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On Birth and Death Processes in Symmetric Random Environment

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We prove a limit theorem for a process in a random one-dimensional medium, which has been considered before as a model for hopping conduction in a disordered medium. To the edge between the two integers j and $(j+1)$ a rate $\lambda_j > 0$ is attached. These $\{\lambda_j\colon j \text{ integral}\}$ are taken as independent, identically distributed random variables, and represent the medium. For given values λ_j, $X(t)$ is a Markov chain in continuous time which jumps from j to $(j+1)$ and from $(j+1)$ to j at the same rate λ_j. We show that in many cases there exists normalizing constants $\gamma(t)$ (which tend to ∞ with t) such that the distribution of $X(t)/\gamma(t)$, or more generally of the whole process $\{X(st)/\gamma(t)\}_{s\geqslant 0}$, converges to a limit as $t\to\infty$. The limit process is continuous and self-similar.

KEY WORDS: Random environment; birth and death process; disordered one-dimensional system; hopping conductivity; limit theorems; invariance principle.

1. INTRODUCTION

We consider the following model: Let $\{\lambda_j\}_{-\infty<j<\infty}$ be a doubly infinite sequence of independent identically distributed random variables with values in $(0, \infty)$. These λ_j represent the random environment. $\{X_t\}_{t\geqslant 0}$ is an integer-valued process which denotes the position of a particle on the lattice $\mathbb{Z}$ at time t. λ_j is the rate at which X_t jumps from j to $j+1$ or from $j+1$ to j, when the environment is fixed. Note that the rate for jumping across the interval $(j, j+1)$ is the same in both directions, so that we should think of λ_j as a number attached to the interval $(j, j+1)$. This model was considered by many authors.[1,3–8,11,21,24] It is believed that the model describes various physical phenomena such as electrical lines of random conductances and

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hopping conduction in disordered media. Multidimensional analogs have also been considered; see Refs. 2 and 18, and for a continuous space analog, Ref. 17.

In a more formal way, we consider the $\{\lambda_j\}$ as random variables on a probability space $(\Omega', \mathfrak{F}', P')$ and denote by $\mathfrak{A} = \sigma\{\lambda_j : -\infty < j < \infty\}$ the σ field generated by the λ_js. Given the λ_js $X(\cdot)$ is a Markov chain whose transition rates are determined by

$$\begin{aligned} &P\{X(t+h) = j+1 \mid X(t) = j, \mathfrak{A}\} = \lambda_j h + o(h) \\ &P\{X(t+h) = j-1 \mid X(t) = j, \mathfrak{A}\} = \lambda_{j-1} h + o(h) \\ &P\{X(t+h) = j \mid X(t) = j, \mathfrak{A}\} = 1 - (\lambda_j + \lambda_{j-1}) h + o(h) \end{aligned} \tag{1}$$

as $h \downarrow 0$, for each j. In the articles mentioned above which deal with the one-dimensional case, the asymptotic behavior of the transition probability $P_n(t) = P\{X_t = n \mid X_0 = 0, \mathfrak{A}\}$ is discussed. These probabilities satisfy the "randomized master equation"

$$\frac{dP_j}{dt} = \lambda_{j-1} P_{j-1} + \lambda_j P_{j+1} - (\lambda_{j-1} + \lambda_j) P_j \tag{2}$$

which is a consequence of (1). A limit theorem for $P_n(t)$ is usually called a "local limit theorem" in the theory of stochastic processes.

The authors of [1] classified the possible distributions μ of λ_0 as follows (in other articles slightly different conditions are assumed):

Class (a): distributions whose density $\rho(\omega)$ satisfies

$$\int_0^\infty \omega^{-1} \rho(\omega)\, d\omega < \infty$$

Class (b): distributions whose density $\rho(\omega)$ satisfies

$$\rho(\omega) \to \text{constant as } \omega \to 0$$

Class (c): distributions whose density $\rho(\omega)$ satisfies

$$\rho(\omega) = \begin{cases} \alpha \omega^{-(1-\alpha)}, & 0 \leqslant \omega \leqslant 1 \\ 0, & \text{otherwise } 0 < \alpha < 1 \end{cases}$$

It is argued in Ref. 1 that the following results should hold:

For Class (a),

$$E'[P_0(t)] \sim c_1 t^{-1/2}, \qquad t \to \infty \tag{3}$$

For Class (b),

$$E'[P_0(t)] \sim c_2 \left(\frac{1}{t} \ln t\right)^{1/2}, \qquad t \to \infty \tag{4}$$

For Class (c),

$$E'[P_0(t)] \sim c_3 t^{-\alpha/(1+\alpha)}, \qquad t \to \infty \tag{5}$$

where c_1, c_2, c_3 are positive constants and E' is the expectation with respect to the measure P', the distribution of the environment. However, the authors of Refs. 1 and 4–8 used a "scaling assumption" for which no mathematical justification is given. Kaijser[15] gave a rigorous derivation of (3) [Class (a)].

Anshelevic and Vologodskii[3] considered a birth and death process $X^{(N)}$ on $\{0, 1/N, ..., (N-1)/N, 1\}$ with 0 and 1 as absorbing states. In Ref. 3 λ_j denotes the rate at which transitions take place from j/N to $(j+1)/N$ and from $(j+1)/N$ to j/N. It is assumed that for some constant $c > 0$

$$\lambda_j \geqslant c \qquad \text{for all } j \text{ and}$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda_j} \text{ exists and is strictly positive.}$$

Let[3]

$$P^{(N)}(\xi, \eta, t) = \begin{cases} NP\{X^{(N)}(N^2 t) = \lfloor N\eta \rfloor \mid X^{(N)}(0) = \lfloor N\xi \rfloor\} & \text{if } N^{-1} \leqslant \xi, \eta < 1 \\ 0 \quad \text{otherwise} \end{cases}$$

It is shown in Ref. 3 that $P^{(N)}(\xi, \eta, t)$ converges (as $N \to \infty$) to the transition density function of Brownian motion on (0, 1) with absorbing barriers at 0 and 1.

In our situation, if $E'\{\lambda_0^{-1}\} < \infty$—which corresponds to Class (a)—it is also possible to show that

$$\lim_{t\to\infty} \sqrt{t}\, P\{X(t) = \lfloor \xi\sqrt{t} \rfloor \mid X(0) = 0, \mathfrak{A}\} = \left(\frac{E\{\lambda_0^{-1}\}}{2\pi}\right)^{1/2} \cdot \exp -\frac{\xi^2}{2} E\{\lambda_0^{-1}\}$$

for almost all environments. However, we have been unable to prove a local limit theorem in the other cases. Here we only prove the global limit theorem given below. $\{Y(t)\}_{t\geqslant 0}$ is a Brownian motion, and $D = D([0, \infty), \mathbb{R})$ is the

[3] $\lfloor a \rfloor$ denotes the largest interger $\leqslant a$.

class of right continuous functions from $[0, \infty)$ to $\mathbb{R}$ which have left limits everywhere. See Ref. 9 for this function space and for the definition of weak convergence. [19] and [26] are the standard references for weak convergence on infinite intervals.

Theorem. (i) If

$$E'\left\{\frac{1}{\lambda_0}\right\} < \infty \tag{6}$$

then in almost all environments $[P']$, the process $\{X_n(t)\} := \{(1/n)\, X(n^2 t)\}$ (conditioned on the environment) converges weakly in D to $\{(E\{\lambda_0^{-1}\})^{-1/2}\, Y(t)\}$ (as $n \to \infty$).

(ii) If there exists a slowly varying function $L_1(\cdot)$ such that

$$\frac{1}{nL_1(n)} \sum_{i=0}^{n} \frac{1}{\lambda_i} \to 1 \qquad \text{in probability} \tag{7}$$

for some slowly varying function $L_1(\cdot)$, then the process $\{X_n(t)\} := \{(1/n)\, X(n^2 L_1(n)\, t)\}$ converges weakly in D to $\{Y(t)\}$.

(iii) Let λ_0^{-1} belong to the domain of attraction of a one-sided stable distribution of index α, $0 < \alpha < 1$, that is, assume that for some slowly varying function $L_2(\cdot)$

$$\{n^{1/\alpha} L_2(n)\}^{-1} \sum_{k=0}^{n} \frac{1}{\lambda_k} \tag{8}$$

converges in law to a one-sided stable distribution of index α. Then the process $\{X_n(t)\} := \{(1/n)\, X(n^{(1+\alpha)/\alpha} L_2(n)\, t)\}$ converges weakly in D to a continuous process X_* as $n \to \infty$. X_* is self-similar with exponent $\alpha/(\alpha + 1)$. (It is defined more explicitly in Section 4.)

Remarks. (i) Case (i), with $E'\{1/\lambda_0\} < \infty$ was already well understood and the results are quite complete [see also Remark (iii) below]. The main novelty of our results is for the cases (ii) and (iii). Note that in these cases we do not obtain any results which hold in almost all environments, but only limit theorems when the randomness of the environment is also taken into account.

(ii) The cases (i)–(iii) correspond to the cases (a)–(c), respectively, of Ref. 1. However, our classes are wider than those of Ref. 1. For instance, in case (b) listed above

$$P\left\{\frac{1}{\lambda_0} \geqslant y\right\} \sim \frac{C}{y}, \qquad y \to \infty$$

for some constant $C > 0$. It follows from the generalized weak law of large numbers (Ref. 13, Theorem VII.7.2) that in this case (7) holds with

$$L_1(n) = C \log n$$

More generally, by Ref. 13, Theorem VII.7.2, (7) is equivalent to

$$\lim_{y \to \infty} \frac{E\{1/\lambda_0; 1/\lambda_0 \leqslant y\}}{yP\{1/\lambda_0 > y\}} = \infty$$

Similarly, it follows from the determination of the domain of attraction of a stable law (Ref. 13, Chap. XVII.5), that (8) holds if and only if

$$P\left\{\frac{1}{\lambda_0} > y\right\} \sim y^{-\alpha} L_3(y), \qquad y \to \infty$$

for some slowly varying function L_3.

(iii) The analog of case (i) in a continuous space setting is already treated by Papanicolaou and Varadhan in Ref. 20.

2. PRELIMINARIES

Lemma 1. Set

$$S(j) = \begin{cases} \sum_{k=0}^{j-1} \lambda_k^{-1}, & j > 0 \\ 0, & j = 0 \\ -\sum_{k=j}^{-1} \lambda_k^{-1}, & j < 0 \end{cases} \tag{9}$$

Then, conditioned on $\mathfrak{A}$, $S(X(t))$ is in natural scale, i.e.,

$$P\{S(X(t)) \text{ hits } \{a, b\} \text{ first at } a \mid S(X(0)) = x, \mathfrak{A}\} = (b - x)/(b - a) \tag{10}$$

where $x, a, b \in S(\mathbb{Z})$, $a < x < b$, $\mathbb{Z}$ is the set of all integers.

Proof. By a well-known result for birth and death processes (cf. Karlin and Taylor Ref. 16, p. 133),

$$P\{X(t) \text{ hits } \{i-1, i+1\} \text{ first at } i-1 \mid X(0) = i, \mathfrak{A}\}$$
$$= \frac{\lambda_{i-1}}{\lambda_{i-1} + \lambda_i} = \frac{S(i+1) - S(i)}{S(i+1) - S(i-1)}, \qquad \text{for every } i \in \mathbb{Z}$$

Thus (9) holds for $x, a, b = S(i), S(i-1), S(i+1)$. It is easy now to prove (9) by induction on $b - a$. ∎

We may assume that the process $\{S(X(t))\}$ is the process obtained by a time change of a Brownian motion $\{Y(t)\}$ on some probability space $(\Omega'', \mathfrak{F}'', P'')$. (cf. Stone[25]). This time change proceeds as follows. Let $L(t, x)$ denote the local time at x of $\{Y(t)\}$ (cf. Ref. 25 or 10, Chap. V.3), and let the measure m be given by

$$m(dx) = \sum_i \delta_{S(i)}(dx)$$

where δ_x is the Dirac measure at $\{x\}$. Set

$$V(t) = \int L(t, x)\, m(dx) = \sum_i L(t, S(i)) \tag{11}$$

Henceforth we define the inverse function of a right continuous function $h(x)$ with left limits on the real line by

$$h^{-1}(x) = \inf\{u : h(u) > t\}$$

Then $\{S(X(t))\}_{t \geqslant 0}$ is equivalent to $\{Y(V^{-1}(t))\}_{t \geqslant 0}$. This follows from the following observations: $Y(V^{-1}(\cdot))$ takes on only values in $\operatorname{supp}(m) = \{S(i), i \in \mathbb{Z}\}$, because only values of t for which $Y(t) \in \operatorname{supp}(m)$ can occur as values of $V^{-1}(s)$ for some s. When $Y(V^{-1}(\cdot))$ leaves $S(i)$ it jumps to $S(i-1)$ or to $S(i+1)$ and

$$\begin{aligned}&P\{Y(V^{-1}(\cdot)) \text{ jumps to } S(i+1) \text{ before it jumps to}\\ &S(i-1) \mid Y(V^{-1}(0)) = S(i), \mathfrak{A}\} = \frac{\lambda_i}{\lambda_i + \lambda_{i-1}} = P\{S(X(t)) \text{ hits}\\ &\{S(i-1), S(i+1)\} \text{ first in } S(i+1) \mid S(X(0)) = i, \mathfrak{A}\}\end{aligned}$$

We can also compare the amount of time which $Y(V^{-1}(\cdot))$ and $S(X(\cdot))$ spend at a point $S(i)$ before jumping to $S(i+1)$ or $S(i-1)$. More precisely we compare the so-called holding times at $S(i)$, i.e., the time between an entry into the state $S(i)$ and the first time thereafter when the process hits $S(i-1)$ or $S(i+1)$. One can show that this holding time has an exponential distribution both for $Y(V^{-1}(\cdot))$ and for $S(X(\cdot))$. If $Y(\cdot)$ hits $S(i)$ at t_1 and the first hitting time of $\{S(i-1), S(i+1)\}$ after t_1 is t_2, then the corresponding holding time for $Y(V^{-1}(\cdot))$ is $V(t_2) - V(t_1) = L(t_2, S(i)) - L(t_1, S(i))$ [since $L(t, S(j)]$ is constant on $[t_1, t_2]$ for all $j \neq i$). Thus the holding time for $Y(V^{-1}(\cdot))$ reduces to an increment of the local time of Y and

$$\begin{aligned}&E''\{Y(V^{-1}(t))\text{s holding time at } S(i) \mid Y(V^{-1}(0)) = S(i), \mathfrak{A}\}\\ &= \frac{(S(i+1) - S(i))(S(i) - S(i-1))}{S(i+1) - S(i-1)} = \frac{1}{\lambda_i + \lambda_{i-1}}\end{aligned}$$

(cf. Stone[25] for the first equality and (9) for the second), and by the theory of birth and death processes

$$E''\{X(t)\text{s holding time at } i \mid X(0)=i, \mathfrak{A}\} = \frac{1}{\lambda_i + \lambda_{i-1}}$$

Finally, both $Y(V^{-1}(\cdot))$ and $S(X(\cdot))$ are right continuous and have left limits. Thus these two processes are equivalent. In the sequel we shall assume that $S(X(t))$ equals $Y(V^{-1}(t))$.

3. PREPARATION FOR THE PROOF OF THE THEOREM

Let D_0 be the set of functions on the real line which have left limits and are right continuous and are increasing.[4] D_0 is equipped with the Skorohod metric d_0. (D_0, d_0) is a separable metric space (cf. Billingsley[9]).

In this section, we consider the case in which λ_0^{-1} belongs to the domain of attraction of an α-stable distribution, $0 < \alpha < 1$. Thus we assume (8), but for simplicity we restrict ourselves to the case $L_2(x) \equiv 1$. Define

$$S_n(x) = n^{-1/\alpha} S(\lfloor nx \rfloor), \qquad n = 1, 2, \ldots, -\infty < x < \infty \tag{12}$$

Then by results of Skorohod,[23] $\{S_n\}$ converges weakly to an α-stable process $\{W\}$. W is defined on some probability space $(\Omega^0, \mathfrak{F}^0, P^0)$. $\{W\}$ is a D_0-valued process and W is strictly increasing with probability 1 on all of $\mathbb{R}$. Indeed by the Ito–Lévy representation (Ref. 14, Chaps. 1.8–1.12), for $x \geqslant 0$

$$W(x) = \sum_{0<y<x} (W(y+) - W(y-))$$

where the jumps at location $(x, W(x+) - W(x-))$ have a Poission distribution on $[0, \infty) \times [0, \infty)$ with density $ds \cdot w^{-\alpha}\, dw$. Since $\int w^{-\alpha}\, dw = \infty$, there are infinitely many jumps in each open time interval. A similar statement holds for $x < 0$.

By the arguments of Skorohod[23] and Dudley,[12] there exists a probability space $(\Omega, \mathfrak{F}, \mu)$ and D_0-valued random variables $\tilde{S}_n$, and $\tilde{W}$ on $(\Omega, \mathfrak{F}, \mu)$ such that $\tilde{S}_n(\cdot)$ converges to $\tilde{W}(\cdot)$ in the Skorohod metric on D_0 for almost all $\omega[\mu]$, and $\tilde{S}_n$ and S_n have the same distribution. Also $\tilde{W}$ and W have the same distribution. For $\omega \in \Omega$, let us define random masures $\tilde{m}_n(dx, \omega)$ and $\tilde{m}_*(dx, \omega)$ on the real line by

$$\int f(x)\, \tilde{m}_n(dx) = \int f(\tilde{S}_n(x, \omega))\, dx \tag{13}$$

[4] We say that f is increasing if $f(t_1) \geqslant f(t_2)$ for $t_1 > t_2$. If $f(t_1) > f(t_2)$ for $t_1 > t_2$ we call f strictly increasing.

and

$$\int f(x)\,\tilde{m}_*(dx) = \int f(\tilde{W}(x,\omega))\,dx \tag{14}$$

for every function $f \in C_0(\mathbb{R})$ (=the set of continuous functions with compact support on the real line).

Lemma 2. $\{Y(\tilde{V}_n^{-1}(t))\}$ converges to $\{Y(\tilde{V}_*^{-1}(t))\}$ in the J_1 topology almost everywhere in $\Omega \times \Omega''$, where

$$\tilde{V}_n(t) = \int L(t,x)\,\tilde{m}_n(dx)$$

and

$$\tilde{V}_*(t) = \int L(t,x)\,\tilde{m}_*(dx)$$

Proof. We shall check that for almost all $\omega[\mu]$ the measures $\tilde{m}_n(\cdot,\omega)$ and $\tilde{m}_*(\cdot,\omega)$ satisfy the conditions (i)–(viii) of Stone's Theorem 1.[25] [Condition (ix) is not needed, since it is only used by Stone for the distribution of functionals which we do not consider.] All these conditions except (iv) are obvious (since we have $k \equiv 0$ and $a = -\infty$, $b = \infty$ in our case). For (iv), we have to check that if $x_n \in \operatorname{supp}(\tilde{m}_n)$ and $x_n \to x_0$, then $x_0 \in \operatorname{supp}(\tilde{m}_*)$. But since $\operatorname{supp}(\tilde{m}_n(\cdot,\omega))$ consists of the values of $\tilde{S}_n(\cdot,\omega)$, we may assume that $x_n = \tilde{S}_n(a_n,\omega)$ and that $a_n \to a_0$ for some $a_0 \in \mathbb{R}$ (recall that $|\tilde{S}_n(t,\omega)| \to \infty$ as $|t| \to \infty$). Since $\tilde{S}_n$ converges to $\tilde{W}$ in the Skorohod metric, there exist continuous functions ξ_n on $(-\infty,\infty)$ which for every compact set K in $(-\infty,\infty)$ satisfy

$$\lim_{n\to\infty} \sup_{t\in K} |t - \xi_n(t)| = 0$$

and

$$\lim_{n\to\infty} \sup_{t\in K} |\tilde{S}_n(t) - \tilde{W}(\xi_n(t))| = 0$$

This implies that $x_0 = \tilde{W}(a_0)$ or $\tilde{W}(a_0-)$. Both of these belong to $\operatorname{supp}(\tilde{m}_*)$, because $\tilde{W}$ belongs to D. Consequently we have

$$\overline{\lim_{n\to\infty}}\ \operatorname{supp}(\tilde{m}_n) \subset \operatorname{supp}(\tilde{m}_*), \qquad \text{almost everywhere } [\mu]$$

Thus for almost all ω Stone's theorem can be applied and the lemma follows. ∎

Set

$$\tilde{T}(x) = \tilde{W}^{-1}(x) \qquad \text{for } x \in R \tag{15}$$

Since $\{\tilde{W}\}$ is strictly increasing, $\tilde{T}(x)$ has continuous increasing paths almost everywhere, $[\mu]$. This implies that for almost all $\omega[\mu]$, $\tilde{S}_n^{-1}$ converges to $\tilde{T}$ uniformly on compact intervals.

Proposition 1. $\{\tilde{S}_n^{-1}(Y(\tilde{V}_n^{-1}))\}$ converges to $\{\tilde{T}(Y(\tilde{V}_*^{-1}))\}$ in the J_1 topology on D a.e., on $\Omega \times \Omega''$.

Proof. By Lemma 2, there exists for almost all (ω, ω'') a continuous one-to-one function θ_n on $[0, \infty)$ such that

$$\lim_{n\to\infty} \sup_{t\in K} |\theta_n(t) - t| = 0$$

and

$$\lim_{n\to\infty} \sup_{t\in K} |Y(\tilde{V}_n^{-1}(t)) - Y(\tilde{V}_*^{-1}(\theta_n(t)))| = 0$$

for every compact subset K in $[0, \infty)$. This implies

$$\begin{aligned}
&\sup_{t\in K} |\tilde{S}_n^{-1}(Y(\tilde{V}_n^{-1}(t))) - \tilde{T}(Y(\tilde{V}_*^{-1}(\theta_n(t)))| \\
&\quad \leqslant \sup_{t\in K} |\tilde{S}_n^{-1}(Y(\tilde{V}_n^{-1}(t))) - \tilde{T}(Y(\tilde{V}_n^{-1}(t)))| \\
&\qquad + \sup_{t\in K} |\tilde{T}(Y(\tilde{V}_n^{-1}(t))) - \tilde{T}(Y(\tilde{V}_*^{-1}(\theta_n(t)))| \to 0
\end{aligned}$$

as $n \to \infty$, since $\tilde{T}$ is continuous on all of $\mathbb{R}$, and $\tilde{S}_n^{-1} \to \tilde{T}$ uniformly on compact sets with probability 1.

Lemma 3. Set $\tilde{X}_*(t) = \tilde{T}(Y(\tilde{V}_*^{-1}(t)))$. Then $\{\tilde{X}_*(t)\}$ is a self-similar process with exponent $\alpha/(\alpha+1)$.

Proof. We use the notation $X \overset{d}{=} Y$ to denote that the variables X and Y have the same distribution.

Since $\tilde{W}$ is an α-stable process, $\{\tilde{W}(x)\} \overset{d}{=} \{\eta^{1/\alpha} \tilde{W}(\eta x)\}$ for every $\eta > 0$. Set

$$\tilde{W}_\eta(x) = \eta^{-1/\alpha} \tilde{W}(\eta x), \qquad \text{for } \eta > 0$$

and

$$\tilde{T}_\eta(x) = \tilde{W}_\eta^{-1}(x)$$

Then

$$\tilde{T}_\eta(x) = \eta^{-1}\tilde{T}(\eta^{1/\alpha}x)$$

Let

$$Y_\gamma(t) = \gamma^{-1}Y(\gamma^2 t), \qquad \gamma \in R$$

then $\{Y_\gamma\}$ is a Brownian motion and its local time $L_\gamma(t, x)$ equals

$$L_\gamma(t, x) = \gamma^{-1}L(\gamma^2 t, \gamma x)$$

Set

$$\tilde{V}_\gamma(t, \tilde{W}) = \int L_\gamma(t, \tilde{W}(x))\,dx = \gamma^{-1}\int L(\gamma^2 t, \gamma\tilde{W}(x))\,dx$$

Then for $\gamma > 0$, $\eta > 0$

$$(Y, \tilde{T}, \tilde{W}, \tilde{V}_*) \stackrel{d}{=} (Y_\gamma, \tilde{T}_\eta, \tilde{W}_\eta, \tilde{V}_\gamma(\cdot, \tilde{W}_\eta))$$

and consequently

$$\begin{aligned}\{\tilde{X}_*(at)\} &= \{\tilde{T}(Y(\tilde{V}_*^{-1}(at)))\} \\ &\stackrel{d}{=} \{\tilde{T}_\eta(Y_\gamma(\tilde{V}_\gamma^{-1}(at, \tilde{W}_\eta)))\}\end{aligned}$$

Now take $\gamma = a^{-1/(\alpha+1)}$, $\eta = a^{-\alpha/(\alpha+1)}$. By a simple algebraic calculation, we see that

$$\begin{aligned}&\tilde{T}_\eta(Y_\gamma(\tilde{V}_\gamma^{-1}(at, \tilde{W}_\eta))) \\ &\qquad a^{\alpha/(\alpha+1)}\tilde{T}(Y(\tilde{V}_*^{-1}(t))) = a^{\alpha/(\alpha+1)}\tilde{X}_*(t)\end{aligned}$$

So it holds that

$$\{\tilde{X}_*(at)\} \stackrel{d}{=} \{a^{\alpha/(\alpha+1)}\tilde{X}_*(t)\} \qquad \text{for } a > 0 \quad \blacksquare$$

Lemma 4. The process $\{\tilde{X}_*(t)\}_{t\geq 0}$ of Lemma 3 is continuous almost everywhere on $\Omega \times \Omega''$.

Proof. Fix $(\omega, \omega'') \in \Omega \times \Omega''$ such that $\tilde{W}(\cdot, \omega)$ is right continuous, has left limits, and is strictly increasing on all of $\mathbb{R}$, and such that $Y(\cdot, \omega'')$ is everywhere continuous, and such that $L(\cdot, \cdot, \omega'')$ has the following properties:

(i) $(t, x) \to L(t, x, \omega'')$ is everywhere continuous;

(ii) $t \to L(t, x, \omega'')$ is increasing for all x;

(iii) $\tilde{V}_*(r_2, \omega, \omega'') > \tilde{V}_*(r_1, \omega, \omega'')$ for all rational intervals (r_1, r_2) which contain a time t with $Y(t, \omega'') \in R(\omega)$, where $R(\omega) =$ closure of the range of $\tilde{W}(\cdot, \omega)$.

As remarked at the beginning of this section $\tilde{W}(\cdot, \omega)$ is almost everywhere $[\mu]$ strictly increasing. It is also well known that for fixed ω (and hence also $R(\omega)$ fixed) $L(\cdot, \cdot, \omega'')$ has the properties (i)–(iii) for almost all ω. Indeed (i) and (ii) are standard (see Ref. 10, Chap. V. 3), and (iii) follows by considering

$$T(r_1) = \inf\{t > r_1 : Y(t, \omega'') \in R(\omega)\}$$

Then $L(T(r_1) + s, Y(T(r_1))) - L(T(r_1), Y(T(r_1))) > 0$ for all $s > 0$ for almost all ω'', by Theorem V. 3.5 in Ref. 10. Then continuity of $(t, x) \to L(t, x)$ shows

$$L(T(r_1) + s, y) - L(T(r_1), y) > 0$$

for all y in some neighborhood of $Y(T(r_1)) \in$ closure of range of $\tilde{W}$. This implies $\tilde{V}_*(T(r_1) + s) > \tilde{V}_*(T(r_1))$.

We leave it to the reader to fill in the measure theoretic details to show that all the above properties hold for almost all pairs $(\omega, \omega'') \in \Omega \times \Omega''$. For an (ω, ω'') with these properties $t \to \tilde{V}_*^{-1}(t, \omega, \omega'')$ is everywhere right continuous and $x \to \tilde{T}(x, \omega)$ is everywhere continuous. Thus $\tilde{X}_*(\cdot, \omega, \omega'')$ is everywhere right continuous and continuous at each continuity point of $\tilde{V}_*^{-1}$. Now let t_0 be a jump point of V_*^{-1} and let $a = \tilde{V}_*^{-1}(t_0 -, \omega, \omega'')$, $b = \tilde{V}_*^{-1}(t_0, \omega, \omega'')$. Then $a < b$ and $\tilde{V}_*(a, \omega, \omega'') = \tilde{V}_*(b, \omega, \omega'') = t_0$. Consequently $Y(t) \notin R(\omega)$ for $a < t < b$. Since $\tilde{W}(\cdot, \omega) \in D$ and $Y(\cdot, \omega'')$ is continuous this can only happen

$$\{Y(t, \omega''): a < t < b\} \subset (\tilde{W}(s_0 -, \omega), \tilde{W}(s_0, \omega))$$

for some jump time s_0 of $\tilde{W}$. But then

$$Y(a, \omega''), Y(b, \omega'') \in [\tilde{W}(s_0 -, \omega), \tilde{W}(s_0, \omega)]$$

and since $\tilde{W}$ is strictly increasing

$$\begin{aligned} s_0 = \tilde{T}(\tilde{W}(s_0 -, \omega)) &\leqslant \tilde{T}(Y(a, \omega'')), \tilde{T}(Y(b, \omega'')) \\ &\leqslant \tilde{T}(\tilde{W}(s_0, \omega)) = s_0 \end{aligned}$$

Thus $\tilde{X}_*$ is continuous at t_0 as well.

4. PROOF OF PART (iii) OF THE THEOREM

Put

$$T(x) = W^{-1}(x), \qquad x \in R \tag{16}$$

$$V_*(t) = \int L(t, W(x))\, dx \tag{17}$$

and

$$X_*(t) = T(Y(V_*^{-1}(t))) \tag{18}$$

Note that

$$Y_n(t) = n^{-1/\alpha} Y(n^{2/\alpha} t)$$

is a Brownian motion with local time $L_n(t, x) = n^{-1/\alpha} L(n^{2/\alpha} t, n^{1/\alpha} x)$. Set

$$V_n(t) = \int L_n(t, S_n(x))\, dx$$

Then we see from (12) and (11) that

$$V_n(t) = n^{-(\alpha+1)/\alpha} V(n^{2/\alpha} t)$$
$$V_n^{-1}(t) = n^{-2/\alpha} V^{-1}(n^{(\alpha+1)/\alpha} t)$$

This implies (see end of Section 2)

$$\begin{aligned} Y_n(V_n^{-1}(t)) &= n^{-1/\alpha} Y(n^{2/\alpha} V_n^{-1}(t)) \\ &= S_n(X_n(t)) \end{aligned} \tag{19}$$

Since $S_n(\cdot)$ has a strictly positive jump at each point k/n, $k \in \mathbb{Z}$, one has

$$|S_n^{-1}(Y_n(V_n^{-1}(t))) - X_n(t)| \leqslant \frac{1}{n} \tag{20}$$

Also

$$\{S_n^{-1}(Y_n(V_n^{-1}(t)))\}_{t \geqslant 0} \stackrel{d}{=} \{\tilde{S}_n^{-1}(Y(\tilde{V}_n^{-1}(t)))\}_{t \geqslant 0}$$

Proposition 1 now shows that $\{X_n\}$ converges weakly on D to $\{X_*\}$, since $\{X_*\} \stackrel{d}{=} \{\tilde{X}_*\}$. Lemmas 3 and 4 show that X_* is self-similar and continuous.

5. PROOF OF PART (ii) OF THE THEOREM

We can apply the same procedure for the proof of part (ii) as used in Sections 3 and 4. (12) should now be replaced by

$$S_n(x) := \frac{1}{nL_1(n)} S(\lfloor nx \rfloor), \qquad n = 1, 2,..., -\infty < x < \infty \tag{21}$$

Now $\{S_n(x)\}$ converges weakly on D to $\{x\}$ as $n \to \infty$. This follows from the fact that $x \to S_n(x)$ is increasing, and $S_n(x) \to x$ in probability for each fixed x. The (deterministic) process $\{x\}$ takes over the role of $\{W(x)\}$ in Sections 3 and 4. Also $T(x) = W^{-1}(x) = x$ in this case, and

$$\sup_{|x| \leqslant K} |S_n^{-1}(x) - x| \to 0 \qquad \text{in probability}$$

for each K as $n \to \infty$. Now set

$$Y_n(t) = \frac{1}{nL_1(n)} Y(tn^2L_1^2(n))$$

$$V_n(t) = (nL_1(n))^{-1} \int L(tn^2L_1^2(n), S_n(x)\, nL_1(n))\, dx$$

$$= (n^2L_1(n))^{-1} V(tn^2L_1^2(n))$$

Then we can show that $\{S_n^{-1}(Y_n(V_n^{-1}))\}$ converges weakly on D to $\{T^{-1}(Y(V_*^{-1}))\}$, with

$$V_*(t) = \int L(t, x)\, dx = t$$

by a repetition of the arguments of Sections 3 and 4. Since $T(x) = x$, $T^{-1}(Y(V_*^{-1})) = Y$, while a simple calculation shows

$$Y_n(V_n^{-1}(t)) = (nL_1(n))^{-1} Y(V^{-1}(tn^2L_1(n))$$

$$= S_n\left(\frac{1}{n} X(n^2L_1(n)t)\right)$$

[compare (19)]. Finally (20) is replaced by

$$\left| S_n^{-1}(Y_n(V_n^{-1}(t))) - \frac{1}{n} X(n^2L_1(n)t) \right| \leqslant \frac{1}{n}$$

Assertion (ii) follows from these observations.

6. PROOF OF PART (i) OF THE THEOREM

When $E'\{\lambda_0^{-1}\} < \infty$, the strong law of large numbers implies

$$S_n(x) := \frac{1}{n} S(\lfloor nx \rfloor \to E' \left\{\frac{1}{\lambda_0}\right\} x \qquad \text{almost everywhere } [P'] \tag{22}$$

Set

$$\beta = E' \left\{\frac{1}{\lambda_0}\right\}$$

Since the increasing function $S_n(x)$ has the continuous strictly increasing limit βx, we have in almost all environments

$$\begin{aligned} S_n(x) &\to \beta x && \text{uniformly on compact sets} \\ \text{and } S_n^{-1}(x) &\to \beta^{-1} x && \text{uniformly on compact sets} \end{aligned} \tag{23}$$

Now set

$$Y_n(t) = n^{-1} Y(n^2 t), \qquad L_n(t, x) = n^{-1} L(n^2 t, nx)$$

and

$$V_n(t) = \int L_n(t, S_n(x))\, dx = n^{-2} V(n^2 t)$$

The analogs of (19) and (20) are this time

$$Y_n(V_n^{-1}(t)) = S_n \left(\frac{1}{n} X(n^2 t)\right)$$

and

$$\left| S_n^{-1}(Y_n(V_n^{-1}(t))) - \frac{1}{n} X(n^2 t) \right| \leqslant \frac{1}{n}$$

One sees directly that, for any fixed environment for which (23) holds $\{S_n^{-1}(Y_n(V_n^{-1}(t)))\}$ converges weakly in D to $\{\beta^{-1} Y(\beta t)\}$ with respect to the P'' measure. Indeed $V_n(\cdot)$ is continuous and increasing and (still with the environment fixed) has the same distribution with respect to P'' as $\int L(t, S_n(x))\, dx$. Thus, for any environment for which (23) holds, $V_n(t)$ converges in law to $\int L(t, \beta x)\, dx = t/\beta$ for each fixed t. This easily implies that for any such environment and any $K < \infty$, $\varepsilon > 0$

$$P''\{\sup_{t \leqslant K} |V_n(t) - t/\beta| \geqslant \varepsilon \qquad \text{or} \qquad \sup_{t \leqslant K} |V_n^{-1}(t) - \beta t| \geqslant \varepsilon\} \to 0 \qquad (n \to \infty)$$

From this the required convergence is immediate. Alternatively, we could have used Theorem 1 of Ref. 25 to prove the above weak convergence, as in Lemma 2 with $\tilde{W}(x) = \beta x$. Statement (i) of the theorem follows since $\beta^{-1} Y(\beta t) \stackrel{d}{=} \beta^{-1/2} Y(t)$.

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