi(n)\}$, $\psi(n + q_s) = \psi(n)$ by the formula

$$(H_\varepsilon^{(s)}(\alpha)\,\psi)(n) = -\varepsilon(\psi(n+1) + \psi(n-1)) + V(n\omega_s + \alpha)\,\psi(n) \qquad (2.1)$$

The spectrum of $H_\varepsilon^{(s)}(\alpha)$ consists of q_s numbers $\lambda_{\varepsilon,1}^{(s)}(\alpha) \leqslant \lambda_{\varepsilon,2}^{(s)}(\alpha) \leqslant \cdots \leqslant \lambda_{\varepsilon,q_s}^{(s)}(\alpha)$, where each $\lambda_{\varepsilon,i}^{(s)}(\alpha)$ is a periodic function of α with period q_s^{-1}. A typical form of these functions is presented in Fig. 1. The fact that they

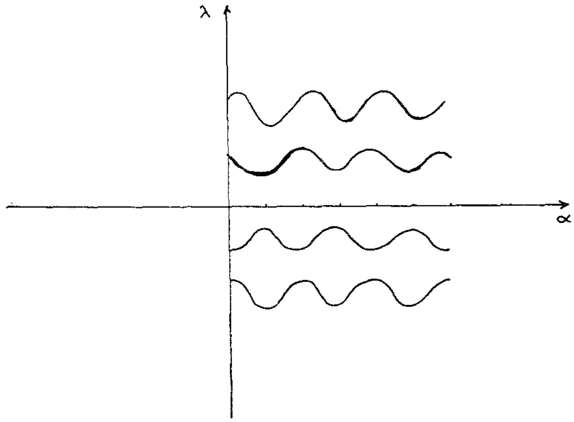

Figure 1

really look like this will be clear from the later analysis. The intervals $(\max_\alpha \lambda_{\varepsilon,i}^{(s)}(\alpha), \min_\alpha \lambda_{\varepsilon,i+1}^{(s)}(\alpha))$ are forbidden zones (f.z.) because there are no α for which $\lambda_{\varepsilon,i}^{(s)}(\alpha)$ fit in these intervals. Generically the number of all f.z. is equal to $q_s - 1$.

This representation for the spectrum is not very convenient for our goals and we shall elaborate another one. For $\varepsilon = 0$ the spectrum of $H_0^{(s)}(\alpha)$ consists of the numbers $V(m\omega_s + \alpha)$, $0 \leqslant m < q_s$. Each e.f. $\psi_{\alpha,m}^{(s)}$ is concentrated at a single point, i.e., $\psi_{\alpha,m}^{(s)} = \delta_m$. If we put $\Lambda_0^{(s)}(\alpha) = V(\alpha)$, then we see that the whole spectrum is given by the formula $\bigcup_{m=0}^{q_s-1} \Lambda_0^{(s)}(\alpha + mq_s^{-1})$.

It is natural to define the essential support (e.s.) of the e.f. δ_m as consisting of the point m. Thus, $\Phi_0^{(s)}(\alpha)$ is the δ-function with the eigenvalue (e.v.) $\Lambda_0^{(s)}(\alpha) = V(\alpha)$.

Denote $\Pi_{k,l} = \{\alpha \mid V(k\omega_s + \alpha) = V(l\omega_s + \alpha)\}$. It follows from our assumptions concerning V that $\Pi_{k,l}$ consists precisely of two points (see Fig. 2). Also, $\Pi_{k,l} = R_{\omega_s}^k \Pi_{0,l-k}$. The structure of e.f. of $H_\varepsilon^{(s)}(\alpha)$ might change from that of $H_0^{(s)}(\alpha)$ in neighborhoods of $\Pi_{k,l}$. Introduce neighborhoods $O_{k,l,i}$ of points $\alpha_i \in \Pi_{k,l}$, $i = 1, 2$, of radii $\rho_{|l-k|}$ in such a way that $O_{k,l,i} = R_{\omega_s}^k O_{0,l-k,i}$ and different neighborhoods are disjoint.

Assume that $\alpha \notin \bigcup_{k,l,i} O_{k,l,i}$. Then e.v. $\bar{\lambda}_{\alpha,j}^{(s)}$, $1 \leqslant j \leqslant q_s$, of the unperturbed operator $H_0^{(s)}(\alpha)$ are sufficiently far from each other, i.e., $|\bar{\lambda}_{\alpha,i}^{(s)} - \bar{\lambda}_{\alpha,j}^{(s)}| \geqslant \chi$, where $\chi > 0$ depends only on the choice of the numbers $\rho_{|l-k|}$. The standard perturbation theory is applied provided that ε is sufficiently small. It gives for small enough ε the existence of q_s normed e.f. $\psi_{\alpha,m}^{(s)}$, $1 \leqslant m \leqslant q_s$, such that $|\psi_{\alpha,m}^{(s)}(n)| \leqslant (\text{const} \cdot \varepsilon)^{\text{dist}(n,m)}$, where dist is the usual distance on the set $0 \leqslant n < q_s$ with periodic boundary conditions. The corresponding e.v. $\lambda_{\alpha,m}^{(s)}$ satisfy the inequalities $|\lambda_{\alpha,m}^{(s)} - V(m\omega_s + \alpha)| \leqslant \text{const} \cdot \varepsilon$. The reader

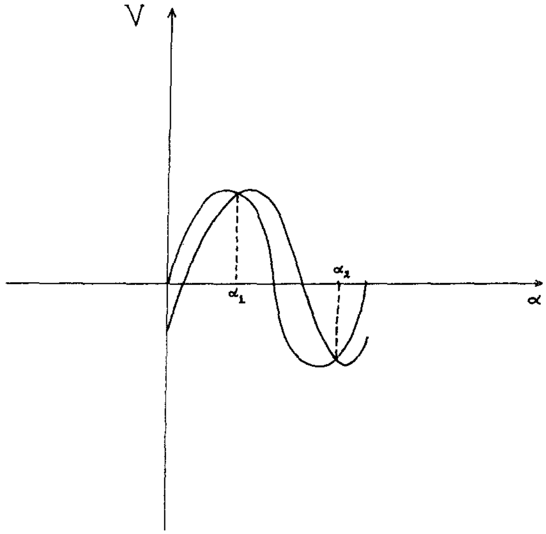

Figure 2

can also derive all these facts concerning $\psi^{(s)}_{\alpha,m}$ from the results of Section 4. Some nontriviality is contained in the fact that the constant does not depend on ε. In deriving the inequality for $\lambda^{(s)}_{\alpha,i}$, one has to use the fact that $|\bar{\lambda}^{(s)}_{\alpha,j_1} - \bar{\lambda}^{(s)}_{\alpha,j_2}| \geqslant \text{const} \cdot |j_1 - j_2|^{-b}$ for some $b > 0$, which follows easily from the assumptions about ω. As before, we define the e.s. of $\psi^{(s)}_{\alpha,m}$ as consisting of the point m, i.e., $Z(\psi^{(s)}_{\alpha,m}) = m$.

Let us take $\alpha \in O_{k,l,i}$. Now the difference between one pair of unperturbed e.v. $\bar{\lambda}^{(s)}_{\alpha,k}$, $\bar{\lambda}^{(s)}_{\alpha,l}$ can be arbitrarily small while all other e.v. $\bar{\lambda}^{(s)}_{\alpha,i}$ are far from each other and from this pair. This pair generates the simplest resonant e.f. whose form might be more complicated. Other e.f. are constructed with the help of the usual perturbation theory and their values decay exponentially outside some point m. For such e.f. $\psi^{(s)}_{\alpha,m}$ we put as before $Z(\psi^{(s)}_{\alpha,m}) = m$.

The analysis of the resonant e.f. is a bit more complicated. We look for two e.f. having the form

$$\psi^{(s)}_{\alpha,k,l,\pm} = A_1 \delta_k + A_2 \delta_l + \cdots$$

where dots denote the projections to the space orthogonal to δ_k, δ_l, which are small enough compared with $A_1^2 + A_2^2$. The graphs of the corresponding e.v. $\lambda_\pm = \lambda^{(s)}_{\alpha,k,l,\pm}$ are given in Fig. 3. One sees explicitly the appearence of the forbidden zones (f.z.) due to the resonances. The formulas of perturbation theory show that the widths of f.z. decay exponentially with the dist(k, l).

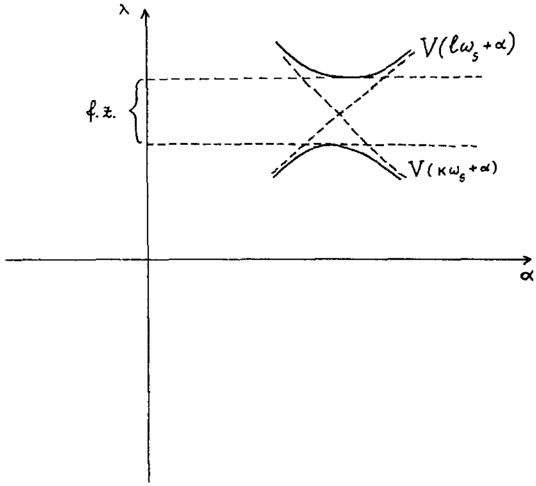

Figure 3

The main technique of this paper is the perturbation theory. It will be applied to functions that are not exact e.f. but are e.f. with some precision or almost e.f. (see later). In other words, sometimes it is more convenient for us to replace the exact e.f. by approximate e.f. if the corresponding error is small enough. According to this approach, at the initial step of the construction we shall consider the exact e.f. if the widths of the f.z. are not too small, or, more precisely, if

$$|k-l| < (2-\delta_1)\ln q_{s0}\cdot\left(\ln\frac{1}{\varepsilon}\right)^{-1}$$

where δ_1 will be specified later. In this case we put

$$Z(\psi^{(s)}_{\alpha,k,l,+}) = Z(\psi^{(s)}_{\alpha,k,l,-}) = \{k, l\}$$

If

$$|k-l| \geqslant (2-\delta)\ln q_{s0}\cdot\left(\ln\frac{1}{\varepsilon}\right)^{-1}$$

then the width of the f.z. is equal to $\min|\lambda_+(\alpha)-\lambda_-(\alpha)| \leqslant q_s^{-2+2\delta_1}$. Here we make the operation that we call the cut of the resonant e.f. Namely, we find the numbers B_i, $1\leqslant i\leqslant 4$, such that

$$\psi^{(s)}_{\alpha,k} = B_1\psi^{(s)}_{\alpha,k,l,+} + B_2\psi^{(s)}_{\alpha,k,l,-} = \delta_k + \cdots$$
$$\psi^{(s)}_{\alpha,l} = B_3\psi^{(s)}_{\alpha,k,l,+} + B_4\psi^{(s)}_{\alpha,k,l,-} = \delta_l + \cdots$$

where dots have the same meaning as above. The functions $\psi_{\alpha,k}^{(s)}$ and $\psi_{\alpha,l}^{(s)}$ are almost e.f. and the error is of the order $\lambda_+ - \lambda_-$.

For approximate e.f. $\psi_{\alpha,k}^{(s)}$ and $\psi_{\alpha,l}^{(s)}$ we put $Z(\psi_{\alpha,k}^{(s)}) = k$ and $Z(\psi_{\alpha,l}^{(s)}) = l$. Now we can define the functions $\Phi^{(s)}(\alpha)$ and $\Lambda^{(s)}(\alpha)$. If $\alpha \notin \bigcup_{l,i} O_{0,l,i}$, we put $\Phi^{(s)}(\alpha)$ to be equal to the exact e.f. $\psi_{\alpha,0}^{(s)}$, for which $Z(\psi_{\alpha,0}^{(s)}) = 0$ and $\Lambda^{(s)}(\alpha)$ is equal to the corresponding e.v. If $\alpha \in O_{0,k,i}$ and $0 < k < (2 - \delta_1) \ln q_s \cdot (\ln 1/\varepsilon)^{-1}$, then $\Phi^{(s)}(\alpha)$ consists of two exact e.f. having the e.s. $\{0, k\}$, and $\Lambda^{(s)}(\alpha)$ consists of two corresponding e.v. Remark that for $\alpha \in O_{-k,0,i}$ the values of $\Phi^{(s)}(\alpha)$ and $\Lambda^{(s)}(\alpha)$ are empty sets. Indeed, for such α the e.f. whose e.s. contains 0 has the e.s. $\{-k, 0\}$. Thus, there are no e.f. for which the e.s. contains O and lies to the right of 0.

If $\alpha \in O_{0,k,i}$ and $k \geqslant (2 - \delta_1) \ln q_s \cdot (\ln 1/\varepsilon)^{-1}$, then we take $\Phi^{(s)}(\alpha)$ to be equal to the approximate e.f. $\psi_{\alpha,0}^{(s)}$, for which $Z(\psi_{\alpha,0}^{(s)}) = 0$. If $\psi_{\alpha,0}^{(s)} = B_1 \psi_{\alpha,0,k,+}^{(s)} + B_2 \psi_{\alpha,0,k,-}^{(s)}$, $B_1^2 + B_2^2 = 1$, we put $\Lambda^{(s)}(\alpha) = B_1^2 \lambda_+ + B_2^2 \lambda_-$. Thus, $\Phi^{(s)}(\alpha)$ and $\Lambda^{(s)}(\alpha)$ are completely defined. The graph of $\Lambda^{(s)}(\alpha)$ is given in Fig. 4.

The function $\Lambda^{(s_0)}(\alpha)$ has the following property.

For each λ from the range of $\Lambda^{(s_0)}$ except the extremal one there are two values α_1, α_2 for which $\lambda = \Lambda^{(s)}(\alpha_1) = \Lambda^{(s)}(\alpha_2)$. The e.s. of corresponding elements of $\Phi^{(s)}(\alpha_1)$ and $\Phi^{()}(\alpha_2)$ are the same.

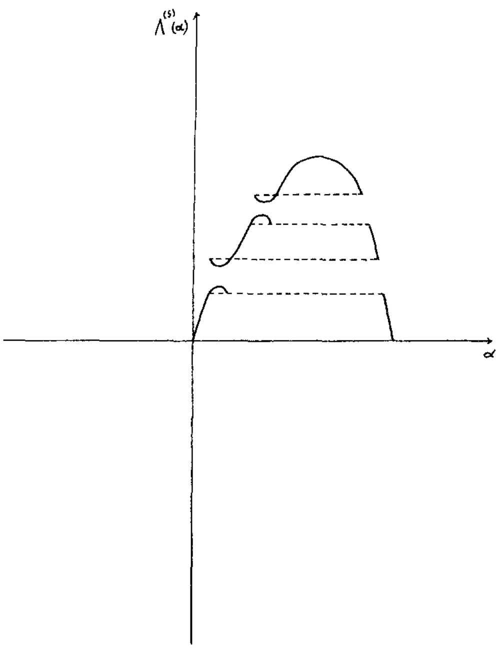

Figure 4

In our construction we consider the transitions $s \to s+1$ and the graphs $\Lambda^{(s)}(\alpha + kq_{s+1})$, $0 \leqslant k < q_{s+1}$. In small neighborhoods of their points of intersection there appear new resonances where the form of the e.f. changes. The main idea of our approach is to follow for these changes.

3. INDUCTIVE ASSUMPTIONS ABOUT THE SPECTRUM OF $H_\varepsilon^{(s)}(\alpha)$

The main content of this paper is an inductive process that makes it possible to construct e.f. of $H_\varepsilon^{(s+1)}(\alpha)$ based on detailed information concerning e.f. and e.v. of $H_\varepsilon^{(s)}(\alpha)$. This information is described in this section.

Denote the e.v. of the operator $H_\varepsilon^{(s)}(\alpha)$ by $\mu_i^{(s)}(\alpha)$, $1 \leqslant i \leqslant q_s$. Generically,

$$\mu_1^{(s)}(\alpha) < \mu_2^{(s)}(\alpha) < \cdots < \mu_{q_s}^{(s)}(\alpha)$$

and each $\mu_i^{(s)}(\alpha)$ is a periodic function of α with the period q_s^{-1}. Intervals $\Pi_i = (\max_\alpha \mu_{i-1}(\alpha), \min_\alpha \mu_i(\alpha))$ are forbidden zones (f.z.). A f.z. is called wide if the length $|\Pi_i| > q_s^{-2+\delta_1}$. Other f.z. are called narrow. They appear as a result of wide resonances.

As in Section 2, we shall deal with another description of the spectrum. Assume that on the axis of the spectrum $r \leqslant q_s$ nonoverlapping segments $[a_l, b_l]$, $a_{l+1} - b_l > q_s^{-2+\delta_1}$, and subsegments $[a_l, c_l]$, $[d_l, b_l]$, $a_l < c_l < d_l < b_l$, are given, and thus the gap (b_l, a_{l+1}) is always wide.

I. For each l, $1 \leqslant l \leqslant r$, four nonoverlapping intervals $\Delta_{l,i}^{(s)} = [\alpha_{l,i}^{(s)}, \beta_{l,i}^{(s)}]$, $1 \leqslant i \leqslant 4$, on the axis of α are given and on each of them a C^2-function $\Lambda_{l,i}^{(s)}$ is defined in such a way that:

1. The following hold:

$$\Lambda_{l,2}^{(s)}(\Delta_{l,2}^{(s)}) = \Lambda_{l,3}^{(s)}(\Delta_{l,3}^{(s)}) = [c_l, d_l]$$

$$\frac{1}{h_s^{(1)} s^2} \leqslant \frac{d\Lambda_{l,2}^{(s)}(\alpha)}{d\alpha} \leqslant h_s^{(1)}, \qquad \left| \frac{d^2 \Lambda_{l,2}^{(s)}(\alpha)}{d\alpha^2} \right| \leqslant h_s^{(2)} q_s^{1/2}$$

$$-h_s^{(1)} \leqslant \frac{d\Lambda_{l,3}^{(s)}(\alpha)}{d\alpha} \leqslant -\frac{1}{h_s^{(1)} s^2}, \qquad \left| \frac{d^2 \Lambda_{l,3}^{(s)}(\alpha)}{d\alpha^2} \right| \leqslant h_s^{(2)} q_s^{1/2}$$

Here $h_s^{(1)}$, $h_s^{(2)}$ are constants and we will see from our construction that they converge to a limit as $s \to \infty$.

2. The segments $\Delta_{l,1}^{(s)}$, $\Delta_{l,4}^{(s)}$ are called resonant zones (r.z.). For each r.z. the moment $s(l, i)$, $i = 1, 4$, of its appearance is defined. Also,

$$\Lambda_{l,1}^{(s)}(\Delta_{l,1}^{(s)}) = [a_l, c_l], \qquad \Lambda_{l,4}^{(s)}(\Delta_{l,4}^{(s)}) = [d_l, b_l]$$

3. The following hold:

$$\frac{d^2 \Lambda_{l,1}^{(s)}}{d\alpha^2} \geqslant (h_s^{(1)})^{-1}\, q_{s(l,1)}^{1/4}, \qquad \frac{d^2 \Lambda_{l,4}}{d\alpha^2} \leqslant -(h_s^{(1)})^{-1}\, q_{s(l,1)}^{1/4}$$

and each value except a_l, b_l is taken precisely at two points.

It is convenient to assume that each function $\Lambda_{l,i}^{(s)}$ is defined everywhere but its value outside $\Delta_{l,i}^{(s)}$ is an empty set.

A typical form of $\Lambda_{l,i}^{(s)}$ is presented at in Fig. 5.

II. For each $\Delta_{l,i}^{(s)}$, nonoverlapping subintervals $\Delta_{l,i,k}^{(s)} \subset \Delta_{l,i}^{(s)}$ are defined, which are called small resonance zones. We shall see that if $\alpha + \omega_s t \notin \bigcup_{l,i,k} \Delta_{l,i,k}^{(s)}$ for all t, $0 \leqslant t < q_s$, then the spectrum of $H_\varepsilon^{(s)}(\alpha)$ is the set

$$\bigcup_{t=0}^{q_s-1} \bigcup_{l,i} \Lambda_{l,i}^{(s)}(\alpha + t\omega_s)$$

which therefore has q points. If $\alpha \in \Delta_{l,i,k}^{(s)}$ for some l, i, k, then the set

$$\bigcup_{t=0}^{q_s-1} \bigcup_{l,i} \Lambda_{l,i}^{(s)}(\alpha + t\omega_s)$$

contains as before q_s points, but some of them are only approximate e.v.

The function $\Lambda^{(s)}(\alpha)$ is defined by the expression

$$\Lambda^{(s)}(\alpha) = \bigcup_{l,i:\alpha \in \Delta_{l,i}^{(s)}} \Lambda_{l,i}^{(s)}(\alpha)$$

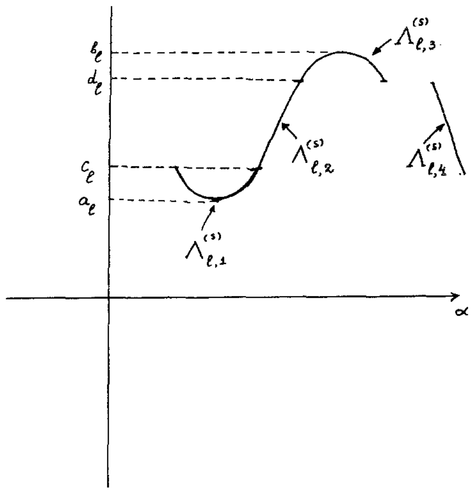

Figure 5

III. Small r.z. $\Delta^{(s)}_{l,i,k}$ are united in pairs $\Delta^{(s)}_{l,i,k}$, $\Delta^{(s)}_{l,i',k'} = \Delta^{(s)}_{l,i,k} + \omega_s t$ for some t, where

$$(2-\delta_1)\ln q_s \cdot \left(\ln \frac{1}{\varepsilon}\right)^{-1} \leqslant t < q_s$$

Moreover, $i = i'$ for $i = 1, 4$ and $i' = i \pm 1$ for $i = 2$ or 3. The value of t is called the width of the resonance corresponding to the pair $\Delta^{(s)}_{l,i,k}$, $\Delta^{(s)}_{l,i',k'}$. Functions $\Lambda^{(s)}_{l,i}$ are monotone on each of these intervals; on one of them it is increasing, while it is decreasing on the other, but $\Lambda^{(s)}_{l,i}(\Delta^{(s)}_{l,i,k}) = \Lambda^{(s)}_{l,i'}(\Delta^{(s)}_{l,i',k'})$. The lengths of the intervals $\Delta^{(s)}_{l,i,k}$, $\Delta^{(s)}_{l,i',k'}$ satisfy the inequalities

$$|\Delta^{(s)}_{l,i,k}|,\ |\Delta^{(s)}_{l,i',k'}| \leqslant q_s^{-3m/2-1/4}$$

where

$$m = \left[\frac{t\ln(1/\varepsilon)}{(2-\delta_1)\ln q_s}\right] - 1$$

Now we shall formulate inductive assumptions concerning e.f.

IV. Assume that $\alpha \notin \bigcup_{l,i,k} \Delta^{(s)}_{l,i,k}$ and $\alpha \in \Delta^{(s)}_{l_0,i_0}$. Then an exact normed e.f. $\psi^{(s)}_{\alpha,l_0,i_0}$ of $H^{(s)}_\varepsilon(\alpha)$ is defined whose e.v. is equal to $\Lambda^{(s)}_{l_0,i_0}(\alpha)$. If $\alpha \in \Delta^{(s)}_{l,i,k}$, then a normed almost e.f. $\psi^{(s)}_{\alpha,l,i,k}$ is defined in a sense to be specified later. For each e.f. $\psi = \psi^{(s)}_{\alpha,l_0,i_0}$ or almost e.f. $\psi = \psi^{(s)}_{\alpha,l,i,k}$ a finite set $Z(\psi) \subset [0, \frac{1}{2}q_s]$, $0 \in Z(\psi)$, is defined, which is called an essential support (e.s.) of ψ. For this set:

(1) $|\psi(n)| \leqslant (C\varepsilon)^{\operatorname{dist}(n,Z(\psi))}$

for those n for which

$$\operatorname{dist}(n, Z(\psi)) \leqslant \frac{3\ln q_s}{2\ln 1/\varepsilon}\left(1 - \frac{c}{\ln\varepsilon}\right)$$

and

(2) $\operatorname{diam}(Z(\psi)) \leqslant C\left(\ln \frac{1}{\varepsilon}\right)^{-1} s$

The values of the constants C and c follow easily from the construction.

Now we can explain the sense in which $\psi^{(s)}_{\alpha,l,i,k}$ is an almost e.f. Namely,

$$|(H^{(s)}_\varepsilon(\alpha)\,\psi^{(s)}_{\alpha,l,i,k})(n) - \Lambda^{(s)}_{l,i}(\alpha)\,\psi^{(s)}_{\alpha,l,i,k}(n)| \leqslant q_s^{-2+\delta_1}$$

for all n for which

$$\operatorname{dist}(n, Z(\psi^{(s)}_{\alpha,l,i,k})) \leqslant \frac{3\ln q_s}{2\ln(1/\varepsilon)}\left(1 - \frac{c}{\ln\varepsilon}\right)$$

V. The e.s. Z does not depend on $\alpha \in \Delta_{l,2}^{(s)} \cup \Delta_{l,3}^{(s)}$ for each $l = 1, 2,..., r$. For $\alpha \in \Delta_{l,1}^{(s)}$ ($\alpha \in \Delta_{l,4}^{(s)}$) the e.s. also does not depend on α and there exists m_1 (m_4) such that $Z = Z' \cup T^{m_i}Z'$, $i = 1, 4$, where Z' is an e.s. corresponding to $\Delta_{l,2}$. It is assumed also that Z and $T^{m_i}Z'$ do not intersect each other.

If ψ is an e.f. or almost e.f. of $H_\varepsilon^{(s)}(\alpha)$, then $T^t\psi$ is an e.f. or almost e.f. of $H_\varepsilon^{(s)}(\alpha + t\omega_s)$. By definition, the e.s. of $T^t\psi$ is $Z(\psi) + t$.

VI. For each phase α put

$$\Phi^{(s)}(\alpha) = \bigcup_{\alpha \in \Delta_{l,i}^{(s)}} (\psi_{\alpha,l,i}^{(s)} \cup \psi_{\alpha,l,i,k}^{(s)})$$

The set of all e.f. or almost e.f.

$$\bigcup_{0 \leqslant t < q_s} \bigcup_{l,i} T^t(\psi_{\alpha,l,i}^{(s)} \cup \psi_{\alpha,l,i,k}^{(s)}) = \bigcup_{0 \leqslant t < q_s} T^t\Phi^{(s)}(\alpha + t\omega_s)$$

is a basis in the space of periodic sequences $\psi = \{\psi(n)\}$, $\psi(n + q_s) = \psi(n)$.

VII. It follows from V that $\#(Z(\psi)) = 2^r$. The number r is called the range of an e.f. or of the interval $\Delta_{l,i}^{(s)}$. It means that the corresponding e.f. appeared as a result of r resonances. If $r \geqslant 1$, then $Z(\psi) = Z' \cup T^m Z'$, where m is the width of the resonance. In fact for the resonances of the rth range we have r numbers $m_1 < m_2 < \cdots < m_r$, where $m = m_r$, m_{r-1} is defined in a similar way for Z' and so on. Also, $\Delta_{l,i}^{(s)} \subset \Delta_{l',i'}^{(s')}$ and $Z' = Z(\psi_{\alpha,l',i'}^{(s')})$ for $\alpha \in \Delta_{l',i'}^{(s')}$ and so on.

In passing from $s \to s+1$ we first construct the functions $\Lambda^{(s+1)}(\alpha)$ and $\Phi^{(s+1)}(\alpha)$, then decompose the λ axis onto corresponding segments and define corresponding $\Delta_{l,i}^{(s+1)}$. The construction gives all needed properties of these functions. An approximate form of $\Lambda^{(s)}(\alpha)$ is presented in Fig. 6. It differs from Fig. 4 by a slightly more complicated form of resonances. Multivaluedness of $\Lambda^{(s)}$ is connected with the resonant e.f. Also, there are intervals where $\Lambda_{l,i}^{(s)}(\alpha)$ is an empty set for all l, i (see Section 2). We show in Fig. 7 two resonant e.f. whose essential support consists of two points $Z = \{0, t\}$. In this situation $H_\varepsilon^{(s)}(\alpha)$ does not have e.f. or almost e.f. whose e.s. is the point t, or the operator $H_\varepsilon^{(s)}(\alpha - \omega_s t)$ does not have e.f. or almost e.f. whose e.s. contains O and lies to the right of 0. This effect was already explained in Section 2.

4. GENERAL THEOREMS OF PERTURBATION THEORY

We consider the operator $H_\varepsilon^{(s)}(\alpha)$ [see (1.1)]. Assume that there are given periodic normed functions $\bar\varphi_{\alpha,i}^{(s)}$, $1 \leqslant i \leqslant q_s$, with the period q_s such that:

(a_0) For each $\bar\varphi_{\alpha,i}^{(s)}$ an e.s. $Z(\bar\varphi_{\alpha,i}^{(s)})$ is defined,

$$\operatorname{diam} Z(\bar\varphi_{\alpha,i}^{(s)}) \leqslant C(\ln 1/\varepsilon)^{-1} s$$

(a_1) The following holds:

$$|\bar{\varphi}_{\alpha,i}^{(s)}(n)| \leqslant (C\varepsilon)^{\operatorname{dist}(n, Z(\bar{\varphi}_{\alpha,i}^{(s)}))}$$

(a_2) $\bar{\varphi}_{\alpha,i}^{(s)}(n) = 0$ if

$$\operatorname{dist}(n, Z(\bar{\varphi}_{\alpha,i}^{(s)})) > \left[\frac{3 \ln q_s}{2 \ln(1/\varepsilon)}\right] + 2$$

(a_3) We have

$$|(\bar{\varphi}_{\alpha,i_1}^{(s)}, \bar{\varphi}_{\alpha,i_2}^{(s)}) - \delta_{i_1 i_2}| \leqslant q_s^{-3/2 - c(\ln \varepsilon)^{-1}}$$

(a_4) For each i a number $\lambda_{\alpha,i}^{(s)}$ is given for which

$$|(H_\varepsilon^{(s)}(\alpha)\, \bar{\varphi}_{\alpha,i}^{(s)})(n) - \lambda_{\alpha,i}^{(s)} \bar{\varphi}_{\alpha,i}^{(s)}(n)| \leqslant q_s^{-2+\delta_1}$$

if

$$\operatorname{dist}(n, Z(\bar{\varphi}_{\alpha,i}^{(s)})) \leqslant \left[\frac{3}{2} \frac{\ln q_s}{\ln(1/\varepsilon)}\right] + 1$$

The properties (a_0), (a_1) show the sense in which the functions $\bar{\varphi}_{\alpha,i}^{(s)}$ are concentrated near e.s. The property (a_3) means that the set of functions $\{\bar{\varphi}_{\alpha,i}^{(s)}\}$ is an almost orthogonal basis. The property (a_4) shows that $\bar{\varphi}_{\alpha,i}^{(s)}$ is an almost e.f.

We denote by $\varphi_{\alpha,i}^{(s)}$ the function that coincides with $\bar{\varphi}_{\alpha,i}^{(s)}$ for those n for which

$$\operatorname{dist}(n, Z(\bar{\varphi}_{\alpha,i}^{(s)})) \leqslant \left[\frac{3 \ln q_s}{2 \ln \varepsilon^{-1}}\right]$$

and is equal to zero for other n. Put $Z(\varphi_{\alpha,i}^{(s)}) = Z(\bar{\varphi}_{\alpha,i}^{(s)})$. We have

$$H_\varepsilon^{(s)}(\alpha)\, \varphi_{\alpha,i}^{(s)} = \lambda_{\alpha,i}^{(s)} \varphi_{\alpha,i}^{(s)} + \Gamma_{\alpha,i}^{(s)} + h_{\alpha,i}^{(s)} \tag{4.1}$$

Here $h_{\alpha,i}^{(s)}(n) = 0$ if

$$\operatorname{dist}(n, Z(\varphi_{\alpha,i}^{(s)})) > \left[\frac{3 \ln q_s}{2 \ln \varepsilon^{-1}}\right] + 1$$

and $|h_{\alpha,i}^{(s)}(n)| \leqslant \text{const} \cdot q_s^{-2+\delta_1}$ for other n. The vector $\Gamma_{\alpha,i}^{(s)}$ is defined as follows:

1. If z is such that

$$\mathrm{dist}(z, Z(\varphi_{\alpha,i}^{(s)})) = \left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right]$$

$$\mathrm{dist}(z+1, Z(\varphi_{\alpha,i}^{(s)})) = \left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right] + 1$$

then

$$\Gamma_{\alpha,i}^{(s)}(z) = -\varepsilon\bar{\varphi}_{\alpha,i}^{(s)}(z+1); \qquad \Gamma_{\alpha,i}^{(s)}(z+1) = -\varepsilon\bar{\varphi}_{\alpha,i}^{(s)}(z)$$

2. If z is such that

$$\mathrm{dist}(z, Z(\varphi_{\alpha,i}^{(s)})) = \left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right]$$

$$\mathrm{dist}(z-1, Z(\varphi_{\alpha,i}^{(s)})) = \left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right] + 1$$

then

$$\Gamma_{\alpha,i}^{(s)}(z) = -\varepsilon\bar{\varphi}_{\alpha,i}^{(s)}(z-1); \qquad \Gamma_{\alpha,i}^{(s)}(z-1) = -\varepsilon\bar{\varphi}_{\alpha,i}^{(s)}(z)$$

In all other cases $\Gamma_{\alpha,i}^{(s)}(n) = 0$. If there are two z for which condition 1 holds, then $\Gamma_{\alpha,i}^{(s)}(n)$ is nonzero at four points and so on.

Definition 1. An almost e.v. $\lambda_{\alpha,i}^{(s)}$ is nonresonant if for all $j \neq i$

$$|\lambda_{\alpha,i}^{(s)} - \lambda_{\alpha,j}^{(s)}| \geqslant \frac{1}{s^{10} + [\mathrm{dist}(Z(\varphi_{\alpha,i}^{(s)}), Z(\varphi_{\alpha,j}^{(s)}))]^{10}} + q_s^{-1 + c(\ln \varepsilon)^{-1}}$$

Theorem 1. Assume that $\lambda_{\alpha,i}^{(s)}$ is a nonresonant e.v. Then the operator $H_\varepsilon^{(s)}(\alpha)$ has an exact e.f. $\psi_{\alpha,i}^{(s)}$, which can be written in the form

$$\psi_{\alpha,i}^{(s)} = \varphi_{\alpha,i}^{(s)} + \delta\psi_{\alpha,i}^{(s)} + \delta\delta\psi_{\alpha,i}^{(s)}$$

where:

(b$_1$) $\delta\psi_{\alpha,i}^{(s)} = \sum_{j \neq i} \frac{(\Gamma_{\alpha,i}^{(s)} + h_{\alpha,i}^{(s)}, \varphi_{\alpha,j}^{(s)})}{\lambda_{\alpha,i}^{(s)} - \lambda_{\alpha,j}^{(s)}} \varphi_{\alpha,j}^{(s)}$

(b$_2$) $|\delta\delta\psi_{\alpha,i}^{(s)}(n)| \leqslant q_s^{-2 + c(\ln \varepsilon)^{-1}}$ for all n

If $\mu_{\alpha,i}^{(s)}$ is the corresponding e.v., then

$$|\mu_{\alpha,i}^{(s)} - [\lambda_{\alpha,i}^{(s)} + (\Gamma_{\alpha,i}^{(s)}, \varphi_{\alpha,i}^{(s)}) + (h_{\alpha,i}^{(s)}, \varphi_{\alpha,i}^{(s)})]| \leqslant q_s^{-2+c}\left(\ln \frac{1}{\varepsilon}\right)^{-1}$$

Proof. Write the e.f. $\psi_{\alpha,i}^{(s)}$ in the form

$$\psi_{\alpha,i}^{(s)} = \varphi_{\alpha,i}^{(s)} + \sum_{j \neq i} x_j \varphi_{\alpha,j}^{(s)}$$

Then for the unknown e.v. μ and the unknown coefficients x_j we have the following set of equations:

$$\mu = \lambda_{\alpha,i}^{(s)} + g_{ii} + h_{ii} + \sum_{j \neq i} x_j (g_{ji} + h_{ji}) \tag{4.2}$$

$$(\mu - \lambda_j - g_{jj} - h_{jj})\, x_j + \sum_{j_1 \neq j,i} x_{j_1} (g_{j_1 j} + h_{j_1 j}) = g_{ij} + h_{ij} \tag{4.3}$$

The coefficients g_{ij}, h_{ij} are found from the expansions

$$\Gamma_{\alpha,i}^{(s)} = \sum g_{ij} \varphi_{\alpha,j}^{(s)}$$

$$h_{\alpha,i}^{(s)} = \sum_j h_{ij} \varphi_{\alpha,j}^{(s)}$$

First we consider (4.3) assuming that μ is a free parameter. Rewrite (4.3) in the operator form:

$$(D - F)\, x = f \tag{4.4}$$

where f is the vector with the components $g_{ij} + h_{ij}$, D is the diagonal matrix with the matrix elements $(\mu - \lambda_j - g_{jj} - h_{jj})\, \delta_{jj_1}$, and F is the matrix with the matrix elements $f_{jj_1} = g_{j_1 j} + h_{j_1 j}$. If D is invertible, we have from (4.4)

$$(I - D^{-1}F)\, x = D^{-1} f \tag{4.5}$$

If the norm $\|D^{-1}F\| < 1$, we can write the solution of (4.5) as the Neumann series

$$x = D^{-1}f + D^{-1}FD^{-1}f + D^{-1}FD^{-1}FD^{-1}f + \cdots \tag{4.6}$$

In Appendix A we show that

$$|h_{ij}| \leqslant q_s^{-2 + \delta_1 - (m-1)(3/2 + c(\ln \varepsilon)^{-1}/2)} \tag{4.7$'$}$$

$$|g_{ij}| \leqslant q_s^{-m[3/2 + c(\ln \varepsilon)^{-1}/2]} \tag{4.7$''$}$$

where m is found from the inequalities

$$(m-1)\left[\frac{3 \ln q_s}{2 \ln \varepsilon^{-1}}\right] \leqslant \operatorname{dist}(Z(\varphi_{\alpha,i}^{(s)}), Z(\varphi_{\alpha,j}^{(s)})) < m \left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right]$$

This immediately implies that if

$$|\mu-\lambda_{\alpha,i}^{(s)}|\leqslant \tfrac{1}{2}q_s^{-1-c(\ln\varepsilon)^{-1}}$$

then

$$\|D^{-1}\|\leqslant \text{const}\cdot q_s^{1-c(\ln\varepsilon)^{-1}}$$

and

$$\|D^{-1}F\|\leqslant \text{const}\cdot q_s^{-1/2-2c(\ln\varepsilon)^{-1}}$$

Thus the solution (4.6) really exists.

The components $\bar{x}_{ij}$ of the vector $D^{-1}f$ have the form $\bar{x}_{ij}=(g_{ij}+h_{ij})/(\mu-\lambda_{\alpha,j}^{(s)})$. We put $g_{ij}=(\Gamma_{\alpha,i}^{(s)},\varphi_{\alpha,j}^{(s)})+g_{ij}^{(1)}$ and $h_{ij}=(h_{\alpha,i}^{(s)},\varphi_{\alpha,j}^{(s)})+h_{ij}^{(1)}$ and define $\delta\psi_{\alpha,i}^{(s)}$ according to (b_1) while the rest is equal to

$$\delta\delta\psi_{\alpha,i}^{(s)}=\sum_j\frac{(\Gamma_{\alpha,i}^{(s)}+h_{\alpha,i}^{(s)},\varphi_{\alpha,j}^{(s)})(\mu-\lambda_{\alpha,i}^{(s)})}{(\lambda_{\alpha,i}^{(s)}-\lambda_{\alpha,j}^{(s)})(\mu-\lambda_{\alpha,j}^{(s)})}\varphi_{\alpha,j}^{(s)}$$
$$+\sum_j\frac{g_{ij}^{(1)}\varphi_{\alpha,j}^{(s)}}{\mu-\lambda_{\alpha,j}^{(s)}}+\sum_j\frac{h_{ij}^{(1)}\varphi_{\alpha,j}^{(s)}}{\mu-\lambda_{\alpha,j}^{(s)}}+\sum_{k=1}^{\infty}((D^{-1}F)^k D^{-1}f,\varphi)$$

where φ means the vector $\{\varphi_{\alpha,j}^{(s)}, j\neq i\}$.

The estimation of each of these terms is straightforward. In the first sum $\sum^{(1)}$ we have only terms for which

$$\operatorname{dist}(Z(\varphi_{\alpha,i}^{(s)}),Z(\varphi_{\alpha,j}^{(s)}))\leqslant \text{const}\cdot\left(\ln\frac{1}{\varepsilon}\right)^{-1}\cdot s$$

The nonresonant condition implies that the denominator is not less than $\text{const}\cdot s$. The total estimation for $\sum^{(1)}$ follows from the fact that for every n the number of almost e.f. $\varphi_{\alpha,j}^{(s)}$ for which $n\in Z(\varphi_{\alpha,j}^{(s)})$ and n is the left boundary of $Z(\varphi_{\alpha,j}^{(s)})$ is not more than $\text{const}\cdot s-20$. This is a direct consequence of (a_3). The estimations of the second and third sums $\sum^{(2)}$ and $\sum^{(3)}$ follow from the nonresonant condition and the estimations (4.7′) and (4.7″). The estimation of the fourth sum $\sum^{(4)}$ is based upon the nonresonant condition and the form of the vector f. In order to get the final estimations, one has to take into account the direct uniform estimations of terms in the fourth sum for $k>1$ and more concrete estimations for $k=1$.

Now denote the solution of (4.3) as $x_j(\mu)$. Obviously it is a continuous function of μ. Putting it into (4.2), we get the equation for μ only. Our estimations for $x_j(\mu)$ and g_{ji}, h_{ji} give easily the existence of at least one solution of (4.2) in the considered neighborhood

$$|\mu-\lambda_{\alpha,i}^{(s)}|\leqslant \tfrac{1}{2}q_s^{-1-c(\ln\varepsilon)^{-1}}$$

Two or more solutions cannot exist, because the appearing e.f. would not be orthogonal, being small perturbations of $\varphi_{\alpha,i}^{(s)}$. ▮

In Theorems 2 and 3 we shall consider a resonant case. First we give the following:

Definition 2. Assume that for two almost e.v. $\lambda_{\alpha,i_1}^{(s)}$ and $\lambda_{\alpha,i_2}^{(s)}$ the following inequalities hold:

$$|\lambda_{\alpha,i_1}^{(s)}-\lambda_{\alpha,i_2}^{(s)}|\leqslant\frac{1}{s^{10}+[\operatorname{dist}(Z(\varphi_{\alpha,i_1}^{(s)}),Z(\varphi_{\alpha,i_2}^{(s)}))]^{10}}+q_s^{-1+c(\ln\varepsilon)^{-1}} \tag{4.8'}$$

$$|\lambda_{\alpha,j}^{(s)}-\lambda_{\alpha,i_k}^{(s)}|\geqslant\frac{1}{s^{10}+[\operatorname{dist}(Z(\varphi_{\alpha,j}^{(s)}),Z(\varphi_{\alpha,i_k}^{(s)}))]^{10}}+q_s^{-1+c(\ln\varepsilon)^{-1}},\qquad k=1,2,\quad j\neq i_1,i_2 \tag{4.8''}$$

We shall call $(\lambda_{\alpha,i_1}^{(s)},\lambda_{\alpha,i_2}^{(s)})$ a simple resonant pair. This means that only one difference of e.v. might be arbitrarily small while all others are large.

Theorem 2. Assume that $(\lambda_{\alpha,i_1}^{(s)},\lambda_{\alpha,i_2}^{(s)})$ is a simple resonant pair. Then the operator $H_\varepsilon^{(s)}(\alpha)$ has two exact e.f., which we denote by $\psi_{\alpha,(i_1,i_2),\pm}^{(s)}$ and write in the form

$$\psi_{\alpha,(i_1,i_2),+}^{(s)}=A_+\varphi_{\alpha,i_1}^{(s)}+B_+\varphi_{\alpha,i_2}^{(s)}+\delta\psi_{\alpha,(i_1,i_2),+}^{(s)}+\delta\delta\psi_{\alpha,(i_1,i_2),+}^{(s)}$$

$$\psi_{\alpha,(i_1,i_2),-}^{(s)}=A_-\varphi_{\alpha,i_1}^{(s)}+B_-\varphi_{\alpha,i_2}^{(s)}+\delta\psi_{\alpha,(i_1,i_2),-}^{(s)}+\delta\delta\psi_{\alpha,(i_1,i_2),-}^{(s)}$$

where:

(c_1) $\delta\psi_{\alpha,(i_1,i_2),+}^{(s)}$

$$=A_+\sum_{j\neq i_1,i_2}\frac{(\Gamma_{\alpha,i_1}^{(s)}+h_{\alpha,i_1}^{(s)},\varphi_{\alpha,j}^{(s)})}{\lambda_{\alpha,i_1}^{(s)}-\lambda_{\alpha,j}^{(s)}}\varphi_{\alpha,j}^{(s)}+B_+\sum_{j\neq i_1,i_2}\frac{(\Gamma_{\alpha,i_2}^{(s)}+h_{\alpha,i_2}^{(s)},\varphi_{\alpha,j}^{(s)})}{\lambda_{\alpha,i_2}^{(s)}-\lambda_{\alpha,j}^{(s)}}\varphi_{\alpha,j}^{(s)}$$

$\delta\psi_{\alpha,(i_1,i_2),-}^{(s)}$

$$=A_-\sum_{j\neq i_1,i_2}\frac{(\Gamma_{\alpha,i_1}^{(s)}+h_{\alpha,i_1}^{(s)},\varphi_{\alpha,j}^{(s)})}{\lambda_{\alpha,i_1}^{(s)}-\lambda_{\alpha,j}^{(s)}}\varphi_{\alpha,j}^{(s)}+B_-\sum_{j\neq i_1,i_2}\frac{(\Gamma_{\alpha,i_2}^{(s)}+h_{\alpha,i_2}^{(s)},\varphi_{\alpha,j}^{(s)})}{\lambda_{\alpha,i_2}^{(s)}-\lambda_{\alpha,j}^{(s)}}$$

(c_2) The remainder terms $\delta\delta\psi^{(s)}_{\alpha,(i_1,i_2),+}$ and $\delta\delta\psi^{(s)}_{\alpha,(i_1,i_2),-}$ satisfy the same estimations as in Theorem 1, i.e.,

$$|\delta\delta\psi^{(s)}_{\alpha,(i_1,i_2),+}(n)|,\ |\delta\delta\psi^{(s)}_{\alpha,(i_1,i_2),-}(n)| \leqslant q_s^{-2+c(\ln\varepsilon)^{-1}}$$

(c_3) The e.v. $\mu^{(s)}_{\alpha,(i_1,i_2),+}$ and $\mu^{(s)}_{\alpha,(i_1,i_2),-}$ satisfy the inequalities

$$\left|\mu^{(s)}_{\alpha,(i_1,i_2),\pm} - \left\{\frac{\lambda^{(s)}_{\alpha,i_1}+\lambda^{(s)}_{\alpha,i_2}}{2} + \frac{g_{i_1i_1}+h_{i_1i_1}+g_{i_2i_2}+h_{i_2i_2}}{2}\right.\right.$$
$$\pm\frac{1}{2}\left[(\lambda^{(s)}_{\alpha,i_1}-\lambda^{(s)}_{\alpha,i_2}+g_{i_1i_1}+h_{i_1i_1}-g_{i_2i_2}-h_{i_2i_2})^2\right.$$
$$\left.\left.\left.+4(g_{i_1i_2}+h_{i_1i_2})(g_{i_2i_1}+h_{i_2i_1})\right]^{1/2}\right\}\right| \leqslant q_s^{-2+c(\ln\varepsilon)^{-1}}$$

(c_4) The two-dimensional matrix

$$\begin{vmatrix} A_+ & B_+ \\ A_- & B_- \end{vmatrix}$$

is an almost orthogonal matrix in the following sense:

$$|A_+^2+B_+^2-1|,\ |A_-^2+B_-^2-1|,\ |A_+A_-+B_+B_-| \leqslant q_s^{-2+c(\ln\varepsilon)^{-1}}$$

Proof. Look for an e.f. ψ in the following form:

$$\psi = A\varphi^{(s)}_{\alpha,i_1} + B\varphi^{(s)}_{\alpha,i_2} + \sum_{j\neq i_1,i_2} x_j\varphi^{(s)}_{\alpha,j}$$

Then for the unknown x_j and the corresponding e.v. μ we have the system of equations

$$A(\lambda^{(s)}_{\alpha,i_1}+g_{i_1i_1}+h_{i_1i_1}-\mu)+B(g_{i_2i_1}+h_{i_2i_1}) + \sum_{j\neq i_1,i_2} x_j(g_{ji_1}+h_{ji_1})=0 \tag{4.9}$$

$$A(g_{i_1i_2}+h_{i_1i_2})+B(\lambda^{(s)}_{\alpha,i_2}+g_{i_2i_2}+h_{i_2i_2}-\mu) + \sum_{j\neq i_1,i_2} x_j(g_{ji_2}+h_{ji_2})=0 \tag{4.10}$$

$$(\lambda^{(s)}_{\alpha,j}+g_{jj}+h_{jj}-\mu)\,x_j + \sum_{j_1\neq i_1,i_2,j} x_{j_1}(g_{j_1j}+h_{j_1j}) + A(g_{i_1j}+h_{i_1j})+B(g_{i_2j}+h_{i_2j})=0 \tag{4.11}$$

Again we first consider (4.11), assuming that A, B, and μ are parameters. Denote by f_{i_1} (f_{i_2}) the vector with components $g_{i_1 j}+h_{i_1 j}$ ($g_{i_2 j}+h_{i_2 j}$) and rewrite (4.11) in the operator form

$$Dx+Fx=Af_{i_1}+Bf_{i_2}$$

or

$$(I+D^{-1}F)\,x=AD^{-1}f_{i_1}+BD^{-1}f_{i_2}$$

Let μ be such that

$$|\mu-\lambda^{(s)}_{\alpha,j}|\geqslant q_s^{-1+c(\ln\varepsilon)^{-1}}$$

for all $j\neq i_1, i_2$. Then the same arguments as in Theorem 1 are applied and we can write the solution of (4.11) as the series

$$\begin{aligned} x&=A\left[D^{-1}f_{i_1}+\sum_{k=1}^{\infty}(-1)^k(D^{-1}F)^k\,D^{-1}f_{i_1}\right]\\ &\quad+B\left[D^{-1}f_{i_2}+\sum_{k=1}^{\infty}(-1)^k(D^{-1}F)^k\,D^{-1}f_{i_2}\right]\\ &=Ay_{i_1}+By_{i_2} \end{aligned} \tag{4.12}$$

where y_{i_1} (y_{i_2}) is the vector

$$D^{-1}f_{i_1}+\sum_{k=1}^{\infty}(-1)^k(D^{-1}F)^k\,D^{-1}f_{i_1}$$

$$\left[D^{-1}f_{i_2}+\sum_{k=1}^{\infty}(-1)^k(D^{-1}F)^k\,D^{-1}f_{i_2}\right]$$

whose components are denoted by $y_{i_1 j}$ ($y_{i_2 j}$). The expressions for $\delta\psi^{(s)}_{\alpha,(i_1,i_2),\pm}$ result if we put instead of x only the first term $AD^{-1}f_{i_1}+BD^{-1}f_{i_2}$. The estimation of the remainder terms is done in the same way as in Theorem 1. The substitution of (4.12) into (4.9) and (4.10) gives the closed system of equations for A, B, and μ:

$$\begin{aligned} &A\left[\lambda^{(s)}_{\alpha,i_1}+g_{i_1 i_1}+h_{i_1 i_1}+\sum_{j\neq i_1,i_2}y_{i_1 j}(g_{j i_1}+h_{j i_1})-\mu\right]\\ &\quad+B\left[g_{i_2 i_1}+h_{i_2 i_1}+\sum_{j\neq i_1,i_2}y_{i_2 j}(g_{j i_2}+h_{j i_2})\right]=0 \end{aligned} \tag{4.9$'$}$$

$$A\left[g_{i_1i_2}+h_{i_1i_2}+\sum_{j\neq i_1,i_2} y_{i_1j}(g_{ji_2}+h_{ji_2})\right]$$

$$+B\left[\lambda^{(s)}_{\alpha,i_2}+g_{i_2i_2}+h_{i_2i_2}+\sum_{j\neq i_1,i_2} y_{i_2j}(g_{ji_2}+h_{ji_2})-\mu\right]=0 \qquad (4.10')$$

Denote the matrix of the system (4.9′), (4.10′) by $S=\|s_{kl}\|$, $k, l=1, 2$. Then

$$\mu^{(s)}_{\alpha,(i_1,i_2),\pm}=\frac{s_{11}+s_{22}}{2}\pm\left[\frac{(s_{11}-s_{22})^2+4s_{12}\cdot s_{21}}{4}\right]^{1/2} \qquad (4.13)$$

From our estimations it follows easily that

$$s_{11}-s_{22}=\lambda^{(s)}_{\alpha,i_1}-\lambda^{(s)}_{\alpha,i_2}+\beta_1$$

$$|\beta_1|\leqslant q_s^{-3/2-2c(\ln\varepsilon)^{-1}}$$

$$|s_{12}|, |s_{21}|\leqslant q_s^{-3/2-2c(\ln\varepsilon)^{-1}}$$

Thus, if

$$|\lambda^{(s)}_{\alpha,i_1}-\lambda^{(s)}_{\alpha,i_2}|>q_s^{-3/2-3c(\ln\varepsilon)^{-1}}$$

then we have in fact a nonresonant situation and the formulas of Theorem 2 pass smoothly into the formulas of Theorem 1.

The form of exact e.f. ψ changes essentially if $\lambda^{(s)}_{\alpha,i_1}$ and $\lambda^{(s)}_{\alpha,i_2}$ are sufficiently close to each other and are far from the other $\lambda^{(s)}_{\alpha,j}$. Assume that

$$|\lambda^{(s)}_{\alpha,i_1}-\lambda^{(s)}_{\alpha,i_2}|\leqslant q_s^{-3/2-3c(\ln\varepsilon)^{-1}}$$

From (4.1) it follows that

$$\begin{aligned}(H^{(s)}_\varepsilon(\alpha)\,\varphi^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2})&\\ &=\lambda^{(s)}_{\alpha,i_1}(\varphi^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2})+(\Gamma^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2})+(h^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2})\\ &=g_{i_1i_2}+h_{i_1i_2}+\lambda^{(s)}_{\alpha,i_1}(\varphi^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2})+\delta g_{i_1,i_2}\\ (H^{(s)}_\varepsilon(\alpha)\,\varphi^{(s)}_{\alpha,i_2}, \varphi^{(s)}_{\alpha,i_1})&\\ &=g_{i_2i_1}+h_{i_2i_1}+\lambda^{(s)}_{\alpha,i_2}(\varphi^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2})+\delta g^{(1)}_{i_1,i_2}\end{aligned}$$

For the remainder terms $\delta g_{i_1,i_2}$ and $\delta g^{(1)}_{i_1,i_2}$ we have the estimations

$$|\delta g_{i_1i_2}|, |\delta g^{(1)}_{i_1i_2}|\leqslant q_s^{-2-c(\ln\varepsilon)^{-1}}$$

The self-adjointness of $H^{(s)}_\varepsilon(\alpha)$ implies

$$(H^{(s)}_\varepsilon(\alpha)\,\varphi^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2})=(H^{(s)}_\varepsilon(\alpha)\,\varphi^{(s)}_{\alpha,i_2}, \varphi^{(s)}_{\alpha,i_1})$$

i.e.,

$$|(g_{i_2 i_1}+h_{i_2 i_1})-(g_{i_1 i_2}+h_{i_1 i_2})| \leqslant q_s^{-2-2c(\ln \varepsilon)^{-1}}$$

Thus

$$s_{12}\cdot s_{21}=(g_{i_1 i_2}+h_{i_1 i_2})^2+\beta_2, \qquad |\beta_2| \leqslant q_s^{-3+3c(\ln \varepsilon)^{-1}}$$

The fact that both roots of (4.13) should be real follows from the self-adjointness of $H_\varepsilon^{(s)}(\alpha)$. ∎

We also need a slightly different version of Theorem 2. Assume that there are k functions $\varphi_{\alpha,i_1}^{(s)}, \varphi_{\alpha,i_2}^{(s)},..., \varphi_{\alpha,i_k}^{(s)}$ such that:

(a$_1$) $Z(\varphi_{\alpha,i_l}^{(s)})=Z(\varphi_{\alpha,i_1}^{(s)})+m_l$

(a$_2$) $m_2 \geqslant \text{const}\cdot\left(\ln\frac{1}{\varepsilon}\right)^{-1}\cdot s, \quad m_{l+1}\geqslant m_l^2, \quad l=2,..., k-1$

(a$_3$) $|\lambda_{\alpha,i_1}^{(s)}-\lambda_{\alpha,i_2}^{(s)}|$

$$\leqslant \frac{1}{s^{10}+[\text{dist}(Z(\varphi_{\alpha,i_1}^{(s)}), Z(\varphi_{\alpha,i_2}^{(s)}))]^{10}}+q_s^{-1+c(\ln \varepsilon)^{-1}}$$

(a$_4$) $|\lambda_{\alpha,i_k}^{(s)}-\lambda_{\alpha,i_l}^{(s)}| \geqslant q_s^{-1+c(\ln \varepsilon)^{-1}}, \quad k=1, 2, \quad l>2$

For other $\lambda_{\alpha,j}^{(s)}$, $j \neq i_1, i_2,..., i_k$, we have the nonresonant inequalities of Definition 1 with respect to all functions $\varphi_{\alpha,i_l}^{(s)}$, $1 \leqslant l \leqslant k$.

Theorem 3. The operator $H_\varepsilon^{(s)}(\alpha)$ has two exact e.f. $\psi_{\alpha,i_1}^{(s)}, \psi_{\alpha,i_2}^{(s)}$ and corresponding e.v. $\mu_{\alpha,i_1}^{(s)}, \mu_{\alpha,i_2}^{(s)}$ given by the same expressions and having the same properties as in Theorem 2.

Proof of this theorem goes in the same way as that of Theorem 2.

Remark. The proofs of Theorems 1–3 can be performed without any changes in the slightly more general situation that the intervals in which $\varphi_{\alpha,i}^{(s)}$ are different from zero depend on i, s in a different way provided that all other inequalities remain valid. For example, for the boundary points we may have only

$$\text{dist}(z_i, Z(\varphi_{\alpha,i}^{(s)})) \sim \frac{3 \ln q_s}{2 \ln 1/\varepsilon}$$

5. APPLICATION OF THE THEOREMS OF SECTION 4

Assume that $\bar{\varphi}_{\alpha,i}^{(s)}$ coincides with an exact e.f. $\psi_{\alpha,i}^{(s)}$ for those n where $\bar{\varphi}_{\alpha,i}^{(s)}$ is different from zero. Then $h_{\alpha,i}^{(s)}=0$ and in the nonresonant case our procedure gives the same $\psi_{\alpha,i}^{(s)}$. Thus we can write

$$\psi_{\alpha,i}^{(s)}=\varphi_{\alpha,i}^{(s)}+\sum_{j\neq i}\frac{(\Gamma_{\alpha,i}^{(s)}, \varphi_{\alpha,j}^{(s)})}{\lambda_{\alpha,i}^{(s)}-\lambda_{\alpha,j}^{(s)}}\varphi_{\alpha,j}^{(s)}+\delta\delta\psi_{\alpha,i}^{(s)} \tag{5.1}$$

Take z for which

$$\operatorname{dist}(z, Z(\varphi_{\alpha,i}^{(s)})) = \left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right]$$

$$\operatorname{dist}(z-1, Z(\varphi_{\alpha,i}^{(s)})) = \left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right] + 1$$

Then $\varphi_{\alpha,i}^{(s)}(z-1) = 0$ and

$$\psi_{\alpha,i}^{(s)}(z-1) = \sum_{j \neq i} \frac{(\Gamma_{\alpha,i}^{(s)}, \varphi_{\alpha,j}^{(s)})}{\lambda_{\alpha,i}^{(s)} - \lambda_{\alpha,j}^{(s)}} \varphi_{\alpha,j}^{(s)}(z-1) + \delta\delta\psi_{\alpha,i}^{(s)}(z-1) \tag{5.2}$$

Recalling the form of $\Gamma_{\alpha,i}^{(s)}$, we rewrite (5.2) as follows:

$$\begin{aligned}\psi_{\alpha,i}^{(s)}(z-1) = {} & -\varepsilon\psi_{\alpha,i}^{(s)}(z-1) \sum_{j \neq i} \frac{\varphi_{\alpha,j}^{(s)}(z)\, \varphi_{\alpha,j}^{(s)}(z-1)}{\lambda_{\alpha,i}^{(s)} - \lambda_{\alpha,j}^{(s)}} \\ & - \varepsilon\psi_{\alpha,i}^{(s)}(z) \sum_{j \neq i} \frac{[\varphi_{\alpha,j}^{(s)}(z-1)]^2}{\lambda_{\alpha,i}^{(s)} - \lambda_{\alpha,j}^{(s)}} + \delta\delta_1\psi_{\alpha,i}^{(s)}(z-1)\end{aligned} \tag{5.3}$$

(where the remainder term $\delta\delta_1\psi_{\alpha,j}^{(s)}$ satisfies the same inequalities as $\delta\delta\psi_{\alpha,j}^{(s)}$). Let us introduce the truncated Green's function

$$G_t^{(s)}(x, y; \lambda, \alpha) = \sum_j \frac{\varphi_{\alpha,j}^{(s)}(x)\, \varphi_{\alpha,j}^{(s)}(y)}{\lambda - \lambda_{\alpha,j}^{(s)}}$$

Then from (5.3) we get

$$\begin{aligned}\psi_{\alpha,i}^{(s)}(z-1) = {} & \frac{-\varepsilon G_t^{(s)}(z-1, z-1; \lambda_{\alpha,i}^{(s)}, \alpha)}{1 + \varepsilon G_t^{(s)}(z, z-1; \lambda_{\alpha,i}^{(s)}, \alpha)} \psi_{\alpha,i}^{(s)}(z) \\ & + \frac{\delta\delta_1\psi_{\alpha,i}^{(s)}(z-1)}{1 + \varepsilon G_t^{(s)}(z, z-1; \lambda_{\alpha,i}^{(s)}, \alpha)}\end{aligned} \tag{5.4}$$

Put $U^{(s)}(z; \lambda, \alpha) = -\varepsilon G_t^{(s)}(z-1, z-1; \lambda, \alpha)(1 + \varepsilon G_t^{(s)}(z, z-1; \lambda, \alpha)$. We have from (5.4)

$$\psi_{\alpha,i}^{(s)}(z-1) = U^{(s)}(z; \lambda_{\alpha,i}^{(s)}, \alpha)\, \psi_{\alpha,i}^{(s)}(z) + \frac{\delta\delta_1\psi_{\alpha,i}^{(s)}(z-1)}{1 + \varepsilon G_t^{(s)}(z, z-1; \lambda_{\alpha,i}^{(s)}, \alpha)} \tag{5.5}$$

This formula has a peculiar meaning. It shows that up to corresponding terms the value $\psi_{\alpha,i}^{(s)}(z-1)$ is obtained from $\psi_{\alpha,i}^{(s)}(z)$ by multiplication by a factor that depends primarily on e.f. or almost e.f. concentrated near the point z, i.e., this factor is a function of the potential in the vicinity of z. A similar representation is valid in resonant cases of Theorems 2 and 3.

Now let us take $n=z$ in (5.1). Then $\psi_{\alpha,i}^{(s)}(z)=\varphi_{\alpha,i}^{(s)}(z)$ and

$$-\varepsilon\psi_{\alpha,i}^{(s)}(z-1)\,G_t^{(s)}(z,z;\lambda_{\alpha,i}^{(s)},\alpha)-\varepsilon\psi_{\alpha,i}^{(s)}(z)\times G_t^{(s)}(z-1,z;\lambda_{\alpha,i}^{(s)},\alpha)+\delta\delta_2\psi_{\alpha,j}^{(s)}(z)=0 \tag{5.6}$$

This gives another representation for $\psi_{\alpha,i}^{(s)}(z-1)$:

$$\psi_{\alpha,i}^{(s)}(z-1)=-\psi_{\alpha,i}^{(s)}(z)\frac{G_t^{(s)}(z-1,z;\lambda_{\alpha,i}^{(s)},\alpha)}{\varepsilon G_t^{(s)}(z,z;\lambda_{\alpha,i}^{(s)},\alpha)}+\frac{\delta\delta_2\psi_{\alpha,j}^{(s)}(z)}{\varepsilon G_t^{(s)}(z,z;\lambda_{\alpha,i}^{(s)},\alpha)} \tag{5.7}$$

There is some visible difference between (5.4), (5.5), and (5.7). In (5.5) the function $U(z;\lambda_{\alpha,i}^{(s)},\alpha)$ is proportional to ε. However, in (5.7), $G_t^{(s)}(z,z;\lambda_{\alpha,i}^{(s)},\alpha)$ takes values of order of unity for typical α, while $G_t^{(s)}(z-1,z;\lambda_{\alpha,i}^{(s)},\alpha)$ typically takes values of order of ε. Thus, ε is also present implicitly in (5.7).

In estimations of scalar products $(\varphi_{\alpha,i}^{(s)},\varphi_{\alpha,j}^{(s)})$ we need other applications of the Theorems of Section 4. Assume that for the operator $H_\varepsilon^{(s)}(\alpha)$ the sequence $\{\bar{\varphi}_{\alpha,i}^{(s)}\}$ is given. Consider now the operator $H_\varepsilon^{(s)}(\alpha')$ with

$$|\alpha-\alpha'|\leqslant q_s^{-2-c(\ln\varepsilon)^{-1}}$$

Then the system of functions $\{\bar{\varphi}_{\alpha,i}^{(s)}\}$ satisfies all needed assumptions with respect to $H_\varepsilon^{(s)}(\alpha')$ and we can use it for the construction of e.f. of $H_\varepsilon^{(s)}(\alpha')$. It is essential that

$$\|\bar{\varphi}_{\alpha,i}^{(s)}-\bar{\varphi}_{\alpha',i}^{(s)}\|\leqslant q_s^{-2-2c(\ln\varepsilon)^{-1}}$$

where $\bar{\varphi}_{\alpha',i}^{(s)}$ is obtained from the exact e.f. by the truncation to a suitable neighborhood of the e.s.

6. AN INDUCTIVE CONSTRUCTION OF NONRESONANT EIGENFUNCTIONS OF $H_\varepsilon^{(s+1)}(\alpha)$

Let us consider the operator $H_\varepsilon^{(s+1)}(\alpha)$. Using $\Phi^{(s)}(0)$, we shall construct a basis in which $H_\varepsilon^{(s+1)}(\alpha)$ is almost diagonal. This should be done with some caution because $\Phi^{(s)}(\alpha)$ is a discontinuous function of α. The simplest way is the following. We consider $T^{-t}\Phi^{(s)}(\alpha+t\omega_{s+1})$ for each t, $0\leqslant t<q_{s+1}$, where T is the shift to the left. Each $\psi\in T^{-t}\Phi^{(s)}(\alpha+t\omega_{s+1})$

has its e.s. starting with t and lying to the right of t. We would like to take as the new basis the union

$$\bigcup_{0\leqslant t<q_{s+1}} T^{-t}\Phi^{(s)}(\alpha+t\omega_{s+1})$$

However, some complications may appear. Indeed, assume that $\alpha\in\Delta_{l,1}^{(s)}$ or $\Delta_{l,4}^{(s)}$ and there are two resonant e.f. $\psi_{\alpha,i_1}^{(s)}$ and $\psi_{\alpha,i_2}^{(s)}$ with the same e.s. $Z=Z'\cup T^{-m}Z'$. As was already explained, for $\alpha'=\alpha+m\omega_s$ there are no corresponding e.f. belonging to $\Phi^{(s)}(\alpha')$.

It might happen that, due to the difference between ω_s and ω_{s+1}, we can have an extra function $\psi^{(s)}\in\Phi^{(s)}(\alpha+m\omega_{s+1})$ which in fact is very close to one of the functions $\psi_{\alpha,i_1}^{(s)}$, $\psi_{\alpha,i_2}^{(s)}$. Certainly this is possible only for α very close to the boundary of $\Delta_{l,1}^{(s)}$ or $\Delta_{l,4}^{(s)}$; more exactly, their distance to the boundaries of these intervals should not be more than const $\cdot\, s^2q_s^{-2}$. In order to avoid this doubling, we proceed as follows: if $\alpha+m\omega_{s+1}\in\Delta_{l,1}^{(s)}$ or $\Delta_{l,4}^{(s)}$ for some m, $0\leqslant m<q_{s+1}$, and some l and the e.s. of the corresponding e.f. or almost e.f. is $Z'\cup T^{-m}Z'$ for a finite subset $Z'\subset\mathbb{Z}^1$ we take both e.f. or almost e.f. belonging to $\Phi^{(s)}(\alpha+m\omega_{s+1})$ and do not take the corresponding e.f. or almost e.f. of $\Phi^{(s)}(\alpha+m\omega_{s+1})$. Thus, the whole set of our functions is contained in

$$\bigcup_{0\leqslant m<q_s} T^{-m}\Phi^{(s)}(\alpha+m\omega_{s+1})$$

but we avoid in this way the functions that are not almost orthogonal to each other. Now the index i labels all selected functions ψ, i.e., $\psi_{\alpha,i}^{(s)}\in T^{-m}\Phi^{(s)}(\alpha+m\omega_{s+1})$. Denote also $m=m_i$, $\alpha_i=\alpha+m_i\omega_{s+1}$. Thus, $T^{m_i}\psi_{\alpha,i}^{(s)}$ is an e.f. or almost e.f. of $H_\varepsilon^{(s)}(\alpha+m_i\omega_{s+1})$. In all further considerations the new phenomenon is the dependence of α_i on i. For each $\psi_{\alpha,i}^{(s)}$ we put

$$\bar\varphi_{\alpha,i}^{(s+1)}(n)=\psi_{\alpha,i}^{(s)}(n),\qquad \text{if}\quad \operatorname{dist}(n, Z(\psi_{\alpha,i}^{(s)}))\leqslant\left[\frac{3\ln q_s}{2\ln 1/\varepsilon}\right]+2$$

$$\bar\varphi_{\alpha,i}^{(s+1)}(n)=0,\qquad \text{for other } n$$

$Z(\bar\varphi_{\alpha,i}^{(s+1)})=Z(\psi_{\alpha,i}^{(s)})$. The approximate e.v. $\lambda_{\alpha,i}^{(s+1)}$ is equal to the corresponding value of $\Lambda^{(s)}(\alpha+m_i\omega_{s+1})$. The validity of (a_4) in Section 4 follows from the definition of an almost e.v. (see §IV in Section 3 and the next section) and from the inequality $q_{s+1}/q_s\leqslant$ const $\cdot\,(s+1)^2$. We may also assume that the inductive process starts with a sufficiently large s and the constant C of §IV in Section 3 does not depend on ε. As in Section 4 we pass to the functions $\varphi_{\alpha,i}^{(s+1)}$.

Lemma 1. The set of functions $\{\varphi_{\alpha,i}^{(s+1)}\}$ is a basis in the space of all periodic sequences of the period q_{s+1}. It satisfies (a_3) of Section 4.

Proof of the lemma is given in Appendix B. In this section we consider a nonresonant case. Theorem 1 of Section 4 gives the existence of an exact e.f. $\psi_{\alpha,i}^{(s+1)}$ of $H_\varepsilon^{(s+1)}(\alpha)$, which can be written in the form

$$\psi_{\alpha,i}^{(s+1)} = \varphi_{\alpha,i}^{(s+1)} + \sum_j \frac{(\Gamma_{\alpha,i}^{(s+1)} + h_{\alpha,i}^{(s+1)}, \varphi_{\alpha,j}^{(s+1)})}{\lambda_{\alpha,i}^{(s)} - \lambda_{\alpha,j}^{(s)}} \varphi_{\alpha,j}^{(s+1)} + \delta\delta\psi_{\alpha,i}^{(s+1)} \quad (6.1)$$

The last term will be treated systematically as a remainder term.

The sum in (6.1) is taken over such j that

$$\operatorname{dist}(Z(\varphi_{\alpha,j}^{(s+1)}), Z(\varphi_{\alpha,i}^{(s+1)})) \leqslant 2\left[\frac{3\ln q_s}{2\ln 1/\varepsilon}\right]$$

As in Section 5, take z for which

$$\operatorname{dist}(z, Z(\varphi_{\alpha,i}^{(s+1)})) = \left[\frac{3\ln q_s}{2\ln 1/\varepsilon}\right]$$

$$\operatorname{dist}(z-1, Z(\varphi_{\alpha,i}^{(s+1)})) = \left[\frac{3\ln q_s}{2\ln 1/\varepsilon}\right] + 1$$

Then $\varphi_{\alpha,i}^{(s+1)}(z-1) = 0$ and from (6.1)

$$\begin{aligned} \psi_{\alpha,i}^{(s+1)}(z-1) = & -\varepsilon\psi_{\alpha,i}^{(s)}(z-1)\, \bar{G}_t^{(s+1)}(z-1, z; \lambda_{\alpha,i}^{(s)}, \alpha) \\ & -\varepsilon\psi_{\alpha,i}^{(s)}(z)\, \bar{G}_t^{(s+1)}(z-1, z-1; \lambda_{\alpha,i}^{(s)}, \alpha) + \delta\delta_3\psi_{\alpha,i}^{(s+1)} \end{aligned} \quad (6.2)$$

Here $\bar{G}_t^{(s+1)}(x, y; \lambda, \alpha)$ are truncated Green's functions constructed with the help of the functions $\varphi_{\alpha,j}^{(s+1)}$.

The difference between (6.2) and the formulas of Section 5 is due to the fact that in (6.2) e.f. or almost e.f. of $H_\varepsilon^{(s)}(\alpha)$ with different α enter. However, this dependence also will be considered as a small correction. We write

$$\psi_{\alpha,i}^{(s+1)}(z-1) = U^{(s)}(z; \lambda_{\alpha,i}^{(s)}, \alpha_i)\, \psi_{\alpha,i}^{(s)}(z) + \delta\delta_4\psi_{\alpha,i}^{(s+1)}(z-1) \quad (6.3)$$

Normally $U^{(s)}(z; \lambda_{\alpha,i}^{(s)}, \alpha)$ takes values of the order of unity and the correction is not more than $q_s^{-2-c(\ln\varepsilon)^{-1}}$. If $\psi_{\alpha,i}^{(s)}(z)$ is different from zero, we can rewrite (6.3) as follows:

$$\psi_{\alpha,i}^{(s+1)}(z-1) = \psi_{\alpha,i}^{(s)}(z)[U^{(s)}(z; \lambda_{\alpha,i}^{(s)}, \alpha_i) + \delta\delta_5\psi_{\alpha,i}^{(s+1)}(z-1)] \quad (6.4)$$

This formula is quite analogous to (5.5). We can now use (5.3)–(5.5) in order to get the values of $\psi_{\alpha,i}^{(s+1)}(z)$ for other z with

$$\operatorname{dist}(z, Z(\psi_{\alpha,i}^{(s+1)})) \sim \left[\frac{3}{2}\frac{\ln q_s}{\ln 1/\varepsilon}\right]$$

Let us consider also (6.1) for the same z as in (6.4). Then

$$\begin{aligned}\psi_{\alpha,i}^{(s+1)}(z) &= \psi_{\alpha_i,i}^{(s)}(z) - \varepsilon\psi_{\alpha_i,i}^{(s)}(z-1)\,\bar{G}_t^{(s+1)}(z, z; \lambda_{\alpha_i,i}^{(s)}, \alpha)\\ &\quad - \varepsilon\psi_{\alpha_i,i}^{(s)}(z)\,\bar{G}_t^{(s+1)}(z-1, z; \lambda_{\alpha_i,i}^{(s)}, \alpha) + \delta\delta_6\psi_{\alpha,i}^{(s+1)}(z)\\ &= \psi_{\alpha_i,i}^{(s)}(z) - \varepsilon\psi_{\alpha_i,i}^{(s)}(z-1)\,G_t^{(s)}(z, z; \lambda_{\alpha_i,i}^{(s)}, \alpha)\\ &\quad - \varepsilon\psi_{\alpha_i,i}^{(s)}(z)\,G_t^{(s)}(z-1, z; \lambda_{\alpha_i,i}^{(s)}, \alpha) + \delta\delta_7\psi_{\alpha,i}^{(s+1)}(z)\end{aligned}$$

In the remainder terms $\delta\delta_6\psi_{\alpha,i}^{(s+1)}$ and $\delta\delta_7\psi_{\alpha,i}^{(s+1)}$ we included the errors arising from $h_{\alpha,i}^{(s)}$ and the differences between $\bar{G}_t^{(s+1)}$ and $G_t^{(s)}$. Using (5.6), we have

$$\psi_{\alpha,i}^{(s+1)}(z) = \psi_{\alpha,i}^{(s)}(z) + \delta\delta_8\psi_{\alpha,i}^{(s+1)}(z) \tag{6.5}$$

Lemma 2:

$$|\delta\delta_8\psi_{\alpha,i}^{(s+1)}(z)| \leqslant q_s^{-2-c(\ln\varepsilon)^{-1}}$$

Proof of this lemma is straightforward. The formula (6.5) shows that $\psi_{\alpha,i}^{(s+1)}(z)$ differs from $\psi_{\alpha,i}^{(s)}(z)$ by a small correction. If

$$|\psi_{\alpha,i}^{(s)}(z)| \geqslant q_s^{-3/2+c(\ln\varepsilon)^{-1}}$$

then

$$\psi_{\alpha,i}^{(s+1)}(z) = \psi_{\alpha,i}^{(s)}(z)[1 + \delta\delta_9\psi_{\alpha,i}^{(s)}(z)] \tag{6.6}$$

Similar arguments work at points n where

$$\operatorname{dist}(n, Z(\psi_{\alpha,i}^{(s)}) \leqslant \operatorname{dist}(z, Z(\psi_{\alpha,i}^{(s)}))$$

They mean that the corrections to e.f. at points n within the considered neighborhoods of the e.s. are of order of $q_s^{-2-c(\ln\varepsilon)^{-1}}$ and thus the values of e.f. converge exponentially rapidly to limits.

7. A MECHANISM OF DECAY OF EIGENFUNCTIONS

In this section we shall derive the exponential decay of the e.f. $\psi_{\alpha,i}^{(s+1)}$. In doing this we shall also get estimations from below for e.f. valid at most points. Assume that for all t, $s_1 \leqslant t \leqslant s_2$, we have a sequence of e.f. $\psi_{\alpha,i}^{(t)}$ and corresponding e.v. $\lambda_{\alpha,i}^{(t)}$, $s_2 - s_1 \sim \rho s$, ρ a constant. In fact, i depends on t, but we do not incorporate that now. In this section we consider a non-resonant case, i.e., we assume that all $\psi_{\alpha,i}^{(t)}$ are constructed with the help of Theorem 1 of Section 4. In fact, the difference between the resonant and nonresonant cases is not important here.

In the interval $[s_1, s_2]$ take those $t^{(l)} \in [s_1, s_2]$, $1 \leqslant l \leqslant r$, where

$$y^{(l)} = \left[\frac{3 \ln q_t(l)}{2 \ln 1/\varepsilon}\right] > y^{(l-1)}$$

and consider the points $z^{(l)}$, $1 \leqslant l \leqslant r$, where $z^{(l+1)} < z^{(l)}$ and $\operatorname{dist}(z^{(l)}, Z(\psi_{\alpha,i}^{(s_1)})) = y^{(l)}$, $z_1 = z^{(1)}$, $z_2 = z^{(l)}$. Assume that $s_1 = t^{(1)}$, $s_2 = t^{(r)}$. We have

$$\begin{aligned}\psi_{\alpha,i}^{(s_2)}(z_2) = {} & \prod_{l=1}^{r-1} \frac{\psi_{\alpha,i}^{(t^{(l)})}(z^{(l)})}{\psi_{\alpha,i}^{(t^{(l-1)})}(z^{(l)}+1)} \\ & \times \prod_{z^{(l+1)} \leqslant z < z^{(l)}} \frac{\psi_{\alpha,i}^{(t^{(l)})}(z)}{\psi_{\alpha,i}^{(t^{(l)})}(z+1)}\, \psi_{\alpha,i}^{(s_1)}(z_1)\end{aligned} \tag{7.1}$$

Some danger comes from factors that are too large or too small. If, for example,

$$|\psi_{\alpha,i}^{(t^{(l)})}(z-1)/\psi_{\alpha,i}^{(t^{(l)})}(z)| \leqslant \tfrac{1}{2}\varepsilon[2 + \max_{\alpha} |V(\alpha)|]^{-1}$$

then from the equation for the e.f. written at $z-1$ we have

$$\begin{aligned}|\psi_{\alpha,i}^{(t^{(l)})}(z-2)/\psi_{\alpha,i}^{(t^{(l)})}(z)| = {} & |\psi_{\alpha,i}^{(t^{(l)})}(z-2)/\psi_{\alpha,i}^{(t^{(l)})}(z-1)| \\ & \times |\psi_{\alpha,i}^{(t^{(l)})}(z-1)/\psi_{\alpha,i}^{(t^{(l)})}(z)|\end{aligned}$$

and

$$\tfrac{1}{4} \leqslant |\psi_{\alpha,i}^{(t^{(l)})}(z-2)/\psi_{\alpha,i}^{(t^{(l)})}(z)| \leqslant 4 \tag{7.2}$$

The same inequality is valid for

$$\psi_{\alpha,i}^{(t^{(l)})}(z^{(l)})/\psi_{\alpha,i}^{(t^{(l-1)})}(z^{(l)}+1)$$

as will be shown later. Now we consider the product

$$I = \prod_{z_2 < z \leqslant z_1} U^{(s_1)}(z; \lambda_1, \alpha)$$

where we put $\lambda_1 = \lambda^{(s_1)}_{\alpha,i_1}$. Recall that (see Section 5)

$$U^{(s)}(z; \lambda, \alpha) = -\varepsilon \frac{G^{(s)}_t(z-1, z-1; \lambda, \alpha)}{1 + \varepsilon G^{(s)}_t(z, z-1; \lambda, \alpha)}$$

Lemma 4. Let $X \subset [z_2, z_1]$ be the set of those z for which

$$\left| G^{(s)}_t(z-1, z-1; \lambda, \alpha) - \frac{1}{\lambda - V((z-1)\,\omega_{s_1} + \alpha)} \right| \leqslant \text{const} \cdot \sqrt{\varepsilon}$$

$$|G^{(s)}_t(z, z-1; \lambda, \alpha)| \leqslant \text{const}$$

Then

$$\text{card}(X) \geqslant (1 - \text{const} \cdot \varepsilon^{1/4})(z_1 - z_2)$$

Before giving the proof of the lemma, we shall derive from it the exponential decay of e.f. In our situation we can apply Theorem 1 and (5.4). This yields

$$\begin{aligned}
\psi^{(s_1)}_{\alpha,i}(z_2) &= \prod_{z_2 \leqslant z < z_1} \frac{\psi^{(s_1)}_{\alpha,i}(z)}{\psi^{(s_1)}_{\alpha,i}(z+1)} \psi^{(s_1)}_{\alpha,i}(z_1) \\
&= \prod_{z \in [z_2, z_1] \cap X} \frac{\psi^{(s_1)}_{\alpha,i}(z)}{\psi^{(s_1)}_{\alpha,i}(z+1)} \\
&\quad \times \prod_{z \in [z_2, z_1] \setminus X} \frac{\psi^{(s_1)}_{\alpha,i}(z)}{\psi^{(s_1)}_{\alpha,i}(z+1)} \psi^{(s_1)}_{\alpha,i}(z_1) \\
&= \Pi_1 \cdot \Pi_2 \cdot \psi^{(s_1)}_{\alpha,i}(z_1)
\end{aligned}$$

By definition, we include in Π_2 the terms where

$$\left| \frac{\psi^{(s_1)}_{\alpha,i}(z)}{\psi^{(s_1)}_{\alpha,i}(z+1)} \right| \leqslant \frac{3}{4} \left[1 + \max_{\alpha} |V(\alpha)|\right]^{-1}$$

and the next ones. Remark that if

$$\left| \frac{\psi^{(s_1)}_{\alpha,i}(z)}{\psi^{(s_1)}_{\alpha,i}(z+1)} \right| \geqslant \frac{8}{\varepsilon} \left[1 + \max_{\alpha} |V(\alpha)|\right]$$

then from the equality for the e.f. with the e.v. E

$$\begin{aligned}
&\left| \frac{\psi^{(s_1)}_{\alpha,i}(z+2)}{\psi^{(s_1)}_{\alpha,i}(z+1)} \right| \\
&\quad = \left| \frac{\psi^{(s_1)}_{\alpha,i}(z+2)}{\psi^{(s_1)}_{\alpha,i}(z)} \right| \cdot \left| \frac{\psi^{(s_1)}_{\alpha,i}(z)}{\psi^{(s_1)}_{\alpha,i}(z+1)} \right|
\end{aligned}$$

$$\leqslant \left[1 + \frac{\max_\alpha |V(\alpha)| + |E|}{\varepsilon} \frac{\varepsilon}{8[1+\max_\alpha |V(\alpha)|]}\right] \frac{\varepsilon}{8}$$

$$\times [1 + \max_\alpha |V(\alpha)|]^{-1}$$

$$\leqslant \frac{5\varepsilon[1+\max_\alpha |V(\alpha)|]^{-1}}{32}$$

$$\leqslant \frac{\varepsilon}{4} [1 + \max_\alpha |V(\alpha)|]^{-1}$$

i.e., these terms enter into Π_2. Therefore, for z belonging to Π_1 we have

$$\frac{\varepsilon}{4[1+\max_\alpha |V(\alpha)|]} \leqslant \left| \frac{\psi_{\alpha,i}^{(s_1)}(z)}{\psi_{\alpha,i}^{(s_1)}(z+1)} \right| \leqslant \frac{4[1+\max_\alpha |V(\alpha)|]}{\varepsilon}$$

and for the whole product Π_2 we have a trivial estimation

$$4^{z_2 - z_1} \left\{ \frac{4[1+\max_\alpha |V(\alpha)|]}{\varepsilon} \right\}^{(z_2 - z_1)\cdot \text{const} \cdot \varepsilon}$$

$$\leqslant |\Pi_2|$$

$$\leqslant 4^{z_1 - z_2} \left\{ \frac{4[1+\max_\alpha |V(\alpha)|]}{\varepsilon} \right\}^{(z_1 - z_2)\cdot \text{const} \cdot \varepsilon} \tag{7.3}$$

which follows from (7.2) if we unite small factors in pairs and, following them, large factors.

From (7.3) we see that for $z+1 \in X$ the value

$$|\psi_{\alpha,i}^{(s_1)}(z)| \geqslant q_s^{-3/2 + \text{const}(\ln \varepsilon)^{-1}}$$

and from (5.4) for such $z+1$

$$\left| \frac{\psi_{\alpha,i}^{(s_1)}(z)}{\psi_{\alpha,i}^{(s_1)}(z+1)} + \frac{\varepsilon}{\lambda_1 - V(z\omega_s + \alpha_1)} \right| \leqslant \text{const} \cdot \varepsilon^2$$

Thus, assuming that $z_2 \in X$, we have the estimations from both sides for $\psi_{\alpha,i}^{(s_1)}(z_2)$:

$$\left(\frac{1}{4}\varepsilon\right)^{z_1 - z_2} \exp\left[-\text{const} \cdot \frac{\varepsilon}{\ln 1/\varepsilon}(z_1 - z_2)\right]$$

$$\leqslant \left| \frac{\psi_{\alpha,i}^{(s_1)}(z_2)}{\psi_{\alpha,i}^{(s_1)}(z_1)} \right|$$

$$\leqslant (4\varepsilon)^{z_1 - z_2} \exp\left[\text{const} \cdot \frac{\varepsilon^{1/4}}{\ln 1/\varepsilon}(z_1 - z_2)\right] \tag{7.4}$$

The eigenfunctions $\psi_{\alpha,i}^{(t)}(z)$ with variable t differ from $\psi_{\alpha,i}^{(s_1)}(z)$ by terms not more than $q_t^{-2-c(\ln\varepsilon)^{-1}}$. This follows again from Theorem 1 and Sections 5 and 6. Thus, we get the exponential decay of the value of $\psi_{\alpha,i}^{(s_2)}(z_2)$ and in the same way the exponential decay in other points.

Proof of Lemma 4. From the definitions,

$$U^{(s_1)}(z; \lambda_1, \alpha) = U^{(s_1)}(z_1, \lambda_1, \alpha + \omega_s(z - z_1))$$

Thus, we have to show that the needed estimations hold for most z. We shall use some information concerning the r.z. (see Sections 3, 4, and 8; in fact, Section 8 is independent of Section 7 and the reader interested in details can read Section 8 before this proof). Each r.z. is an interval $\Delta_{l,i}^{(s)}$, $i = 1, 4$, on the α axis. Let us denote its range and width by r and m. We have a chain of intervals

$$\Delta_{l_1,i_1}^{(1)} \supset \Delta_{l_2,i_2}^{(s^{(2)})} \supset \cdots \supset \Delta_{l_{m-1},i_{m-1}}^{(s^{(m-1)})} \supset \Delta_{l_m,i_m}^{(s^{(m)})} = \Delta_{l,i}^{(s_1)}$$

where $s_1 \geqslant s^{(m)} > s^{(m-1)} > \cdots > s^{(2)} > s^{(1)} = Fs$ are the numbers of steps of our procedure where the e.s. of the corresponding functions changes. The next interval $\Delta_{l_k,i_k}^{(s^{(k)})}$ is contained in the intersection

$$\Delta_{l_{k-1},i_{k-1}}^{(s^{(k-1)})} \cap R_{\omega_{s^{(k-1)}}}^{m_k}(\Delta_{l_{k-1},i_{k-1}}^{(s^{(k-1)})})$$

Thus,

$$m_k \geqslant |\Delta_{l_{k-1},i_{k-1}}^{(s^{(k-1)})}|^{-1/4}$$

as can be easily derived from the properties of ω; here $|\cdot|$ means the length. Also it follows from the construction that

$$|\Delta_{l_k,i_k}^{(s^{(k)})}| \leqslant (\text{const} \cdot \varepsilon)^m$$

and thus

$$|\Delta_{l,i}^{(s_1)}| \leqslant (\text{const} \cdot \varepsilon)^{\text{diam } Z}$$

$$\text{card}\{\Delta_{l,i}^{(s_1)} | \text{ diam } Z = t\} \leqslant \exp(\text{const} \cdot t^{\gamma})$$

for some $\gamma < 1$. In fact, the last inequality can be given more exactly.

Let us call the r.z. $\Delta_{l,i}^{(s_1)}$ large if $|\Delta_{l,i}^{(s_1)}| \geqslant \varepsilon s^{-\mathscr{D}}$, where $\mathscr{D}$ will be specified later. Other r.z. are called small. Small r.z. have the following important property:

A. For each small r.z. $\Delta_{l,i}^{(s_1)}$ the number of z among $z_2 \leqslant z \leqslant z_1$ for which $\alpha + \omega_s(z - z_1) \in \Delta_{l,i}^{(s)}$ is not more than 1.

The large r.z. have the following property:

B. The number of z among $z_2 < z \leqslant z_1$ for which

$$\alpha + \omega_s(z - z_1) \in \bigcup_{|l| \leqslant 2 \operatorname{diam} Z} R^l_{\omega_s} \Delta^{(s_1)}_{l,i}$$

is not more than

$$2(\operatorname{diam} Z)[(z_1 - z_2)\, |\Delta^{(s_1)}_{l,i}| + \ln^3 s]$$

Here Z is the e.s. of the corresponding e.f. for $\alpha \in \Delta^{(s_1)}_{l,i}$.

Property B and previous estimations yield:

C. The number of z among $z_2 < z \leqslant z_1$ for which

$$\alpha + \omega_s(z - z_1) \in \bigcup_{|l| \leqslant 2 \operatorname{diam} Z} R^l_{\omega_s} \Delta^{(s_1)}_{l,i}$$

for at least one large r.z. is not more than

$$\text{const} \cdot \varepsilon(z_1 - z_2) + \exp\{\text{const} \cdot (\ln s_1)^{\gamma_1}\}$$

for some $\gamma_1 < 1$.

Indeed, this number is estimated from above by the sum

$$(z_1 - z_2) \sum{}' \, |\Delta^{(s_1)}_{l,i}| + N \ln^3 s_1$$

where $\sum'$ is the sum over large r.z. and N is the total number of large r.z. From what was said above it follows that $N \leqslant \exp[(\ln s_1)^{\gamma_1}]$ for some $\gamma_1 < 1$ and $\sum' |\Delta^{(s_1)}_{l,i}| \leqslant \text{const} \cdot \varepsilon$.

Let us take z not satisfying property C. We have $U^{(s_1)}(z; \lambda_1, \alpha) = -\varepsilon u^{(s_1)}(z; \lambda_1, \alpha)$,

$$u^{(s_1)}(z; \lambda_1, \alpha) = G^{(s_1)}_t(z-1, z-1; \lambda_1, \alpha)[1 + \varepsilon G^{(s_1)}_t(z-1, z; \lambda_1, \alpha)]^{-1}$$

We shall investigate the first factor; the second one is treated in a similar way. Let us write

$$\begin{aligned} G^{(s_1)}_t&(z-1, z-1; \lambda_1, \alpha) \\ &= \sum_j \frac{[\varphi^{(s_1)}_{\alpha,j}(z-1)]^2}{\lambda_1 - \lambda^{(s_1)}_{\alpha,j}} \\ &= \sum_{r \geqslant 0} \; \sum_{j:\, \operatorname{dist}(Z(\varphi^{(s_1)}_{\alpha,j}), z-1) = r} \frac{[\varphi^{(s_1)}_{\alpha,j}(z-1)]^2}{\lambda_1 - \lambda^{(s_1)}_{\alpha,j}} \end{aligned} \tag{7.4}$$

First we consider the term with $r=0$. Our assumptions and definitions give that either α is outside all r.z. or it is contained in a small r.z. $\Delta_{l,i}^{(s_1)}$.

In the first case the value $r=0$ corresponds to only one $\varphi_{\alpha,j}^{(s_1)}$ for which $Z(\varphi_{\alpha,j}^{(s_1)})=z-1$ and thus $\varphi_{\alpha,j}^{(s_1)}(z-1)\geqslant 1-\text{const}\cdot\varepsilon$. Our procedure also shows that the shift in the e.v. from the unperturbed e.v. $V(z\omega_s+\alpha)$ is not more than $\text{const}\cdot\varepsilon$.

In the second case our procedure gives the existence of two resonant functions $\varphi_{\alpha,j_1}^{(s_1)}$ and $\varphi_{\alpha,j_2}^{(s_1)}$ whose e.s. has the form

$$Z=(z-1)\cup(z-1+m), \qquad m\geqslant \mathscr{D}\,\text{const}\cdot\left(\ln\frac{1}{\varepsilon}\right)^{-1}\ln s_1$$

These functions can be written in the form

$$\varphi_{\alpha,j_1}^{(s_1)}=a_{11}\psi_1+a_{12}\psi_2$$
$$\varphi_{\alpha,j_2}^{(s_1)}=a_{21}\psi_1+a_{22}\psi_2$$

where the matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

differs from an orthogonal matrix to an error that is not more than $(\text{const}\cdot\varepsilon)^m\leqslant s^{-\text{const}\cdot\mathscr{D}}$ and the functions ψ_1 and ψ_2 are concentrated near $(z-1)$ and $(z-1+m)$, respectively, in the sense that

$$\psi_1(z-1)\geqslant 1-\text{const}\cdot\varepsilon, \qquad |\psi_1(n)|\leqslant(\text{const}\cdot\varepsilon)^{\text{dist}(n,z-1)}$$
$$\psi_2(z-1+m)\geqslant 1-\text{const}\cdot\varepsilon$$
$$|\psi_2(n)|\leqslant(\text{const}\cdot\varepsilon)^{\text{dist}(n,z-1+m)}$$

From these estimations and the nonresonant condition it follows easily that

$$\frac{[\varphi_{\alpha,j_1}(z-1)]^2}{\lambda_1-\lambda_{\alpha,j_1}^{(s_1)}}+\frac{[\varphi_{\alpha,j_2}(z-1)]^2}{\lambda_1-\lambda_{\alpha,j_2}^{(s_1)}}$$
$$=\frac{a_{11}^2+a_{21}^2}{\lambda_1-\lambda_{\alpha,j_1}^{(s_1)}}\psi_1^2(z-1)+\text{an error}$$

and the error is not more than $s^{-\text{const}\cdot\mathscr{D}}$. Thus, again the value of the sum differs from $1/\{\lambda_1-V[(z-1)\,\omega_s+\alpha]\}$ to an error not more than $\text{const}\cdot\varepsilon$.

Our next step is to prove that the rest of the sum (7.4) with $r>0$ is relatively small. The main idea is to show that for typical $z-1$ the denominators decrease more slowly than the numerators, thus making all terms smaller and smaller.

To give more precise arguments, let us consider for each r those $(z-1)$ for which

$$\begin{aligned}&\min |\lambda_1 - \lambda^{(s_1)}_{\alpha,j}| \geqslant \sqrt{\varepsilon}\, r^{-10}\\ &j\colon\ \operatorname{dist}(Z(\varphi^{(s_1)}_{\alpha,j}), z-1) = r\end{aligned} \tag{7.5}$$

The same arguments as above show that in this case

$$\begin{aligned}J &= \left| \sum_{j:\, \operatorname{dist}(Z(\varphi^{(s_1)}_{\alpha,j}), z-1) = r} \frac{[\varphi^{(s_1)}_{\alpha,j}(z-1)]^2}{\lambda_1 - \lambda^{(s_1)}_{\alpha,j}} \right|\\ &\leqslant (\text{const}\cdot\varepsilon)^{r-1}\sqrt{\varepsilon}\cdot(r-1)^{10}\end{aligned}$$

Therefore $\sum_{r\geqslant 1} J_r \leqslant \text{const}\cdot\sqrt{\varepsilon}$. We have to investigate only the cardinality of $(z-1)$ for which (7.5) holds. Recall the structure of the multivalued function $\Lambda^{(s_1)}$. Its range is a union of intervals separated by f.z. The length of each interval is not more than

$$\text{const}\cdot q_{s_1}^{-1-c(\ln\varepsilon)^{-1}}$$

Introduce neighborhoods $O_r(\lambda_1)$ whose radii are contained between $\sqrt{\varepsilon}\, r^{-10}$ and $2\sqrt{\varepsilon}\, r^{-10}$ and whose endpoints are endpoints of f.z. For such neighborhoods, $(\Lambda^{(s)})^{-1}(O_r(\lambda_1)) = A_r(\lambda_1)$ is a union of two intervals. As follows from the inductive hypothesis of Section 3, $|A_r(\lambda_1)| \leqslant \text{const}\cdot\varepsilon^{1/4} r^{-5}$. If $z-1$ is such that

$$\alpha + t\omega_{s_1} \notin A_r(\lambda_1) \qquad \text{for all} \quad t,\ |t-(z-1)| \leqslant r \tag{7.6}$$

and $r \geqslant 1$, then (7.5) is valid. The cardinality of z, for which $\alpha + t\omega_s \in A_r(\lambda_1)$ for at least one t, $|t-(z-1)| \leqslant r$, is not more than $\text{const}\cdot\varepsilon^{1/4} r^{-4}(z_1 - z_2)$. Thus, provided that $r \leqslant (z_1 - z_2)^{3/4}$, the cardinality of $(z-1)$ for which (7.5) is violated for at least one t and $r \leqslant (z_1 - z_2)^{3/4}$ is more than $\text{const}\cdot \varepsilon^{1/4}(z_1 - z_2)$. If $r > (z_1 - z_2)^{3/4}$ and $\alpha + t\omega_s \in A_r(\lambda_1)$ for some t, $|t-(z-1)| \leqslant r$, then it would violate the nonresonance condition. ∎

The estimation of the product in (7.1) is the key to the proof of the total exponential decay of e.f. We decompose the whole interval $[z, \bar{z}]$ onto subintervals $[z_i, z_{i+1}]$, $z = z_0$, $z_{r+1} = \bar{z}$, $1 \leqslant i \leqslant r$, and z lies to the left of the e.s.; $\bar{z}$ is the left boundary of $Z(\psi^{(s)}_{\alpha,i_1})$. In each subinterval we include the dependence of the e.v. and the phase on the step of the procedure into the remainder term. The estimation of I in Lemma 4 shows that

$$\frac{\ln |I|}{z_1 - z_2} \sim \ln\varepsilon + \frac{1}{z_1 - z_2}\sum{}' \ln \frac{1}{|\lambda_1 - V[(z-1)\,\omega_s + \alpha]|} \tag{7.7}$$

where $\sum'$ means that the summation goes over z for which the absolute value of the corresponding logarithm is not more than $\ln(1/\varepsilon) + \text{const}$. The statement of Lemma 4 shows that the frequency of such z is large. Thus, the last term in (7.7) is bounded and for sufficiently small ε the main term is $\ln \varepsilon$. This gives the exponential decay at final points z of the considered neighborhoods. This character of decay is not changed in other points because the corrections are of the order of $q_s^{-2-c(\ln \varepsilon)^{-1}}$, as is explained in Section 5. Such smallness has no influence on the smallness of the main estimation of the e.f.

The formula (7.7) is close in spirit to the well-known Thouless formula for the Liapunov exponent.[32] In fact, (7.7) gives, in the main order in ε, the Thouless formula. Apparently the decay of e.f. is determined completely by the Thouless formula, but it depends on terms of higher order in ε, for which it is difficult to follow.

8. AN INDUCTIVE CONSTRUCTION AND EXPONENTIAL DECAY OF THE RESONANT EIGENFUNCTIONS

We use the same basis $\{\varphi_{\alpha,i}^{(s+1)}\}$ as in the beginning of Section 6 (see Lemma 1) and corresponding e.v. or almost e.v. $\lambda_{\alpha,i}^{(s+1)}$. We shall discuss in this section the following topics.

1. The behavior under the transition $s \to s+1$ of the already constructed f.z.
2. The appearance and the width of the new f.z.
3. The construction of small r.z. and corresponding almost e.f.
4. The exponential decay of the resonant e.f. and almost e.f.

Before doing this we introduce some notations. We consider for each t the points $\bar{\alpha} = \bar{\alpha}(t, l, i)$, $1 \leqslant i \leqslant 4$, where $\Lambda_{l,i}^{(s)}(\bar{\alpha}) = \Lambda_{l,i}^{(s)}(\bar{\alpha} + t\omega_{s+1})$ and their neighborhoods $O(\bar{\alpha})$ and $U(\bar{\alpha})$, $U(\bar{\alpha}) \subset O(\bar{\alpha})$, whose radii are equal, respectively, to $q_{s+1}^{-3/2}$ and $(s^5 + \frac{1}{2}t^5)^{-1}$. Let us show that the neighborhoods $U(\bar{\alpha})$ appearing for different t and fixed l, i do not intersect each other. Indeed, if for some $\bar{\alpha}_1, \bar{\alpha}_2$ we would have $U(\bar{\alpha}_1) \cap U(\bar{\alpha}_2) \neq \varnothing$, then $|\bar{\alpha}_1 - \bar{\alpha}_2| \leqslant 2q_{s+1}^{-3/2}$. Denoting by t_1, t_2 the corresponding shifts for $\bar{\alpha}_1, \bar{\alpha}_2$, we can write

$$|\Lambda_{l,i}^{(s)}(\bar{\alpha}_2 + m_2\omega_{s+1}) - \Lambda_{l,i}^{(s)}(\bar{\alpha}_1)| = |\Lambda_{l,i}^{(s)}(\bar{\alpha}_2) - \Lambda_{l,i}^{(s)}(\bar{\alpha}_1)| \leqslant \text{const} \cdot q_{s+1}^{-3/2}$$

Therefore, either

$$\operatorname{dist}(\bar{\alpha}_2 + t_2\omega_{s+1}, \bar{\alpha}_1) \leqslant \text{const} \cdot q_{s+1}^{-3/2}$$

or

$$\operatorname{dist}(\bar{\alpha}_2 + t_2\omega_{s+1}, \bar{\alpha}_1 + t_1\omega_{s+1}) \leqslant \text{const} \cdot q_{s+1}^{-3/2}$$

in view of the inductive assumptions §I.1 in Section 3. Thus, either

$$\operatorname{dist}(t_2\omega_{s+1}, \mathbb{Z}^1) \leqslant \text{const} \cdot q_{s+1}^{-3/2}$$

or

$$\operatorname{dist}((t_2 - t_1)\,\omega_{s+1}, \mathbb{Z}^1) \leqslant \text{const} \cdot q_{s+1}^{-3/2}$$

Both inequalities contradict properties of ω_{s+1} and this implies the desired result.

For each α and l, i take the maximal set $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_k$, $\bar{\alpha}_b = \bar{\alpha}_b(t_b, l, i)$, $1 \leqslant b \leqslant k$, such that $\alpha \in O(\bar{\alpha}_1) \cap O(\bar{\alpha}_2) \cap \cdots \cap O(\bar{\alpha}_k)$, $|O(\bar{\alpha}_b)| = 2(s^5 + \frac{1}{2}t_i)^{-5}$, $t_1 < t_2 < \cdots < t_k$. It follows easily from the construction (see also Section 9) that $t_{i+1} \geqslant t_i^2$. Thus, we can use Theorem 3 and the inequality $k \leqslant \text{const} \cdot \ln s$.

1. Behavior of already constructed f.z. Consider the case when $\lambda_{\alpha,i_1}^{(s+1)}$, $\lambda_{\alpha,i_2}^{(s+1)} \in \Lambda^{(s)}(\alpha + t\omega_{s+1})$ with the same t, i.e., the corresponding exact e.f. that generated $\varphi_{\alpha,i_1}^{(s+1)}$, $\varphi_{\alpha,i_2}^{(s+1)}$ were resonant e.f. Then Theorems 2–4 give the form of the exact e.f. for $H_\varepsilon^{(s+1)}(\alpha)$. Comparing the expressions for the e.v. written at the sth step à la Section 5 and the expression written at the $(s+1)$th step, which takes into account the perturbation terms $h_{ij}^{(s+1)}$, we immediately see that the error in the e.v. $\mu_{\alpha,i_1}^{(s+1)}$, $\mu_{\alpha,i_2}^{(s+1)}$ is not more than $q_{s+1}^{-2-2c(\ln\varepsilon)^{-1}}$. This shows also that if the width of f.z. is bigger than $q_s^{-2+\delta_1}$ and $\delta_1 \gg c(\ln 1/\varepsilon)^{-1}$, then the position and the size of the new f.z. change relatively little.

The e.s. of the new e.f. is not changed and the formulas for the values of e.f. at new points are the same, up to some remainder terms, as in the nonresonant cases. Thus, the mechanism for the exponential decay of e.f. investigated in Section 7 works in the same manner.

2. Appearence of the new f.z. According to our construction (III in Section 3), we define at the sth step small r.z. if $t \geqslant (2 - \delta_1) \ln q_s \cdot (\ln 1/\varepsilon)^{-1}$. In passing from $s \to s+1$ there might appear t for which

$$(2 - \delta_1) \ln q_s \left(\ln \frac{1}{\varepsilon}\right)^{-1} \leqslant t < (2 - \delta_1) \ln q_{s+1} \left(\ln \frac{1}{\varepsilon}\right)^{-1}$$

For these t take $\bar{\alpha} = \bar{\alpha}(t, l, i)$ and the neighborhood $O(\bar{\alpha})$. For $\alpha \in O(\bar{\alpha})$ we can use Theorems 2 and 3, which give the expressions for the new resonant e.f. and e.v. We have to investigate the width of the new f.z. and in particular to show that it is different from zero. This would mean that in our problem there are no potentials with a finite number of f.z., provided that ε is small enough.

The values of the exact e.v. are given by formula (4.1) or its analogs in Theorem 3. The function $s_{11}-s_{12}$ changes its sign with the change of α in view of our inductive assumptions. Thus, the width of the new f.z. is determined by the product $s_{12}\cdot s_{21}$. We showed in Section 4 that $s_{12}\cdot s_{21}=(g_{i_1i_2}+h_{i_1i_2})^2+\delta_2$, where $|\delta_2|\leqslant q_s^{-2-2c(\ln\varepsilon)^{-1}}$. Also, it was already estimated that $|h_{i_2i_1}|\leqslant q_s^{-2-c(\ln\varepsilon)^{-1}}$. We have

$$g_{i_1i_2}=(\Gamma^{(s)}_{\alpha,i_1},\varphi^{(s)}_{\alpha,i_2})+\delta'_2,\qquad |\delta'_2|\leqslant q_s^{-2-c(\ln\varepsilon)^{-1}}$$

Denote by z and $z+1$ the points lying between $Z(\varphi^{(s)}_{\alpha,i_1})$ and $Z(\varphi^{(s)}_{\alpha,i_2})$ where $\Gamma^{(s)}_{\alpha,i_1}$ is different from zero. Then

$$(\Gamma^{(s)}_{\alpha,i_1},\varphi^{(s)}_{\alpha,i_2})=-\varepsilon[\varphi^{(s)}_{\alpha,i_1}(z)\,\varphi^{(s)}_{\alpha,i_2}(z+1)+\varphi^{(s)}_{\alpha,i_1}(z+1)\,\varphi^{(s)}_{\alpha,i_2}(z)]\qquad(8.1)$$

Rewrite (8.1) using (6.4):

$$(\Gamma^{(s)}_{\alpha,i_1},\varphi^{(s)}_{\alpha,i_2})$$
$$=-\varepsilon\varphi^{(s)}_{\alpha,i_1}(z)\,\varphi^{(s)}_{\alpha,i_2}(z)[U^{(s)}(z+1;\lambda^{(s)}_{\alpha,i_2},\alpha)+U^{(s)}(z;\lambda^{(s)}_{\alpha,i_1},\alpha)]+\delta_3\qquad(8.2)$$

where δ_3 is again a remainder term. Now we shall use the remark made in Section 4. Namely, the boundary where we make the cutoff of $\varphi^{(s)}_{\alpha,i}$ is not rigid. We can replace z by any point z' in the interval

$$|z-z'|\leqslant o(1)\left(\ln\frac{1}{\varepsilon}\right)^{-1}s,\qquad o(1)\to 0\quad\text{as}\quad\varepsilon\to 0$$

and all estimations will remain valid. As was shown in Section 7, among these points z' a relatively dense set consists of points z' where

$$|\varphi^{(s)}_{\alpha,i_1}(z')|\geqslant q_s^{-3/2+c(\ln\varepsilon)^{-1}}$$
$$|\varphi^{(s)}_{\alpha,i_2}(z')|\geqslant q_s^{1/2-\delta_1+\ln c(\ln\varepsilon)}$$

Here one has to use the inequalities

$$\left|\operatorname{dist}(z',Z(\varphi^{(s)}_{\alpha,i_1}))+\frac{3}{2}\ln q_s\,(\ln\varepsilon)^{-1}\right|\leqslant\ln q_s\left(\ln\frac{1}{\varepsilon}\right)^{-1}o(1)$$

$$\left|\operatorname{dist}(z',Z(\varphi^{(s)}_{\alpha,i_2})+\left(\frac{1}{2}-\delta_1\right)\ln q_s\,(\ln\varepsilon)^{-1}\right|$$
$$\leqslant\ln q_s\left(\ln\frac{1}{\varepsilon}\right)^{-1}o(1),\qquad o(1)\to 0\quad\text{as}\quad\varepsilon\to 0$$

and the estimations from below of $\varphi^{(s)}_{\alpha,i_1}$, $\varphi^{(s)}_{\alpha,i_2}$, which were in fact derived in Section 7 and are also valid here because the formulas for the values of e.f. at new points are the same in the resonant and nonresonant cases. Also, it was shown in Section 7 that for a relatively dense set of z' for each function $U^{(s)}(z'+1; \lambda^{(s)}_{\alpha,i_2}, \alpha)$, $U^{(s)}(z'; \lambda^{(s)}_{\alpha,i_1}, \alpha)$ we have

$$0 < \text{const} \leqslant |U^{(s)}(z'+1; \lambda^{(s)}_{\alpha,i_2}, \alpha) + U^{(s)}(z; \lambda^{(s)}_{\alpha,i_1}, \alpha)| \leqslant \text{const}$$

Thus, if in (8.2) we take a typical z' for which all written estimations hold, then we get the estimation of the width of the new r.z., which is not less than $q_s^{-2+\delta_1+o(1)}$. For the e.f. $\psi^{(s+1)}_{\alpha,i}$ we put

$$Z(\psi^{(s+1)}_{\alpha,i}) = Z(\varphi^{(s)}_{\alpha,i_1}) \cup T' Z(\varphi^{(s)}_{\alpha,i_2})$$

3. Construction of small r.z. and corresponding almost e.f. For each t, l, i,

$$t \geqslant (2-\delta_1) \ln q_{s+1} \left(\ln \frac{1}{\varepsilon}\right)^{-1}$$

we consider $\bar{\alpha}(t, l, i)$ and take neighborhoods $O_1(\bar{\alpha})$ of radius $q_s^{-(2-\delta_1)-3m/2}$ where

$$\left(2-\delta_1+\frac{3}{2}m\right) \ln q_s \left(\ln \frac{1}{\varepsilon}\right)^{-1} \leqslant t < \left[2-\delta_1+\frac{3}{2}(m+1)\right] \ln q_s \left(\ln \frac{1}{\varepsilon}\right)^{-1}$$

These neighborhoods do not overlap for fixed l, i. We can use Theorem 2 or 3 of Section 4 for the construction of exact e.f. Now we shall perform an operation that can be called a cut of the e.f. Namely, the exact e.f. is written in the form

$$\begin{aligned}
\psi^{(s+1)}_{\alpha,(i_1,i_2),+} = A_+ &\left(\varphi^{(s)}_{\alpha,i_1} + \sum_{j \neq i_1,i_2} \frac{(\Gamma^{(s)}_{\alpha,i_1} + h^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,j})}{\lambda^{(s)}_{\alpha,i_1} - \lambda^{(s)}_{\alpha,j}} \varphi^{(s)}_{\alpha,j}\right) \\
&+ B_+ \left(\varphi^{(s)}_{\alpha,i_2} + \sum_{j \neq i_1,i_2} \frac{(\Gamma^{(s)}_{\alpha,i_2} + h^{(s)}_{\alpha,i_2}, \varphi^{(s)}_{\alpha,j})}{\lambda^{(s)}_{\alpha,i_2} - \lambda^{(s)}_{\alpha,j}} \varphi^{(s)}_{\alpha,j}\right) + \delta\delta\psi^{(s)}_{\alpha,(i_1,i_2)} \\
\psi^{(s+1)}_{\alpha,(i_1,i_2),-} = A_- &\left(\varphi^{(s)}_{\alpha,i_1} + \sum_{j \neq i_1,i_2} \frac{(\Gamma^{(s)}_{\alpha,i_1} + h^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,j})}{\lambda^{(s)}_{\alpha,i_1} - \lambda^{(s)}_{\alpha,j}} \varphi^{(s)}_{\alpha,j}\right) \\
&+ B_- \left(\varphi^{(s)}_{\alpha,i_2} + \sum_{j \neq i_1,i_2} \frac{(\Gamma^{(s)}_{\alpha,i_2} + h^{(s)}_{\alpha,i_2}, \varphi^{(s)}_{\alpha,j})}{\lambda^{(s)}_{\alpha,i_2} - \lambda^{(s)}_{\alpha,j}} \varphi^{(s)}_{\alpha,j}\right) + \delta\delta\psi^{(s)}_{\alpha,(i_1,i_2)}
\end{aligned}$$

where $\delta\delta\psi^{(s)}_{\alpha,(i_1,i_2),\pm}$ are the remainder terms. We take as almost e.f. the expressions

$$\varphi^{(s+1)}_{\alpha,i_1} = \varphi^{(s)}_{\alpha,i_1} + \sum_{j\neq i_1} \frac{(\Gamma^{(s)}_{\alpha,i_1} + h^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,j})}{\lambda^{(s)}_{\alpha,i_1} - \lambda^{(s)}_{\alpha,j}} \varphi^{(s)}_{\alpha,j}$$

$$= A_+ \psi^{(s+1)}_{\alpha,(i_1,i_2),+} + A_- \psi^{(s+1)}_{\alpha,(i_1,i_2),-} + \cdots$$

$$\varphi^{(s+1)}_{\alpha,i_2} = \varphi^{(s)}_{\alpha,i_2} + \sum_{j\neq i_2} \frac{(\Gamma^{(s)}_{\alpha,i_2} + h^{(s)}_{\alpha,i_2}, \varphi^{(s)}_{\alpha,j})}{\lambda^{(s)}_{\alpha,i_2} - \lambda^{(s)}_{\alpha,j}} \varphi^{(s)}_{\alpha,j}$$

$$= B_+ \psi^{(s)}_{\alpha,(i_1,i_2),+} + B_- \psi^{(s)}_{\alpha,(i_1,i_2),-} + \cdots$$

where dots mean the remainder terms. For the approximate e.v. we take

$$\frac{1}{A_+^2 + A_-^2} [A_+^2(\lambda^{(s)}_{\alpha,i_1} + h^{(s)}_{i_1,i_1} + g_{i_1,i_1}) + A_-^2(\lambda^{(s)}_{\alpha,i_2} + h^{(s)}_{i_2i_2} + g_{i_2i_2})]$$

$$\frac{1}{B_+^2 + B_-^2} [B_+^2(\lambda^{(s)}_{\alpha,i_1} + h^{(s)}_{i_1i_1} + g_{i_1i_1}) + B_-^2(\lambda^{(s)}_{\alpha,i_2} + h^{(s)}_{i_2i_2} + g_{i_2i_2})]$$

4. The exponential decay of the resonant e.f. is investigated in the same manner as in the resonant case, because the formulas for the continuation of the exact e.f. are the same up to some correcting terms.

9. AN OUTLOOK ON THE WHOLE INDUCTIVE PROCEDURE AND THE ANALYSIS OF THE PROPERTIES OF $\Lambda^{(s+1)}(\alpha)$

As noted in Section 1, our analysis is based upon the renormalization group approach. Take $H^{(s+1)}_\varepsilon(\alpha)$. For any x, $-\frac{1}{2}q_{s+1} \leqslant x \leqslant \frac{1}{2}q_{s+1}$, we remark that on the interval $|z-x| \leqslant \text{const}\cdot s$ the operator $H^{(s+1)}_\varepsilon(\alpha)$ is close to the operator $T^{-x}H^{(s)}_\varepsilon(\alpha + x\omega_{s+1})\,T^x$ and the difference is of an order of magnitude not more than $q_s^{-2-c(\ln\varepsilon)}$. This follows from the properties of the rotation number ω and the smoothness of the potential. If, in view of the inductive assumptions, we know the structure and the localization properties of e.f. or almost e.f. of all operators $H^{(s)}_\varepsilon(\alpha)$, then we make the cutoff of all these functions considering their restrictions to the intervals of order $\frac{3}{2}\ln q_s\,(\ln 1/\varepsilon)^{-1}$ centered at the e.s. of the functions. If the values of e.f. or almost e.f. decay as $(C\varepsilon)^n$, where n is the distance to the e.s., then for typical n the values of e.f. or almost e.f. at the ends of the intervals are of the order $q_s^{-3/2-c(\ln\varepsilon)^{-1}}$. This important statement is proven in Section 7 and the corresponding inequalities are in fact valid from both sides at most points. In passing from $s \to s+1$ we consider for each x the operator

$H_\varepsilon^{(s)}(\alpha+\omega_{s+1}x)$ and take e.f. or almost e.f. $\psi^{(s)}_{\alpha+\omega_{s+1}x,j}$ for which the e.s. lies to the right of zero and contains zero. From our procedure it follows that

$$\operatorname{diam}(Z(\psi^{(s)}_{\alpha,j}))\leqslant \text{const}\cdot\left(\ln\frac{1}{\varepsilon}\right)^{-1} s$$

and therefore their choice is free of any contradictions arising from periodic boundary conditions. Then we take $T^{-x}\psi^{(s)}_{\alpha+\omega_{s+1}x,j}$ and make its cutoff in the manner described above: Denote by $\varphi^{(s+1)}_{\alpha,j}$ the functions of the basis thus obtained. We have

$$H_\varepsilon^{(s+1)}(\alpha)\,\varphi^{(s+1)}_{\alpha,j}=\lambda^{(s+1)}_{\alpha,j}\varphi^{(s+1)}_{\alpha,j}+\Gamma^{(s+1)}_{\alpha,j}+h^{(s+1)}_{\alpha,j}$$

The vector $\Gamma^{(s+1)}_{\alpha,j}$ is concentrated at quite a few points, normally four, and there has typically small values of order $q_{s+1}^{-3/2\pm c(\ln\varepsilon)^{-1}}$. The vector $h^{(s+1)}_{\alpha,j}$ is different from zero on the interval where $\varphi^{(s+1)}_{\alpha,j}$ is different from zero, i.e., on the interval whose length is of the order of const $\cdot$ $(\ln 1/\varepsilon)^{-1}\, s$ and there takes values not more than $q_s^{-2-c(\ln\varepsilon)^{-1}}$. Thus, $h^{(s+1)}_{\alpha,j}$ is also a "local vector."

Now we construct exact e.f. of $H_\varepsilon^{(s+1)}(\alpha)$ using the formulas of the perturbation theory derived in Section 4. The simplest situation arises in the nonresonant case, which takes place for most α. For a given $\varphi^{(s+1)}_{\alpha,i}$ in the nonresonant case the difference $|\lambda^{(s+1)}_{\alpha,i}-\lambda^{(s+1)}_{\alpha,j}|$ can become small only if the distance between $Z(\varphi^{(s+1)}_{\alpha,i})$ and $Z(\varphi^{(s+1)}_{\alpha,j})$ is sufficiently large. The formulas of Section 4 give in the nonresonant case the representation of the exact e.f. in the form

$$\psi^{(s+1)}_{\alpha,i}=\varphi^{(s+1)}_{\alpha,i}+\delta\varphi^{(s+1)}_{\alpha,i}+\delta\delta\varphi^{(s+1)}_{\alpha,i}$$

The remainder term $\delta\delta\varphi^{(s+1)}_{\alpha,i}$ is small everywhere and only plays the role of a small correction. The term $\delta\varphi^{(s+1)}_{\alpha,i}$ gives the main correction. It is small on the set where $\varphi^{(s+1)}_{\alpha,i}$ is different from zero and coincides with the e.f. at the boundary. More precisely, it has there an order not more than $q_s^{-2-c(\ln\varepsilon)^{-1}}$. Its form is essential near the boundary, where it really shows how the process of continuation of the e.f. looks. We have derived in Sections 4–6 formulas that imply that the value of the e.f. at the next point is equal to the product of the value of the e.f. at the previous point and some factor U that depends essentially on the e.f. concentrated near this point plus some corrections. As a result, we get that in the Anderson localization regime with exponential decay of e.f. the values of e.f. can be represented as some products of "local" functions, plus small corrections.

These correcting terms, which are always small, become important when the value of the e.f. is anomalously small. Using the equality for the e.f., we see immediately in this case that the values of the e.f. in the neighboring points are almost the same up to the sign. This gives a possibility of obtaining a convenient expression for the continued e.f. that is "uniformly good" for all values of U, because the e.f. is too small or too large when U becomes too large or too small at the corresponding points. Using this fact, we get in Section 7 estimations of e.f. from below.

In the resonant case some eigenvalues $\lambda_{\alpha,j}^{(s+1)}$ may be too close to each other compared with the distances to their e.s. It is important that only two of them may be too close. This is a direct consequence of the fact that the potential V takes each value at not more than two points. Now one has to distinguish two cases. In the first case for $\lambda_{\alpha,i}^{(s+1)}$ there are no other $\lambda_{\alpha,j}^{(s+1)}$ that might be too close to it. Here we use Theorem 3, which shows that the exact e.f. looks as in the nonresonant case. If there is one pair of e.v. $\lambda_{\alpha,i_1}^{(s+1)}$, $\lambda_{\alpha,i_2}^{(s+1)}$ whose difference is too small (in fact, less than $q_s^{-1+c(\ln \varepsilon)^{-1}}$, then two e.f. might appear of a new form, which are, in the main order, linear combinations of $\varphi_{\alpha,i_1}^{(s+1)}$, $\varphi_{\alpha,i_2}^{(s+1)}$. The coefficients of these linear combinations depend on the distance between the e.s. of $\varphi_{\alpha,i_1}^{(s+1)}$ and $\varphi_{\alpha,i_2}^{(s+1)}$ and the matrix of the coefficients is close to an orthogonal two-dimensional matrix. Two eigenvalues of corresponding e.f. differ by a number also depending on the same distance. The intervals on the spectral axis between these numbers give rise to forbidden zones (f.z. or gaps). We choose the value of the parameter measuring the distance between e.s. in such a way that its length is of order $q_s^{-2+\delta_1}$. Here δ_1 must be much greater than $c(\ln 1/\varepsilon)^{-1}$, but much less than 1. In this case the perturbations appearing for increasing s do not destroy the f.z. and at the same time are so small that we still can use the formulas of perturbation theory. The values of new e.f. in the resonant case are obtained with the help of the same "local functions" as in the nonresonant case. Therefore the analysis of the decay of the resonant e.f. is the same as in the nonresonant case. If the distance between the e.s. of resonant e.f. is too great, then it is technically more convenient to pass to approximate e.f. that are e.f. up to precision $q_s^{-2+\delta_1}$. This is achieved by an operation of taking a cut of e.f. The width of the gap here is as small as the correction to the potential under the transition $s \to s+1$ and therefore we have no possibility to estimate it in a sufficiently precise way.

Thus, our construction gives directly the new functions $\Lambda^{(s+1)}(\alpha)$, $\Phi^{(s+1)}(\alpha)$. We have to investigate their properties. As was shown in Section 8, the boundaries of already constructed f.z. shift only a relatively small distance. From Section 8 the character of appearence of new f.z. also follows. This gives us new intervals $[a_l, b_l]$ (see Section 3). In order to construct other points c_l, d_l we have to investigate the smooth properties of $\Lambda^{(s+1)}(\alpha)$. This is done below.

For simplicity we consider the situation of Theorem 2. The equation for the new eigenvalue μ has the form [see (4.9′), (4.13)]

$$[\bar{\Lambda}^{(s)}(\alpha)-\mu+b_{11}(\alpha,\mu)][\bar{\Lambda}^{(s)}(\alpha)-\mu+b_{22}(\alpha,\mu)] + [\bar{g}_{12}(\alpha)+b_{12}(\alpha,\mu)][\bar{g}_{21}(\alpha)+b_{21}(\alpha,\mu)]=0 \tag{9.1}$$

Here we use the following notations:

$$\bar{\Lambda}^{(s)}(\alpha)=\lambda^{(s)}_{\alpha,i_1}+g_{i_1i_1}+h_{i_1i_1}$$

$$\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})=\lambda^{(s)}_{\alpha,i_2}+g_{i_2i_2}+h_{i_2i_2}$$

$$\bar{g}_{12}(\alpha)=g_{i_2i_1}+h_{i_2i_1}, \qquad \bar{g}_{21}(\alpha)=g_{i_1i_2}+h_{i_1i_2}$$

$$b_{kl}(\alpha,\mu)=\sum y_{i_kj}(g_{ji_l}+h_{ji_l}), \qquad k,l=1,2$$

We recall that y_{ij} also depend on μ. All terms b_{kl} and their derivatives will be treated as small corrections. Differentiating (9.1), we have, with $\mu'=d\mu/d\alpha$,

$$\left(\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}-\mu'+\frac{\partial b_{11}}{\partial\alpha}+\frac{\partial b_{11}(\alpha,\mu)}{\partial\mu}\mu'\right)[\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})-\mu+b_{22}]$$
$$+[\bar{\Lambda}^{(s)}(\alpha)-\mu+b_{11}]\left(\frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}-\mu'+\frac{\partial b_{22}}{\partial\alpha}+\mu'\frac{\partial b_{22}}{\partial\mu}\right)$$
$$+\left(\frac{d\bar{g}_{12}}{d\alpha}+\frac{\partial b_{12}}{\partial\alpha}+\frac{\partial b_{12}}{\partial\mu}\mu'\right)[\bar{g}_{21}(\alpha)+b_{21}(\alpha,\mu)]$$
$$+[\bar{g}_{12}(\alpha)+b_{12}(\alpha,\mu)]\left(\frac{d\bar{g}_{21}(\alpha)}{d\alpha}+\frac{\partial b_{21}}{\partial\alpha}+\mu'\frac{\partial b_{21}}{\partial\mu}\right)=0$$

We derive from it the expression for μ':

$$\mu'=\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}\,\frac{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})-\mu}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu} + \frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}\,\frac{\bar{\Lambda}^{(s)}(\alpha)-\mu}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu}+\cdots \tag{9.2}$$

where the dots mean again terms of smaller order. This expression shows that, in the main order, μ' is a linear combination of $d\bar{\Lambda}^{(s)}(\alpha)/d\alpha$ and $d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})/d\alpha$ with the weights

$$p=\frac{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})-\mu}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu}$$

$$q=\frac{\bar{\Lambda}^{(s)}(\alpha)-\mu}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu}$$

Also, from (4.13) and Section 8 it follows that $p \geqslant 0$, $q \geqslant 0$, $p+q=1$. Now for the second derivative we have the expression

$$\mu'' = \frac{d^2\bar{\Lambda}^{(s)}(\alpha)}{d\alpha^2} \frac{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})-\mu}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu}$$
$$+\frac{d^2\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha^2} \frac{\bar{\Lambda}^{(s)}(\alpha)-\mu}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu}$$
$$+2\left(\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}-\mu'\right)\left(\frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}-\mu'\right)$$
$$\times\frac{1}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)}+\cdots$$

Again the dots mean small corrections. Further,

$$\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}-\mu' = q\left(\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}-\frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}\right)+\cdots$$
$$\frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}-\mu' = p\left(\frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}-\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}\right)+\cdots$$

Putting these expressions into the previous formula, we get

$$\mu'' = \frac{d^2\bar{\Lambda}^{(s)}(\alpha)}{d\alpha^2} \frac{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})-\mu}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu}$$
$$+\frac{d^2\bar{\Lambda}^{()}(\alpha)}{d\alpha^2} \frac{\bar{\Lambda}^{(s)}(\alpha)-\mu}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu}$$
$$-2pq\left(\frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}-\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}\right)^2$$
$$\times\frac{1}{\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu}+\cdots \tag{9.3}$$

The most important term in this expression is

$$-2pq\left(\frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}-\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}\right)^2 \frac{1}{\bar{\Lambda}^{()}(\alpha+m\omega_{s+1})+\bar{\Lambda}^{(s)}(\alpha)-2\mu} \tag{9.4}$$

The difference

$$\left|\frac{d\bar{\Lambda}^{(s)}(\alpha+m\omega_{s+1})}{d\alpha}-\frac{d\bar{\Lambda}^{(s)}(\alpha)}{d\alpha}\right| \geqslant \text{const}\cdot s^{-5}$$

as follows from the inductive hypothesis I in Section 3. Consider the branch of μ that is bigger than $\frac{1}{2}[\bar{A}^{(s)}(\alpha+m\omega_{s+1})+\bar{A}^{(s)}(\alpha)]$, i.e., the sign in the square root giving the expression for μ (see Section 4) is positive. Then the minimal value of μ defines a boundary a_l. We define c_l in such a way that $\min(p, q) \geqslant q_{s+1}^{-3}$ and c_l is the largest number for which this property holds. If α', α'' are such that $A^{(s+1)}(\alpha')=A^{(s+1)}(\alpha'')=c_l$, then

$$\left|\frac{dA^{(s+1)}(\alpha')}{d\alpha}-\frac{dA^{(s)}(\alpha')}{d\alpha}\right| \leqslant q_s^{-3/2-c(\ln\varepsilon)^{-1}}$$

$$\left|\frac{dA^{(s+1)}(\alpha'')}{d\alpha}-\frac{dA^{(s)}(\alpha'')}{d\alpha}\right| \leqslant q_s^{-3/2-c(\ln\varepsilon)^{-1}}$$

In a similar way one defines d_{l-1}. On the interval (α', α'') the second derivative

$$d^2A^{(s+1)}/d\alpha^2 \geqslant q_{s+1}^{1/2-c(\ln\varepsilon)^{-1}}$$

This is an immediate consequence of (9.4). Thus we have defined completely the graph $A^{(s+1)}(\alpha)$, the function $\Phi^{(s+1)}(\alpha)$, and the points $a_l<c_l<d_l<b_l$. All other needed properties follow easily from the construction.

The construction also gives for almost all α the limiting function $A(\alpha)=\lim_{s\to\infty} A^{(s)}(\alpha)$, which takes values in a Cantor set of positive measure. The lengths of f.z. decay exponentially with the number labeling their appearence, or, which is the same, with the diameter of the e.s. The existence of the function $\Phi(\alpha)$ also follows easily from the whole process. We give now the complete formulation of the main theorem.

Main Theorem. Assume that V is a C^2-function on S^1 having one nondegenerate minimum and one nondegenerate maximum. Then for sufficiently small ε:

(a_1) The integrated density of states is a noncomplete Cantor devil's staircase.

(a_2) For almost all α the operator $H_\varepsilon(\alpha)$ has a pure point spectrum with exponentially decaying e.f.

10. SOME GENERALIZATIONS

The described technique is applied without any changes to Jacobi matrices where off-diagonal terms depend quasiperiodically on n. Such cases appear in the analysis of the Fröhlich–Peierls model (I. M. Krichever, personal communication). If we consider the Schrödinger difference operators with potentials having two or more basic frequencies, then apparently the complete Anderson localization also holds for sufficiently small ε, but the integrated density of states has no gaps and is absolutely continuous. We also hope that our technique will work for the localization problems in the kicked-rotator model in the theory of quantum chaos (see Ref. 36).

APPENDIX A. ESTIMATION OF THE COEFFICIENTS h_{ij}, g_{ij}

Let us write $(\varphi^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2}) = \delta_{i_1 i_2} + c'_{i_1 i_2}$. Then $c'_{i_1 i_2} = 0$ if

$$\operatorname{dist}(Z(\varphi^{(s)}_{\alpha,i_1}), Z(\varphi^{(s)}_{\alpha,i_2})) \geqslant \text{const} \cdot \left(\ln \frac{1}{\varepsilon}\right)^{-1} s$$

For other i_1, i_2 we have

$$|c'_{i_1 i_2}| \leqslant q_s^{-3/2 - c(\ln \varepsilon)^{-1}}$$

from (a_3). Thus, if $\|(\varphi^{(s)}_{\alpha,i_1}, \varphi^{(s)}_{\alpha,i_2})\| = I + C'$, then

$$\|\delta_{i_1 i_2} + c_{i_1 i_2}\| = (I + C')^{-1} = I - C' + (C')^2 + \cdots$$

and we easily get

$$|c_{i_1 i_2}| \leqslant q_s^{-[3/2 - c(\ln \varepsilon)^{-1}/2]m}$$

if

$$(m-1)\left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right] \leqslant \operatorname{dist}(Z(\varphi^{(s)}_{\alpha,i_1}), Z(\varphi^{(s)}_{\alpha,i_2})) < m\left[\frac{3 \ln q_s}{2 \ln 1/\varepsilon}\right]$$

Write down the expansion

$$h^{(s)}_{\alpha,i} = \sum h_{ij} \varphi^{(s)}_{\alpha,j}$$

The coefficients

$$h_{ij} = \sum_{j_1} (\delta_{jj_1} + c_{jj_1})(h^{(s)}_{\alpha,i}, \varphi^{(s)}_{\alpha,j_1})$$

In the last sum only the terms for which

$$\operatorname{dist}(Z(\varphi^{(s)}_{\alpha,j_1}), Z(\varphi^{(s)}_{\alpha,i})) \leqslant \text{const} \cdot \left(\ln \frac{1}{\varepsilon}\right)^{-1} \ln q_s$$

may give a nonzero contribution to the last sum. Their absolute value is not more than $q_s^{-2+\delta_1}$ and the total number of such terms is not more than const $\cdot s^2$. These remarks and the previous estimation of $c_{i_1 i_2}$ give the needed estimation of h_{ij} [see (4.7')]. In the same way one gets (4.7").

APPENDIX B. PROOF OF LEMMA 1, SECTION 6

First we remark that the sequence $\{\varphi_{\alpha,i}^{(s+1)}\}$ is almost orthogonal and satisfies (a_3), Section 4. This is obvious if the e.s. are sufficiently far from each other. In the opposite case the shifts of $\varphi_{\alpha,i_1}^{(s+1)}$ and $\varphi_{\alpha,i_2}^{(s+1)}$ are e.f. or almost e.f. of $H_\varepsilon^{(s)}(\alpha_{i_1})$ and $H_\varepsilon^{(s)}(\alpha_{i_2})$ and

$$|\alpha_{i_1} - \alpha_{i_2}| \leqslant \text{const} \cdot s^3 q_s^{-2}$$

If $\varphi_{\alpha,i_1}^{(s+1)}$ is a nonresonant e.f., then the estimation

$$|(\varphi_{\alpha,i_1}^{(s+1)}, \varphi_{\alpha,i_2}^{(s+1)})| \leqslant q_s^{-3/2 - c(\ln \varepsilon)^{-1}}$$

follows easily from the theorems of perturbation theory (see Sections 4 and 5). In the case of resonant e.f. one has to consider $\varphi_{\alpha,i_1}^{(s+1)}$ and other corresponding resonant e.f. and write down the expansion of $\varphi_{\alpha,i_2}^{(s+1)}$ over all $\{\psi_{\alpha,i}^{(s)}\}$. Again the needed estimation follows from the perturbation theory. We omit the details of these calculations.

Now we have to show that $\{\varphi_{\alpha,i}^{(s+1)}\}$ is a basis. Assume that this is wrong and that there exists $\bar\psi = \{\psi(n)\}$, $0 \leqslant n < q_{s+1}$, for which

$$(\bar\psi, \varphi_{\alpha,i}^{(s+1)}) = \sum_n \bar\psi(n)\, \varphi_{\alpha,i}^{(s+1)}(n) = 0$$

for all i and $\sum_n |\bar\psi(n)|^2 = 1$. Let $\delta_m = \{\delta_{mn}\}$. Then we can write $\delta_m = \sum_j' d_{mj} \psi_{\alpha_m,j}^{(s)} + \chi_m$, where $\alpha_m = \alpha + m\omega_{s+1}$ and $\sum'$ is taken over those $\psi_{\alpha_m,j}^{(s)}$ for which the distance between m and the e.s. is not more than const $\cdot$ $(\ln 1/\varepsilon)^{-1} s$. We have the estimations

$$\|\chi_m\| \leqslant q_s^{-3/2 - c(\ln \varepsilon)^{-1}}, \qquad \left| \sum_j{}' |c_{mj}|^2 - 1 \right| \leqslant q_{s+1}^{-3/2 - 2c(\ln \varepsilon)^{-1}}$$

Further,

$$\bar\psi = \sum \bar\psi(m)\, \delta_m = \sum_m \sum_j{}' \bar\psi(m)\, c_{mj} \psi_{\alpha_m,j}^{(s)} + \chi$$

where for χ we have the trivial estimation

$$\|\chi\| \leqslant q_{s+1}^{-1/2 - 2c(\ln \varepsilon)^{-1}}$$

We rewrite the expression for $\bar{\psi}$ as follows:

$$\bar{\psi} = \sum_m \sum_j{}' \bar{\psi}(m)\, c_{mj} \psi^{(s)}_{\alpha_m, j} + \chi = \sum d_j \varphi^{(s+1)}_{\alpha, j} + \chi'$$

where we replace $\psi^{(s)}_{\alpha_m, j}$ by the corresponding $\varphi^{(s+1)}_{\alpha, j}$ and include the difference in χ'. We have

$$\left| \sum d_j^2 - 1 \right| \leqslant q_s^{-1/2 - 4c(\ln \varepsilon)^{-1}}, \qquad \|\chi'\| \leqslant q_s^{-1/2 - 4c(\ln \varepsilon)^{-1}}$$

But the last inequalities contradict to orthogonality conditions $(\bar{\psi}, \varphi^{(s+1)}_{\alpha, j}) = 0$ for all j. ∎

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REFERENCES

1. N. Mott and W. D. Twose, *Adv. Phys.* **10**:107 (1961).
2. R. E. Borland, *Proc. R. Soc. Lond. A* **274**:529 (1963).
3. I. Ya. Goldsheid and S. A. Molchanov, *Doklady Akad. Nauk SSSR* **230**:761–764 (1976).
4. Ya. Goldsheid, S. Molchanov, and L. Pastur, *Funct. Anal. Appl.* **11**:1 (1977).
5. H. Kunz and B. Souillard, *Commun. Math. Phys.* **78**:201 (1980).
6. S. Molchanov, *Math. Izv. USSR* **42** (1978).
7. R. Carmona, *Duke Math. J.* **49**:191 (1982).
8. B. Souillard, Spectral properties of discrete and continuous random Schrödinger operators: A review, Preprint (IMA, 1986).
9. J. Fröhlich and T. Spencer, *Commun. Math. Phys.* **88**:151 (1983).
10. J. Fröhlich, F. Martinelli, E. Scoppola, and T. Spencer, *Commun. Math. Phys.* **101**:21 (1985).
11. G. Jona-Lasinio, F. Martinelli, and E. Scoppola, *J. Phys. A* **17**:73 (1984).
12. F. Delyon, Y. Levy, and B. Souillard, *Commun. Math. Phys.* **100**:463 (1985).
13. B. Simon, M. Taylor, and T. Wolff, *Phys. Rev. Lett.* **54**:1589 (1985).
14. F. Wegner, *Z. Phys. B* **44**:9 (1981).
15. R. Carmona, A. Klein, and F. Martinelli, Anderson localization for Bernoulli and other singular potentials, *Commun. Math. Phys.*, to appear.
16. R. Merlin, K. Bajema, R. Clarke, F. Y. Juang, and P. K. Bhattachanja, *Phys. Rev. Lett.* **55**:1768 (1985).
17. A. Bondeson, E. Ott, and T. Antonsen, Jr., *Phys. Rev. Lett.* **55**:2103 (1985).

18. M. Kohmoto, L. P. Kadanoff, and Chao Tang, *Phys. Rev. Lett.* **50**:1870 (1983).
19. P. A. Kalugin, A. Yu. Kitaev, and L. S. Levitov, Electronic spectrum of one-dimensional quasi-crystals, Chernogolovka, Preprint (1986).
20. F. Delyon and D. Petritis, *Commun. Math. Phys.* **103**:441–444 (1986).
21. E. I. Dinaburg and Ya. G. Sinai, *Funct. Anal. Appl.* **9**:279 (1975).
22. E. D. Belokolos, *Theor. Math. Phys.* **25**:176 (1975).
23. Ya. G. Sinai, *Funct. Anal. Appl.* **19**:42 (1985).
24. H. Rüssmann, *Ann. N. Y. Acad. Sci.* **357**:90 (1980).
25. J. Bellisard, R. Lima, and D. Testard, *Commun. Math. Phys.* **88**:207 (1983).
26. J. Avron and B. Simon, *Commun. Math. Phys.* **82**:101–120 (1982); *Duke Math. J.* **50**:369–391 (1983); *Bull. Am. Math. Soc.* **6**:81–86 (1982).
27. R. A. Johnson and J. Moser, *Commun. Math. Phys.* **84**:403–438 (1982).
28. J. Moser and J. Pöschel, An extension of a result by Dinaburg and Sinai on quasi-periodic potentials, Preprint, ETH, Zürich (1983).
29. S. Aubry and G. André, *Ann. Israel Phys. Soc.* **3**:133 (1980).
30. S. Aubry and P. Y. Le Daeron, *Physica 8D*:381–422 (1983).
31. B. Simon, *Adv. Appl. Math.* **3**:463 (1982).
32. L. A. Pastur and A. L. Figotin, *J. Math. Phys.* **25**:774 (1984).
33. L. A. Pastur and A. L. Figotin, *Commun. Math. Phys.* **95**:401 (1984).
34. B. Simon, *Ann. Phys. N. Y.* **158**:415 (1984).
35. J. Pöschel, *Commun. Math. Phys.* **88**:447 (1983).
36. J. Bellisard, Stability and instability in quantum mechanics, Preprint, CNRS, Luminy (1984).