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Noise-Induced Order[1]

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A new noise effect on chaos in one-dimensional mappings is reported. The transition from chaotic behavior to ordered behavior induced by external noise is observed in a certain class of one-dimensional mappings. This transition is clearly shown in terms of the Lyapunov number, entropy, power spectrum, and the nature of orbits.

KEY WORDS: One-dimensional mapping; chaos; noise; power spectrum; entropy.

1. INTRODUCTION

How is the character of chaos influenced by external noise? There are several contributions to this problem in the case of the logistic model.(1,2) In the logistic model, external noise induces the transition from the periodic behavior to the chaotic behavior. There appears a broadening of the invariant density and of the power spectrum and an increase in the Lyapunov number.

We study here other types of one-dimensional maps which exhibit a very different response from the one obtained in the logistic model. In our maps, one of which is directly connected to the real chemical reaction, the Belousov–Zhabotinsky reaction, external noise destabilizes the chaos and produces some kind of order. The transition to the order is indicated by sharpening of power spectrum, abrupt decrease of entropy, appearance of negative Lyapunov number, and localization of orbit.

[1] In our opinion, the new phenomenon which is reported in this paper may be called "periodicity." However, we here call this simply "order," following the comment of one of the referees, because the theory that is expected to justify our opinion is not yet completed. The theory based on the global analysis of maps will be completed in the near future.

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The new transition reported here suggests two important points: (1) Some chaotic orbit is unstable for noise and (2) randomness of the deterministic chaos is definitely different from that of noise. The former aspect indicates that periodic solutions observed in real experiments might in fact be chaos if the external noise did not exist. The latter aspect means that the "randomness" of the noise in the logistic model may induce the "randomness" of the orbit, leading to chaos; on the other hand, the "randomness" of the noise in some class of maps destroys the "randomness" of chaos, lowering its "randomness."

In Section 2, we describe the models investigated here and the calculation techniques used. Numerical results are found in Section 3 for the model of the B–Z reaction and in Section 4 for the logistic model. In Section 5, the mechanism of the transition from the chaotic behavior to the ordered behavior is explained and two other models are constructed to illustrate the mechanism. Section 6 contains the summary and discussion.

2. METHODS

We investigate four models with additive noise equally distributed in the interval $[-\sigma, \sigma]$, where σ is the maximum value of the noise level.

One of the models[3] was proposed in relation to the B–Z reaction and successfully explained the bifurcation sequence obtained in experiments. This model with an additive bifurcation parameter is written as follows:

$$x < 0.125, \qquad f(x) = \left[-(0.125 - x)^{1/3} + .50607357\right]\exp(-x) + b$$

$$0.125 < x < 0.3, \qquad f(x) = \left[(x - 0.125)^{1/3} + 0.50607357\right]\exp(-x) + b$$

$$0.3 < x, \qquad f(x) = 0.121205692 \times \left[10x \exp(-\tfrac{10}{3}x)\right]^{19} + b.$$

The new transition from the chaotic behavior to the ordered behavior as the noise level is increased was first found in this model, so this model is thoroughly investigated in this paper.

The next model is the well-known logistic model:[4]

$$f(x) = ax(1 - x)$$

The investigation of this model is done for comparison with the former model. The transition mentioned above is not observed in the logistic model.

Two other models which have steepness control parameters are presented in Section 5 where the motivation for adopting these models is also explained.

For a systematic investigation, we use the concept of the Markov maps.[5] Let x^* be the point where $df(x^*)/dx = 0$. The Markov maps in the model are the maps where $f^{(n)}(x^*)$ is mapped into one of the unstable periodic points. For the logistic model, it is known that these maps have asbolutely continuous invariant measure.[6] On the other hand, for maps which violate the Schwarz condition, this is not generally true. However, we expect some of the Markov maps are chaotic even when the Schwarz condition is broken. Thus, we compare the models by comparing the same type of Markov map (with positive Lyapunov number) in each model. This enables us to investigate more or less systematically.

For the numerical calculations we used double precision on the FACOM-M200 at the Kyoto University Data Processing Center.

3. NUMERICAL RESULTS ON THE B–Z MODEL

First of all we investigate the B–Z model at the bifurcation parameter value $b = 0.023288\ldots$, which corresponds to the Markov map of type $RLLL.R$.[3]

Our first observation is that the originally positive Lyapunov number[8] changes to a negative one as the noise level is increased. This surprising fact is shown in Fig. 1.

The Lyapunov number is the indicator of the orbital instability in the case without noise. However, that the Lyapunov number is an appropriate indicator in the presence of large noise is very questionable as the concept of orbits is lost. Alternatively, we study the entropy of this system viewed as an information source.

Regarding the R–L sequence of an orbit as the product of an information source, we can calculate its entropy H.[7] We call this quantity the entropy of a system composed of a deterministic dynamics and noise.

As shown in Fig. 2, the entropy abruptly decreases as the noise level is increased.

To check whether or not our phenomenon is observable in experiments, we study the power spectrum. The results are shown in Fig. 3. In Fig. 3a the power spectrum in the case without noise is shown. Figures 3b–3f are the cases where the noise level is gradually increased. For relatively large noise, the sharp peak which implies some kind of order appears. The Fig. 3f where the noise level is maximum cannot be distinguished from the figure for the map in which a 6-periodic orbit is super stable with the same noise level. We call this phenomenon noise-induced order.

[3] Let x^* be the point for which $df(x^*)/dx = 0$. We associate a letter R to a point x when it is in the region $> x^*$, otherwise we associate a letter L. Then for every sequence of points (whether it is an orbit or not) we can associate a sequence of letters consisting of R and L. The type of a Markov map is defined as the R–L sequence associated with the orbit starting from x^*. For part of sequence associated with the unstable periodic orbit, only sequence corresponding to one period is written after the period.

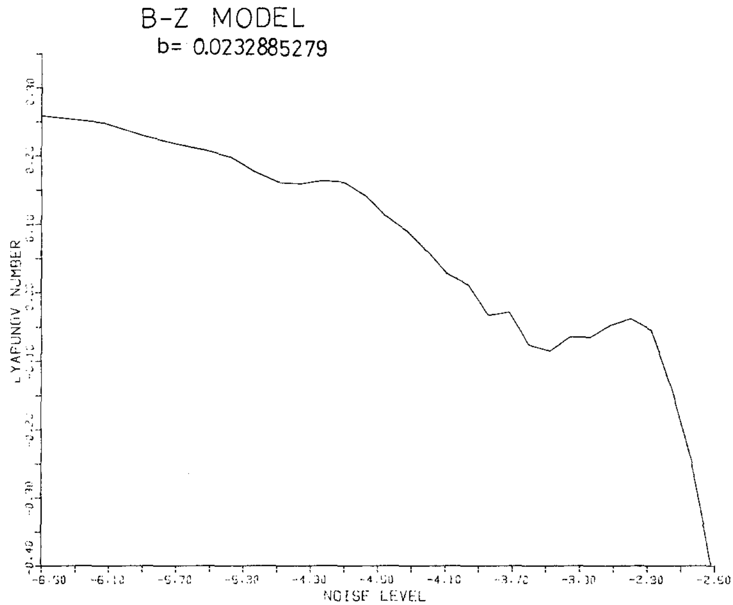


Fig. 1. Lyapunov number vs. noise at $b = 0.0232885279$ for the B–Z model. This is the Markov map of the type *RLLL.R*.

This seems to give a new mechanism for the formation of order, namely, order can be formed from chaos by thermal fluctuation, and this order reflects the periodicity appearing at a different value of bifurcation parameter. The formation of order as seen in nucleation(9,10) or symmetry breaking(11) is represented by transitions between multiple states existing

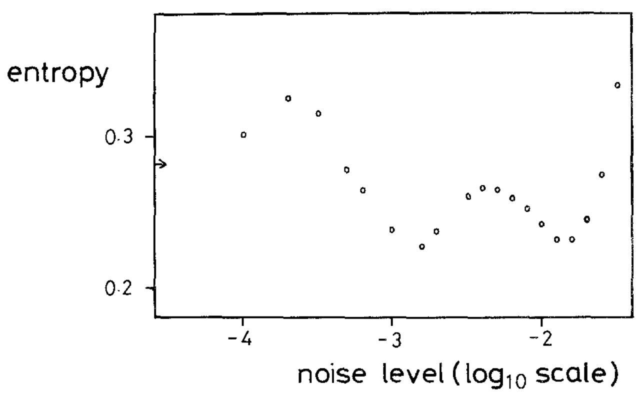
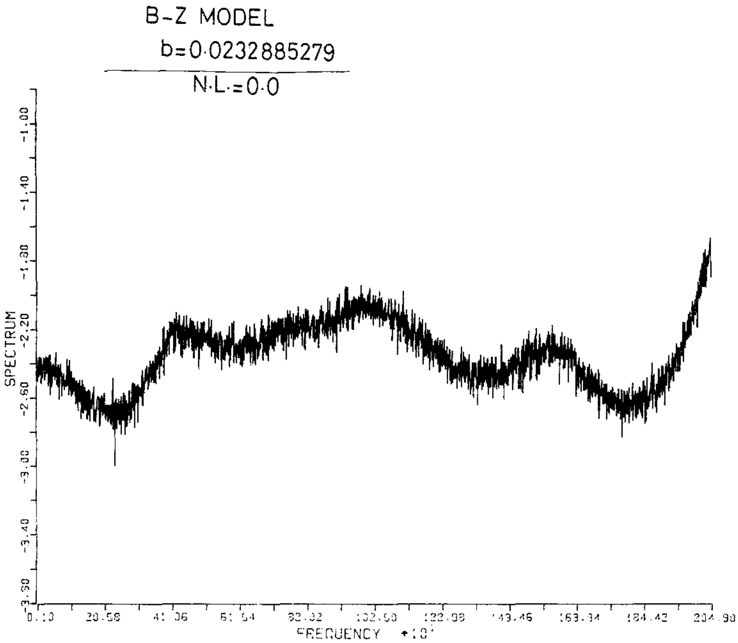
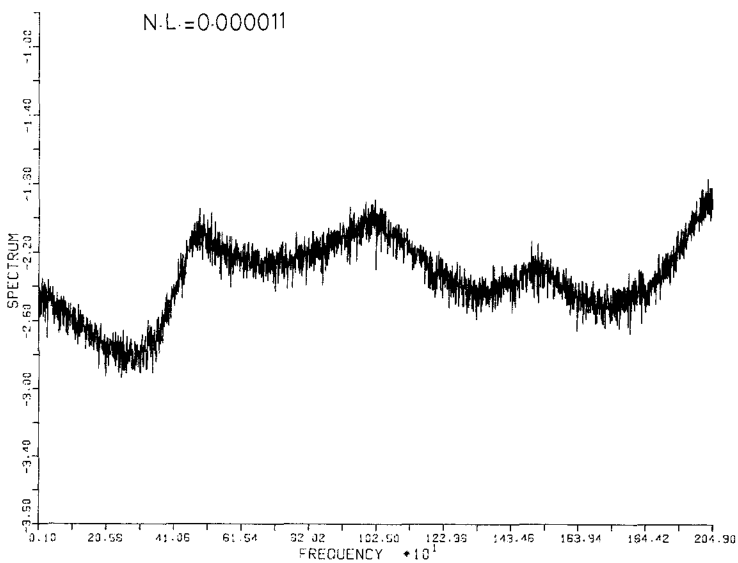


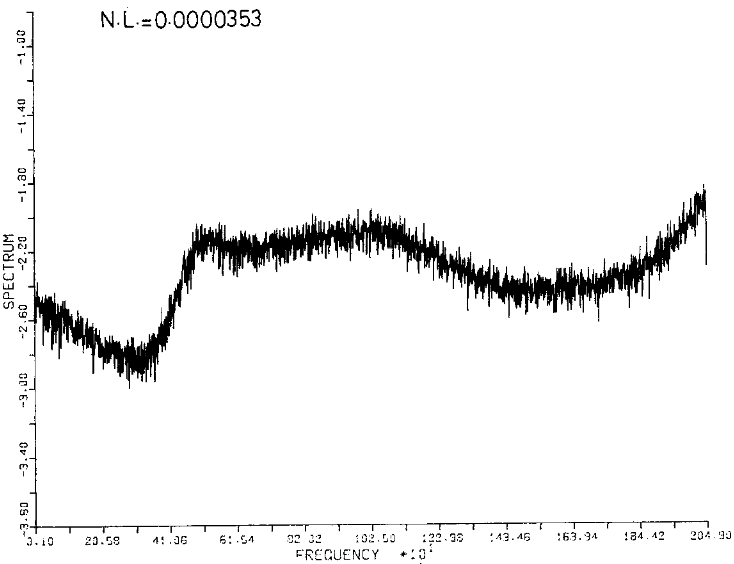
Fig. 2. Entropy of the same map as in Fig. 1. The arrow indicates the value of entropy in the case which is noiseless but for round-off error ($\sim 10^{-16}$).


B-Z MODEL
b=0.0232885279
N.L.=0.0
SPECTRUM
FREQUENCY *10^1

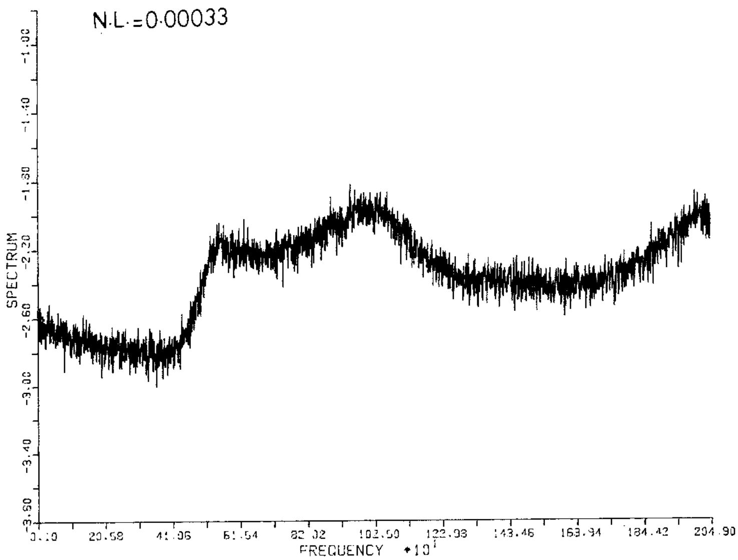
(a)


N.L.=0.000011
SPECTRUM
FREQUENCY *10^1

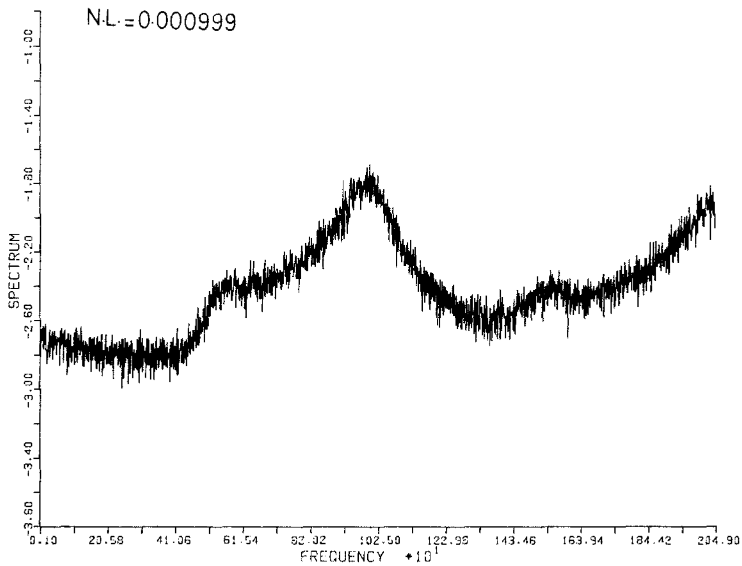
(b)


N·L·=0·0000353
SPECTRUM
FREQUENCY *10

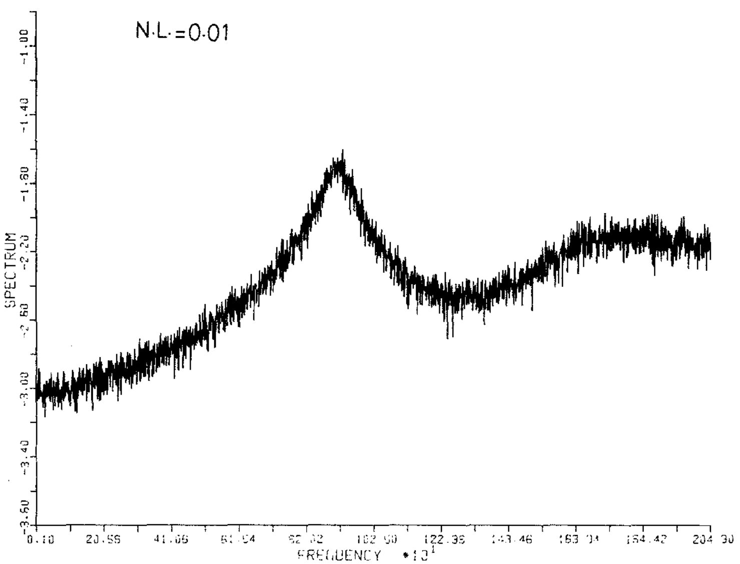
(c)


N·L·=0·00033
SPECTRUM
FREQUENCY *10

(d)



(e)



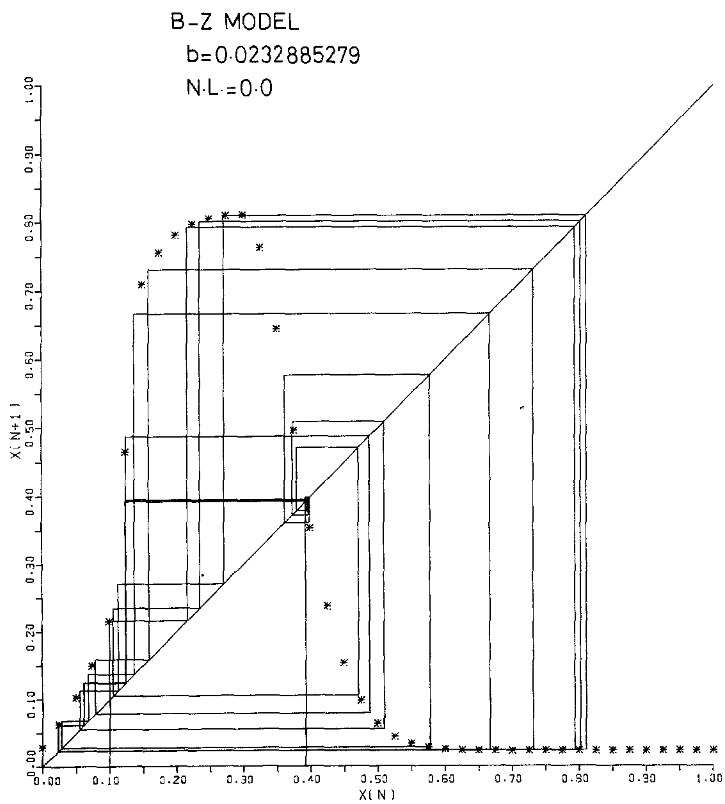
(f)

Fig. 3. Power spectrum (FFT) of orbits of the same map as in Fig. 1 with various noise levels: (a) without noise (only round-off error), noise level is (b) 1.1×10^{-5}, (c) 3.53×10^{-5}, (d) 3.3×10^{-4}, (e) 9.99×10^{-4}, (f) 1.0×10^{-2}.

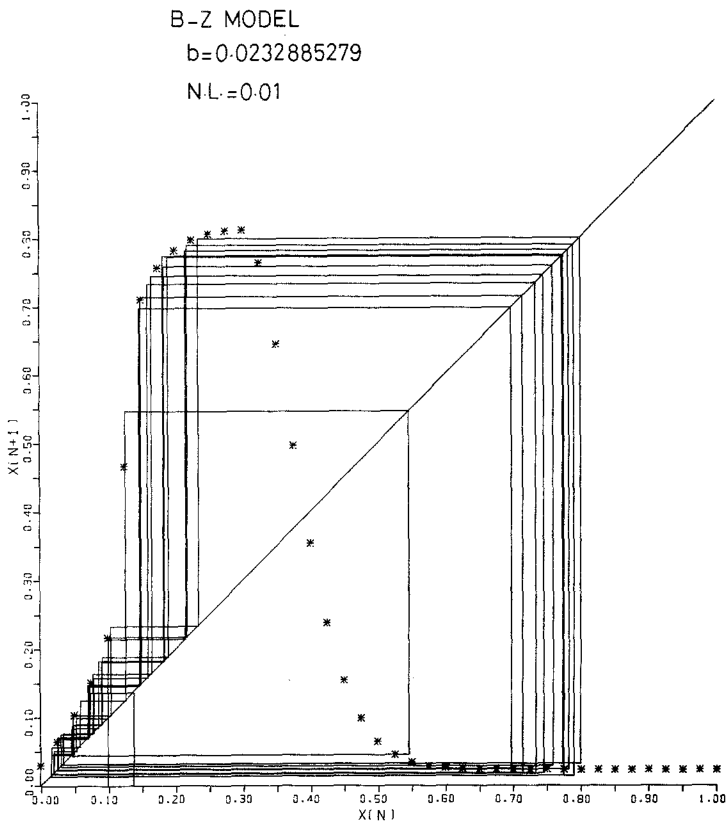
simultaneously at certain value of the bifurcation parameter. In our case, coexistence of phases is not observed.

We also observe that the nature of the orbit in the noisy case differs from that in the noiseless case (see Fig. 4). In the case without noise, the orbit concentrates around the unstable fixed point at $x = 0.3929\ldots$, reflecting the character of this Markov map. On the other hand, in the noisy case the orbit seldom visits the neighborhood of the fixed point but rather visits the neighborhood of the critical point x^*. The mechanism for this is explained in Section 5.

This change in the orbits appears more clearly in the invariant density, which is presented in Fig. 5. It is characteristic that a peak near the fixed point in the absence of noise shifts to the region near the critical point in the presence of noise. This makes the Lyapunov number negative and the overall structure of the orbit similar to that of the 6-periodic orbit with noise.



(a)



(b)

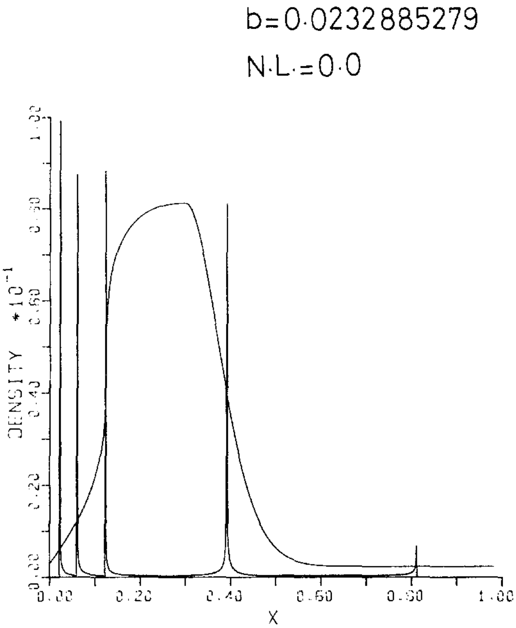
Fig. 4. Orbits in the same map as in Fig. 1. (a) Noiseless case and (b) the case with noise level 1.0×10^{-2}. First 50 points of the orbit starting from $x = 0.1$ are depicted, joined by a solid line. The stars represent the map.

We confirmed that the same features appear also in other types of chaos. As one example, we show this kind of transition in the type *RLLLL.R* Markov map in Fig. 6.

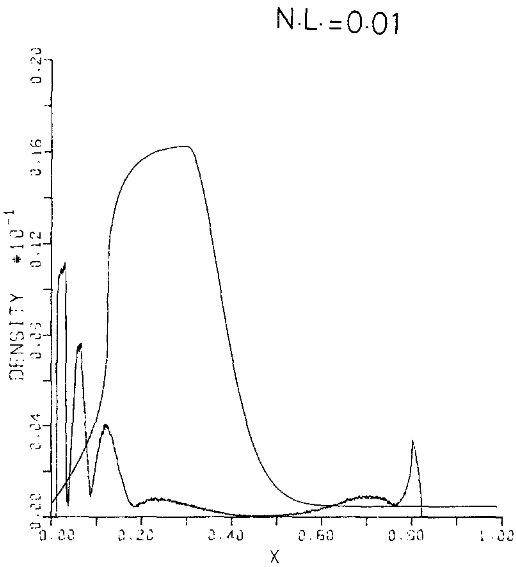
In the next section, we calculate the same quantities as those in this section for the logistic model.

4. NUMERICAL RESULTS ON THE LOGISTIC MODEL

In this section, we report that the transition discussed in the previous section is not observed in the logistic model. Moreover, the mechanism of the transition (see Section 5) suggests that it is unlikely to exist in the logistic model.

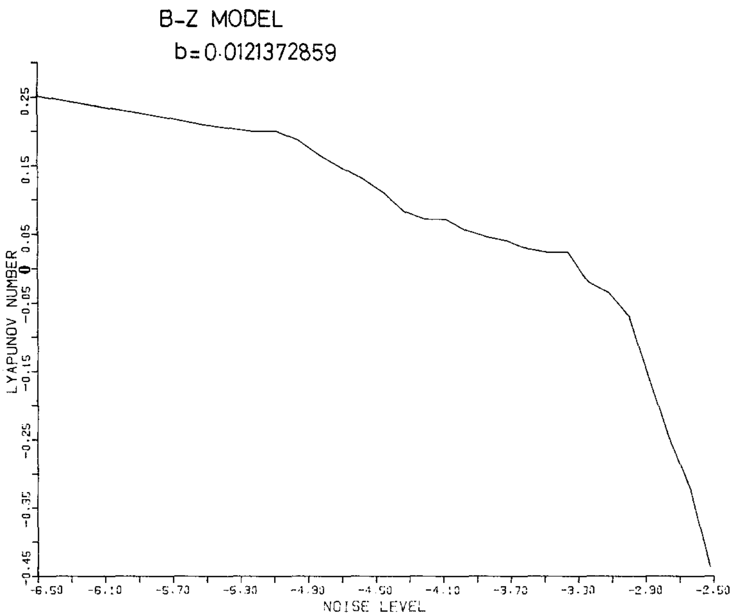


(a)



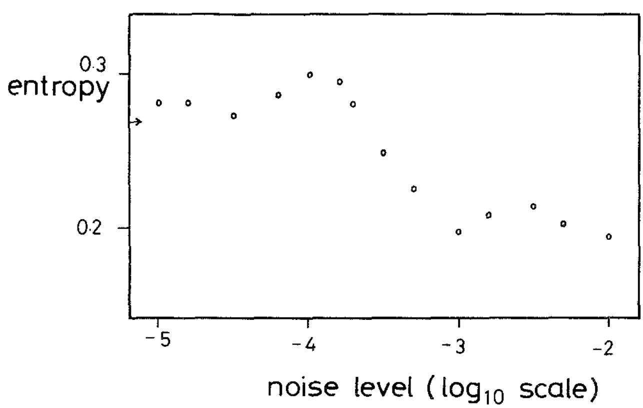
(b)

Fig. 5. Invariant density of the same map as in Fig. 1. (a) Noiseless case and (b) the case with noise level 1.0×10^{-2}.



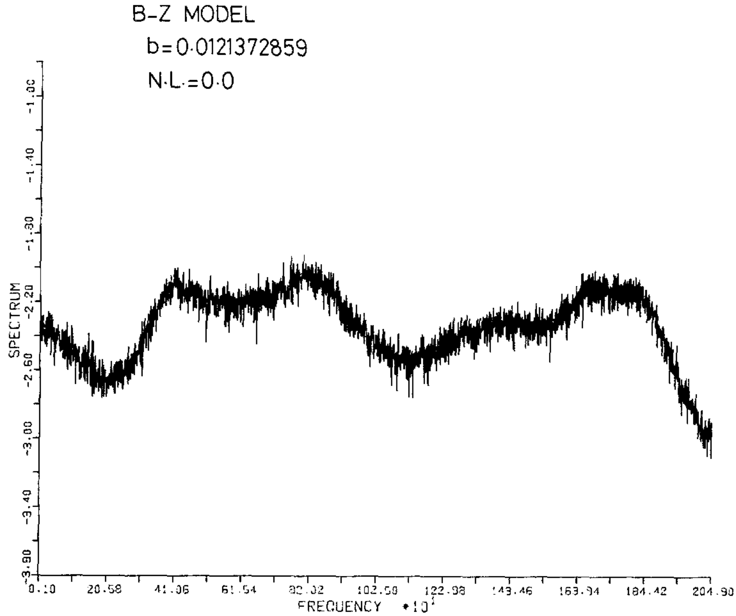
(a)

Fig. 6. The characteristics indicating noise-induced order of the B–Z model at $b = 0.0121372859$. This is the Markov map of the type *RLLLL.R*. (a) Lyapunov number vs. noise.



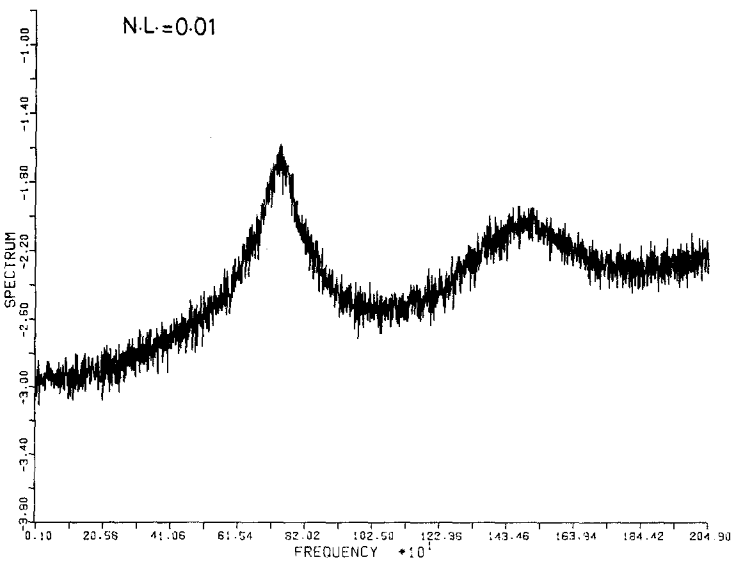
(b)

(b) Entropy vs. noise. The arrow indicates the value of entropy in the case which is noiseless but for round-off error (-10^{-16}).



(c)

(c) Power spectrum (FFT) in the case without noise.



(d)

(d) Power spectrum (FFT) in the case with noise level 1.0×10^{-2}.

The Lyapunov number increases as the noise level is increased, for example, for the case of the Markov map *RLLL.R* the Lyapunov number 0.62419 in the absence of noise becomes slightly larger, 0.646998, at the noise level 0.01. (See Mayer-Kress and Haken[1] for a description of Lyapunov number vs. noise level in the logistic model.)

The power spectrum shows only very weak peak as shown in Fig. 7. The orbit does not show any noticeable periodicity and the density merely broadens. (See Fig. 8 for the orbits and Fig. 9 for the density.)

So, we inquire in what class of maps the transition from chaos to order occurs.

5. NOISE-DESTABILIZED CHAOS

The numerical calculations presented in the previous sections show definitely that the chaos is unstable against external random perturbations in some class of maps. We now investigate the mechanism of this instability.

Our explanation is based on the shift of the peak appearing in the invariant density of Fig. 5. This shift occurs as a result of the splitting of the peak at fixed point ($x = 0.3929\ldots$) due to the smearing of the peaks on the left-hand side of the critical point ($x = 0.3$).

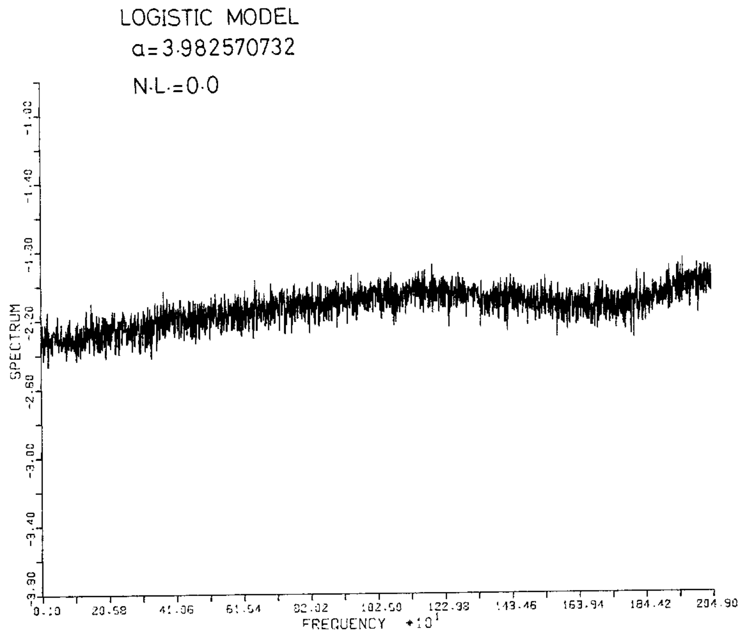
Consider the slope on the left-hand side of the critical point in the B–Z model. If we assume a uniform density $P_n(x)$ in this region of the map, the next iteration $P_{n+1}(x)$ of this density function has clearly two peaks as depicted in Fig. 10. The probability measure of the region between these two peaks is very small. Thus, if the starting density $[P_n(x)]$ is uniform enough, the probability of an orbit visiting this region is very small. As seen from Fig. 5, this type of chaotic orbit uses the region near the unstable fixed point ($x = 0.3929\ldots$) frequently. This is due to the fact that as the B–Z reaction map has wide flat regions and narrow steep regions, it is necessary for chaotic orbits to stay in the steep region for a long time. However, if the noise smears the peaks of the density on the left-hand side of the critical point, the orbit seldom visits the unstable fixed point, as described earlier.

Thus, the chaos is destroyed by noise, and as a result, the Lyapunov number turns negative. This is our explanation of the phenomenon noise-induced order.

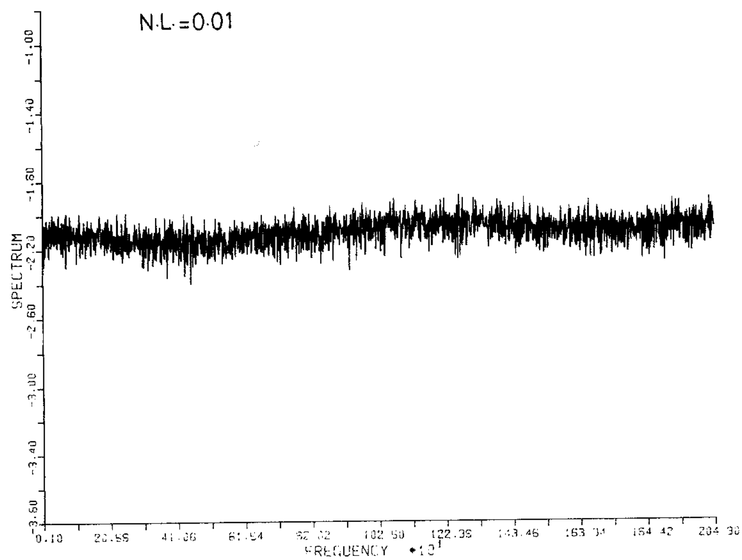
To confirm this explanation, we construct two families of models which each have a parameter which controls the steepness of the slope on the left-hand side of the map.

One of them is

$$f(x) = 0.8 \times (0.3)^{-\alpha} \times e^{\alpha} x^{\alpha} \exp\left(-\frac{\alpha}{0.3}x\right) + b$$



(a)



(b)

Fig. 7. Power spectrum (FFT) of logistic model at $a = 3.982570732$. This is the Markov map of type $RLLL.R$. (a) Noiseless case. (b) The case with noise level 1.0×10^{-2}.

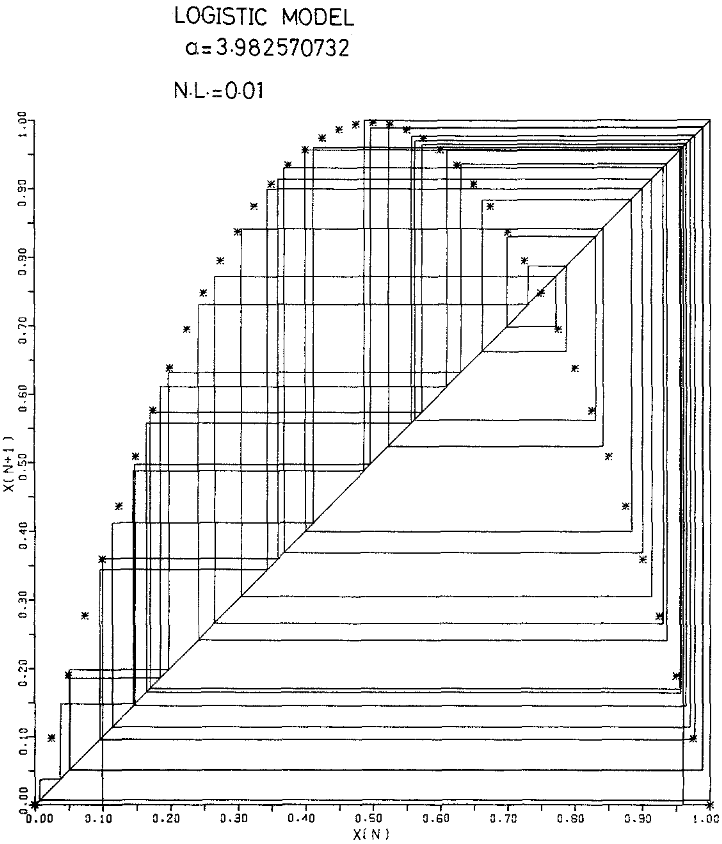


Fig. 8. Orbit of the same map as in Fig. 7. Noise level is 1.0×10^{-2}.

We call it the exponential model. As the control parameter α is varied, steepness is increased and the stable region about the critical point becomes small.

The other map is

$$f(x) = \text{const} \times \{\arctan[\beta(x - 0.2)] + \arctan(0.2\beta)\}/[1 + (2x)^{19}] + b$$

We call it the tangent model.[4] In this model as the parameter β is increased, the feature of the map on the right-hand side of the critical point does not change, but the steepness of the left slope is increased.

For these maps, we calculate the same features as those for the B–Z model. These calculations confirm the mechanism explained above. Typical examples are shown in Figs. 11 and 12. When α and β are small, namely,

[4] Const is chosen so as to give $f(x^*) = 0.8$.

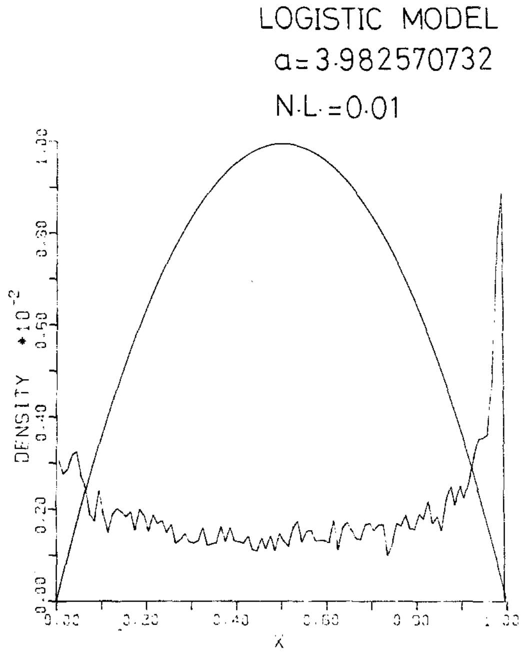


Fig. 9. Invariant density of the same map as in Fig. 7. Noise level is 1.0×10^{-2}.

the left slope of the map is less steep, the transition—noise-induced order—is not observed. On the other hand, when α and β are large, namely, the left slope is large enough, the transition is observed as expected.

6. SUMMARY AND DISCUSSIONS

The transition from chaos to order was observed as the noise level was increased. We call it noise-induced order. This kind of transition was attributed to the steepness of the map. However, global analysis of maps is necessary for understanding this transition more deeply and more quantitatively. This formulation is now in progress. It was also found that the logistic model was not of the type demonstrating this phenomenon.

The transition which was observed here in one-dimensional maps may be observed in flow systems. For example, in the Belousov–Zhabotinsky reaction, the nature of orbits in three-dimensional concentration space seems to suggest this.(12–14) There, extremely localized orbits exist and orbital delocalization(3) due to the saddle or the unstable manifold of the saddle is observed. Furthermore, in the Lorenz plot obtained from experimental data corresponding to chaos,(3) the density of the points in that plot is zero near the fixed point. Therefore, this situation should be considered noise-induced order, as the characteristics mentioned in Section 3 indicate.

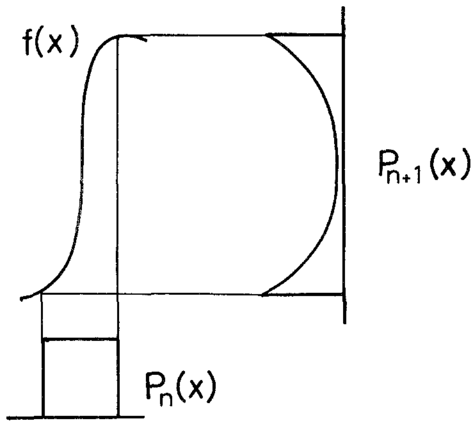


Fig. 10. The mechanism of the peak splitting (see text).

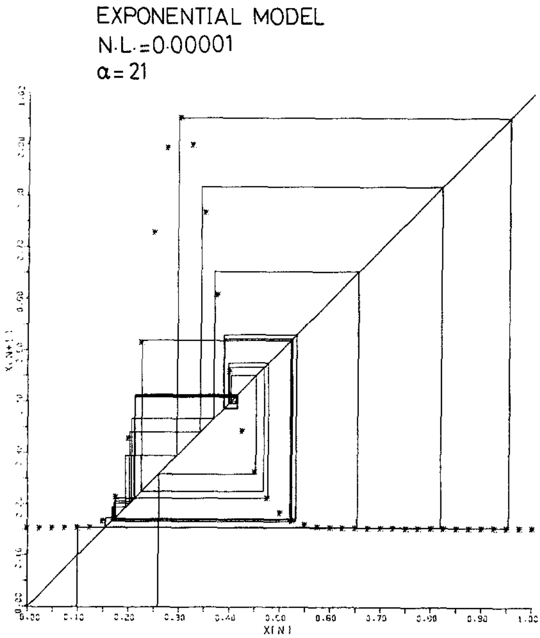
Thus it is expected that the noise-induced order would appear also in flow systems. We are now investigating in this direction.

Furthermore, our noise-induced order would give a new insight for results obtained by computer calculations. It has been believed that poor precision can produce chaos, for example, it is possible that a calculation with single precision produces chaos but one with double precision produces periodicity at the same value of the bifurcation parameter. However, our present observations indicate the opposite possibility. Namely, it is possible that a calculation with double precision produces chaos but one with single precision produces periodicity at the same value of the bifurcation parameter.

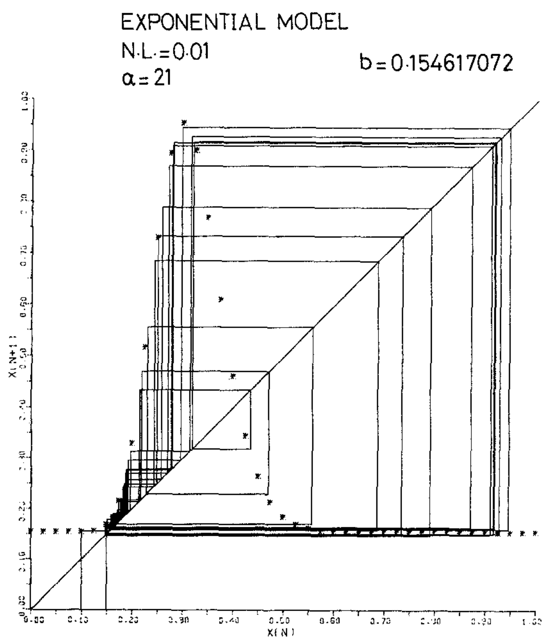
Finally, we should add one more result of calculations, supposing the following question from the practical point of view. Suppose that a periodic solution with noise is obtained in some experiment. Then how do we decide whether the system is chaotic or periodic in the noiseless case, i.e., the ideal case? As shown in Fig. 13, if the chaos exists in the ideal case, the peak of the power spectrum is continuously shifted as the bifurcation parameter is continuously varied. On the other hand, if the periodic solution exists in the ideal case, other peaks, for example, those of subharmonics can appear as the bifurcation parameter is continuously varied. Thus, we can distinguish the above two cases by studying the bifurcation at the same noise level.

NOTE ADDED IN PROOF

For the effects of noise on period doubling systems, see J. P. Crutchfield and B. A. Huberman (*Phys. Lett.* **77A**:407 (1980)) which was the first systematic study of external fluctuations on such systems and

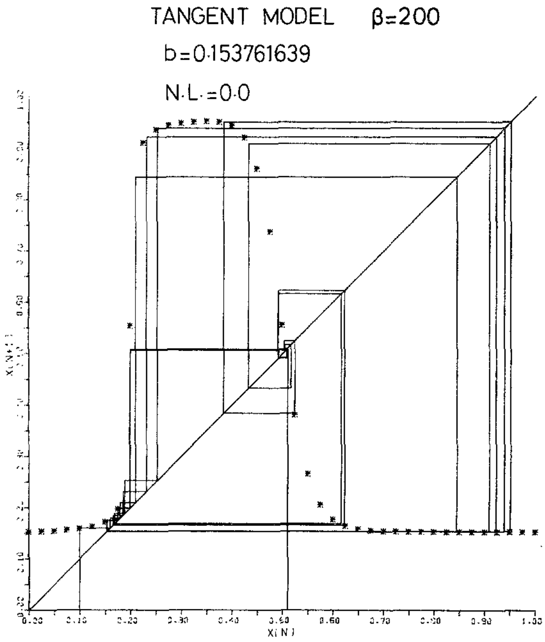


(a)



(b)

Fig. 11. Orbits indicating noise-induced order of the exponential model $\alpha = 21$ at $b = 0.154617072$. This is the Markov map of the type *RLLL.R*. (a) Noise level 1.0×10^{-5}. (b) Noise level 1.0×10^{-2}.



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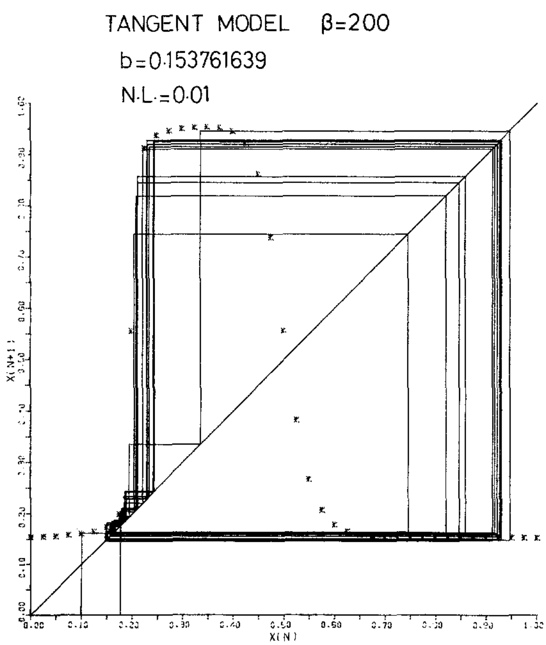

(b)

Fig. 12. Orbits of the tangent model $\beta = 200$ at $b = 0.153761639$. This is the Markov map of the type *RLLL.R*. (a) Noiseless case. (b) Noise level is 1.0×10^{-2}.

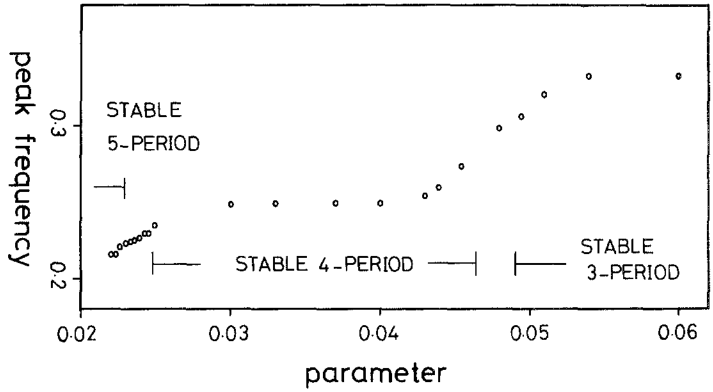

Fig. 13. The frequency of the peak in the power spectrum vs. the bifurcation parameter b in the B–Z model. Noise level is 1.0×10^{-2}, but the noise level dependence of the frequency is not noticeable.

reports the discovery of the bifurcation gap. See also T. Ohno (Publication RIMS Kyoto Univ., to be published) for the mathematical study of the noisy map.

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