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Statistical Macrodynamics of Large Dynamical Systems. Case of a Phase Transition in Oscillator Communities

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A model dynamical system with a great many degrees of freedom is proposed for which the critical condition for the onset of collective oscillations, the evolution of a suitably defined order parameter, and its fluctuations around steady states can be studied analytically. This is a rotator model appropriate for a large population of limit cycle oscillators. It is assumed that the natural frequencies of the oscillators are distributed and that each oscillator interacts with all the others uniformly. An exact self-consistent equation for the stationary amplitude of the collective oscillation is derived and is extended to a dynamical form. This dynamical extension is carried out near the transition point where the characteristic time scales of the order parameter and of the individual oscillators become well separated from each other. The macroscopic evolution equation thus obtained generally involves a fluctuating term whose irregular temporal variation comes from a deterministic torus motion of a subpopulation. The analysis of this equation reveals order parameter behavior qualitatively different from that in thermodynamic phase transitions, especially in that the critical fluctuations in the present system are extremely small.

KEY WORDS: Large dissipative system; population of limit cycle oscillators; order parameter; phase transition via mutual entrainment; approximate invariant measure; dynamical extension of self-consistent equation; critical slowing down; anomalous critical fluctuation.

1. INTRODUCTION

When a pair of limit cycle oscillators with different natural frequencies are coupled, they often come to oscillate with an identical frequency.[1] This is called mutual synchronization or mutual entrainment, and is commonly met in many scientific areas, including nonlinear optics, electrical

Dedicated to Ilya Prigogine on the occasion of his 70th birthday.

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engineering, fluids, chemical reactions, physiology, and so on, i.e., wherever limit cycle oscillations arise at all.[2] It may as often happen that their frequencies remain independent of each other, and in that case the system as a whole should exhibit a quasiperiodic motion. Admittedly, coupled oscillatory processes could also generate many other complicated behaviors, such as frequency locking in various frequency ratios and chaotic motion.[3] However, insofar as the mutual coupling is sufficiently weak and the oscillators are sufficiently similar, we have only two possibilities, one-to-one synchronization and quasiperiodicity.[4] The dynamics of a loosely coupled pair of oscillators is thus almost trivially simple, and this fact will be a great advantage in extending the scope of study to systems of infinitely many oscillators in weak mutual contact.

Physiological implications of these kinds of oscillator communities were fully appreciated by Winfree,[5,6] who was the first to point out the possibility that large oscillator populations with a frequency distribution exhibit a peculiar collective organization or phase transition.[5] In the phase transitions he described the system changes from a macroscopically quiescent phase to a collectively oscillating phase at some critical coupling strength. This is quite dramatic and a number of related theoretical works have since appeared.[7-11] Is it right to say that Prigogine's concept of *time order*,[12] which refers to the spontaneous emergence of macroscopic rhythms in nonequilibrium open systems, found its finest example in this transition phenomenon, all the more so because the cooperative implications of the word would be hard to capture solely by the Hopf bifurcation idea? Is it also right to say that the concept of *order through fluctuations*[13] has now acquired an even deeper implication, because the fluctuations here need no longer be imagined as something supplementary to the dynamics, but should rather be considered as something inevitably arising from the governing deterministic law of motion?

It seems that much of fresh significance beyond physiological relevance could be derived from Winfree's important finding (in 1967) after our experience of the great advances in the field of nonlinear dynamics over the last two decades. This belief is confirmed by the fact that oscillator populations, if properly modeled, constitute typical "complex systems" (in the modern usage of the word), representative examples of which include cellular automata[14] and lattices of coupled mappings.[15] The theory developed below was partly motivated by such newer trends in nonlinear dynamics. We shall deal with a transition phenomenon exhibited by a large dynamical system of dissipative nature, and try to make clear some statistical mechanical aspects of the transition. Our primary concern is to reduce the microscopic dynamics, i.e., the dynamics at the level of the individual oscillators, to that at a macroscopic level, or, in other words, to extract an order parameter evolution equation in a closed form whereby some fluctuations may necessarily be involved.

By statistical mechanics we do not mean the traditional one, because what we have to deal with is totally unlike the Hamiltonian system. The stochasticity involved is by no means extrinsic, but comes from the complicated nature of the solutions to a large set of ordinary differential equations. In dynamical system language, we expect that an ergodic motion on a high-dimensional attractor is going on, and if this is the case, all statistical properties could in principle be determined from the corresponding invariant measure. However, a naive application of the invariant measure to statistical calculations would simply produce long-time averages, by which the most crucial feature of temporal order would be lost. Fortunately, in the present theory, using a rotator model under mean-field coupling, such a problem does not arise if one constructs an approximate invariant measure in a self-consistent manner, and one will find how the invariant measure idea and temporal symmetry breaking can be reconciled with each other.

This paper is organized as follows (see also Fig. 1 for the construction of the present theory). In Section 2 we introduce a rotator model with mean field coupling as an extreme simplification of a large oscillator population with frequency distribution. In the same section we review a previous theory concerning steady states, but with stronger emphasis on the mechanical basis of the system statistics than in the previous work. Our theory leads to a self-consistent equation for steady state values of a suitably defined order parameter, and predicts the existence of a critical condition for the onset of collective oscillation. The term steady states here refers to periodic oscillations as well as quiescent states, because the order parameter as we define it later takes a constant value in each case. The number density distributions as a function of phase and of coupling-modified frequency is also obtained from this theory.

A notable feature of our system is that it clearly splits into two subsystems in the presence of collective oscillation, namely a synchronized part of the population and a desynchronized one. It is only the synchronized part that contributes to the amplitude of the steady oscillation. In Section 3 our self-consistent equation for the order parameter is generalized into a dynamical form so that one can study the approach to (departure from) steady states. Since the present system has no conserved quantities of additive nature, dynamical reduction seems possible only near the transition point where the time scale of macrovariables becomes distinctively longer than those of microvariables. The idea underlying the derivation of the time-dependent self-consistent equation is reminiscent of the dynamical reduction in gas kinetics, such as the determination of transport coefficients

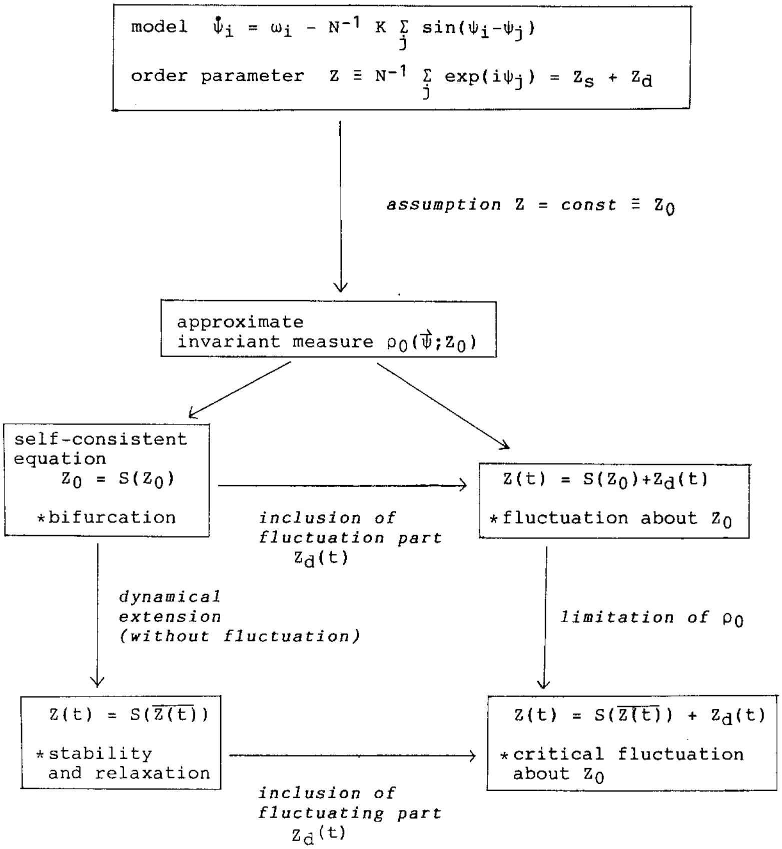

Fig. 1. Construction of the present theory. For notation, refer to the text. Asterisks indicate specific problems to be discussed.

from the Boltzmann equation[16] or the derivation of the Navier–Stokes equation from lattice-gas cellular automata.[17] In all these problems the small deviation of the number density distribution from its local equilibrium form is crucial in generating the evolution of macrovariables.

We also analyze the resulting evolution equation to learn the stability of steady states. The stability here is not mechanical, but rather statistical in nature. This is reflected in the fact that our steady states always involve some fluctuations. The relaxation of the order parameter will be found to be anomalously slow, and even slower than in the usual critical slowing down. This is due to the fact that the fundamental time scale of the system, which is given by the microscopic time scale of the subpopulation relevant to the order parameter dynamics, is by no means a constant parameter, but is strongly dependent on the evolving order parameter value itself.

Section 4 deals with order parameter fluctuations around steady states. The fluctuations can be calculated from the approximate invariant measure already found in Section 2. Such a theory, however, ignores the possibility that the synchronized part of the population could also participate in the order parameter fluctuations, and this effect might be important near the critical point. Section 5 is thus devoted to an improvement of the theory presented in Section 4. The improvement will be achieved not by looking for a correct invariant measure beyond the one obtained in Section 2, but by analyzing the order parameter equation derived in Section 3 with an additional stochastic term (which originates from the desynchronized subpopulation). This term, whose statistical properties have already been revealed in Section 4, is neglected in Section 3 because its simple average is vanishing. We discuss critical fluctuations from this stochastic evolution equation and find that they are unexpectedly weak, and, moreover, the angular fluctuation of the complex order parameter does not diverge. These results are in remarkable contrast to ordinary phase transitions, and their origin will be clarified. A few remarks on the present theory as viewed in a somewhat broader perspective are given in the final section.

2. MODEL SYSTEM AND ITS STATISTICAL STEADY STATES

In this section, we start with the definition of our model system and then give an outline of a theory to find its macroscopic steady states (which are of statistical nature). The steady state theory to be presented here is basically the same as the one developed earlier by Kuramoto,[4,9] except that here it is more clearly stated how the statistics of our system is based on the underlying deterministic law of motion.

The model considered is a population of a large number of similar elements which we call active rotators. An active rotator refers to a phase description of a limit cycle oscillator, and due to its extreme simplicity it has conveniently been employed in the study of collective behavior of large populations in the form of either aggregates or extended tissues.[5,7,9,18–21] In its simplest version, our rotator free from external disturbance obeys the equation

$$d\phi/dt = \omega \tag{2.1}$$

where ϕ represents the phase (mod 2π) of the oscillator, and ω is its natural angular frequency. Suppose that infinitely many such rotators come into mutual contact. Then the model we propose is given by

$$d\phi_i/dt = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\phi_j - \phi_i), \qquad i = 1, 2,..., N \tag{2.2}$$

where N is sufficiently large, and $\Gamma_{ij}(\phi)$ are 2π-periodic functions of ϕ. Equation (2.2) is invariant under the simultaneous translations $\phi_i \to \phi_i + \phi_0$ ($i = 1, 2,..., N$), where ϕ_0 is an arbitrary constant. The above model may look somewhat heuristic, but it can actually be derived perturbatively from a general system of coupled ordinary differential equations

$$\frac{d\mathbf{X}_i}{dt} = \mathbf{F}_i(\mathbf{X}_i) + \sum_{j=1}^{N} \mathbf{G}_{ij}(\mathbf{X}_i, \mathbf{X}_j) \tag{2.3}$$

describing coupled limit cycle oscillators.[4,21] If in Eq. (2.3) the coupling terms $\mathbf{G}_{ij}$ are small and the dependence of $\mathbf{F}_i$ on i is weak (i.e., the oscillators are similar to each other), then the phases are found to be the only relevant variables. As was argued previously,[4,21] a natural definition of phase and a lowest order perturbation theory lead to a great contraction of Eq. (2.3), and one can obtain a universal dynamical equation in the form of Eq. (2.2).

Throughout the present paper, we will be concerned with a mean field model. This is a particularly simple system defined by a special form of interaction as

$$\Gamma_{ij}(\phi) = \frac{K}{N} \sin \phi, \qquad i, j = 1, 2,..., N \tag{2.4}$$

Since in this model the individual oscillators interact with the other $N-1$ oscillators with uniform strength, how they are distributed in real space is completely irrelevant. The coupling constant K is assumed to be positive, so that any pair of oscillators may favor minimizing their phase difference rather than maximizing it. The natural frequency ω_i are distributed according to the number density distribution $g(\omega)$ defined by

$$g(\omega) = \frac{1}{N} \sum_{j=1}^{N} \delta(\omega_j - \omega) \tag{2.5}$$

which is normalized. For the sake of simplicity, $g(\omega)$ is assumed to be symmetric about some frequency ω_0 and to approach a sufficiently smooth function of ω as N goes to infinity. Instead of ϕ_i, it is more convenient to work with new variables ψ_i defined by

$$\psi_i = \phi_i - \omega_0 t \tag{2.6}$$

Upon reassignment of the notation ω_i to $\omega_i - \omega_0$, the model equation takes the form

$$\frac{d\psi_i}{dt} = \omega_i - \frac{K}{N} \sum_{j=1}^{N} \sin(\psi_i - \psi_j) \tag{2.7}$$

where the symmetric distribution $g(\omega)$ is now centered about zero.

Macroscopic states may most conveniently be characterized by the complex order parameter Z defined by

$$Z(t) = |Z(t)|\, e^{i\Theta(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\psi_j(t)} \tag{2.8}$$

This quantity may alternatively be expressed as

$$Z(t) = \int_0^{2\pi} n(\psi, t) e^{i\psi}\, d\psi \tag{2.9}$$

where $n(\psi, t)$ is the number density of the oscillators with phase ψ at time t:

$$n(\psi, t) = \frac{1}{N} \sum_{j=1}^{N} \delta(\psi_j(t) - \psi) \tag{2.10}$$

The great advantage of our mean field model from a mathematical point of view is that Eq. (2.7) reduces formally to a noninteracting system

$$\frac{d\psi_i}{dt} = \omega_i - K|Z| \sin(\psi_i - \Theta) \tag{2.11}$$

Before proceeding to the main part of the theory, it would be appropriate to give a brief speculative discussion as to the kind of steady states to be realized in our system. Suppose that $n(\psi, t)$ approached irreversibly to a certain steady distribution. Due to the aforementioned continuous rotation symmetry inherent in our system, one may expect the existence of a uniform (and hence time-independent) state $n = (2\pi)^{-1}$ representing the steady state of highest symmetry. Obviously, the corresponding order parameter value is vanishing, implying the absence of collective oscillation. Such a state is quite probable when the mutual coupling is sufficiently weak; for stronger coupling, n could become nonuniform as a result of a symmetry-breaking instability. Again from the rotational symmetry of the system, such nonuniform n should propagate steadily like

$$n(\psi, t) = n(\psi - \Omega t) \tag{2.12}$$

as long as no further symmetry-breaking instability occurs. Steady propagation of the distribution implies steady rotation of Z, or

$$Z(t) = |Z|\, e^{i(\Omega t + \Theta_0)} \tag{2.13}$$

where $|Z|$, Ω, and Θ_0 are constants. The system then behaves as a giant oscillator. Incidentally, the order parameter frequency Ω should vanish in the present special model, because by symmetry we find no reason why Z should prefer one direction of rotation to the other; recall that $g(\omega)$ is symmetric about zero, so that the set of equations (2.7) as a whole remains invariant when the signs of all ψ_i are reversed simultaneously.

Assuming that the system on a macroscopic scale approaches a steady state of constant Z, we will now show how this order parameter value is found theoretically. It is seen that under constant Z, Eq. (2.11) can be solved explicitly for each ψ_i, where the solutions still depend on the unknown constant Z. The entire solution set $(\psi_1, \psi_2,..., \psi_N)$ then determines the distribution $n(\psi)$, and its insertion into Eq. (2.10) yields an exact self-consistent equation for Z. This is the way in which the steady state problem is solved macroscopically. It may be questioned, however, how we can say that $n(\psi)$ thus constructed actually becomes stationary as $t \to \infty$, for Eq. (2.11) does not always allow for a time-independent solution. This point will be examined in further detail below.

Equation (2.11) with constant Z clearly divides the system into two subpopulations, one satisfying the condition $|\omega_i/KZ| \leqslant 1$ and the other $|\omega_i/KZ| > 1$. The first group consists of oscillators whose motions are synchronized to the self-generated collective oscillation and will be called the S group; the second group represents the desynchronized part of the population, and will be called the D group. If we define the coupling-modified frequency $\tilde{\omega}_i$ of the ith oscillator by a long-time average of $d\psi_i/dt$, or by

$$\tilde{\omega}_i = \lim_{T \to \infty} \frac{1}{T} \{\psi_i(t_0 + T) - \psi_i(t_0)\} \tag{2.14}$$

then $\tilde{\omega}_i$ are vanishing for the S group and nonvanishing for the D group. In what follows, the indices s and d attached to some quantities refer to S and D groups, respectively. For instance, the quantities n and Z, which are the most basic ones, may be decomposed as

$$n = n_s + n_d \tag{2.15a}$$

$$Z = Z_s + Z_d \tag{2.15b}$$

where

$$Z_{s,d}(t) = \int_0^{2\pi} n_{s,d}(\psi, t) e^{i\psi}\, d\psi \tag{2.16}$$

The respective contributions to the order parameter from the two subsystems are now considered.

S Group. Let the oscillators of this group be numbered as $i = 1, 2, \ldots, N_s$, where

$$N_s = N \int_{-K|Z|}^{K|Z|} g(\omega)\, d\omega \tag{2.17}$$

The fraction

$$N_s/N \equiv r \tag{2.18}$$

therefore measures the degree of frequency condensation into the zero frequency, and may be regarded as another order parameter. From Eq. (2.17) it is clear that r is proportional to $|Z|$ while these quantities remain small.

The phases of S-group oscillators approach fixed points $\psi_{i0}(Z)$, where

$$\psi_{i0}(Z) = \Theta + \sin^{-1}(\omega_i/K|Z|) \tag{2.19}$$

the stability of which is easy to confirm; another fixed point, which appears at $\pm\pi - \psi_{i0}$ ($\psi_{i0} \gtrless 0$), is unstable. Though rather trivial, the group S therefore forms an equilibrium measure

$$\rho_{s0}(\boldsymbol{\psi}_s) = \delta(\boldsymbol{\psi}_s - \boldsymbol{\psi}_{0s}) \tag{2.20}$$

on an N_s-dimensional torus T^{N_s}, where the vector notations

$$\boldsymbol{\psi}_s = (\psi_1, \psi_2, \ldots, \psi_{N_s}) \tag{2.21a}$$

$$\boldsymbol{\psi}_{s0} = (\psi_{10}, \psi_{20}, \ldots, \psi_{N_s 0}) \tag{2.21b}$$

have been used. This measure allows us to express the equilibrium phase distribution $n_{s0}(\psi)$ as

$$\begin{aligned} n_{s0}(\psi; Z) &= \int_0^{2\pi} d\boldsymbol{\psi}_s \rho_{s0}(\boldsymbol{\psi}) \frac{1}{N} \sum_{j=1}^{N} \delta(\psi_j - \psi) \\ &= \int_{-K|Z|}^{K|Z|} d\omega\, g(\omega)\, \delta\left(\Theta + \sin^{-1}\left(\frac{\omega}{K|Z|}\right) - \psi\right) \\ &= g(K|Z| \sin(\psi - \Theta))\, K|Z| \cos(\psi - \Theta) \end{aligned} \tag{2.22}$$

where $\int_0^{2\pi} d\boldsymbol{\psi}_s$ stands for $\int \cdots \int_0^{2\pi} \prod_{j=1}^{N_s} d\psi_j$. The same result for n_{s0} may more easily be reached from the identity

$$n_{s0}(\psi)\, d\psi = g(\omega)\, d\omega \tag{2.23}$$

and the ψ–ω relation (2.19) or

$$\omega = K|Z| \sin(\psi - \Theta) \tag{2.24}$$

D Group. There are $N - N_s \equiv N_d$ oscillators belonging to this group. Their phases are unlocked and change monotonically with t at the rate $v_i(\psi_i)$, where

$$v_i(\psi_i) = \omega_i - K|Z| \sin(\psi_i - \Theta) \tag{2.25}$$

The coupling-modified frequency of the ith oscillator becomes

$$\tilde{\omega}_i = \frac{2\pi}{\int_0^{2\pi/\omega_i} dt} = \frac{2\pi}{\int_0^{2\pi} d\psi/v_i(\psi)} = (\omega_i^2 - |KZ|^2)^{1/2} \tag{2.26}$$

We now come back to our previous question concerning the possibility for D-group oscillators to form a stationary phase distribution. It is essential to notice that these oscillators, if viewed as an N_d-dimensional dynamical system, undergo an ergodic motion on T^{N_d}. Here, of course, $\tilde{\omega}_i$ are supposed to be rationally independent. The motion on T^{N_d} is governed by the equation

$$d\boldsymbol{\psi}_d/dt = \mathbf{v}_d(\boldsymbol{\psi}_d) \tag{2.27}$$

where

$$\boldsymbol{\psi}_d = (\psi_{N_s+1}, \psi_{N_s+2}, \ldots, \psi_N) \tag{2.28a}$$

$$\mathbf{v}_d = (v_{N_s+1}, v_{N_s+2}, \ldots, v_N) \tag{2.28b}$$

If we introduce an ensemble composed of desynchronized subpopulations of an identical mechanical structure, then Eq. (2.27) is equivalent to the following evolution equation for the corresponding density distribution $\rho_d(\boldsymbol{\psi}_d, t)$ on T^{N_d}:

$$\partial \rho_d/\partial t = -\mathrm{div}(\mathbf{v}\rho_d) \tag{2.29}$$

The normalized equilibrium density $\rho_{d0}(\boldsymbol{\psi}_d)$ is thus most naturally chosen as

$$\rho_{d0}(\boldsymbol{\psi}_d) = \prod_{j=N_s+1}^{N} \tilde{\omega}_j/2\pi |v_j(\psi_j)| \tag{2.30}$$

Analogously to the case of $n_{s0}(\psi)$, the equilibrium phase distribution $n_{d0}(\psi)$ is obtained as

$$\begin{aligned} n_{d0}(\psi; Z) &= \int_0^{2\pi} d\boldsymbol{\psi}_d\, \rho_{d0}(\boldsymbol{\psi}_d) \frac{1}{N} \sum_{j=N_s+1}^{N} \delta(\psi_j - \psi) \\ &= \frac{1}{\pi} \int_{K|Z|}^{\infty} d\omega\, g(\omega) \frac{\omega(\omega^2 - |KZ|^2)^{1/2}}{\omega^2 - |KZ \sin(\psi - \Theta)|^2} \end{aligned} \tag{2.31}$$

Having thus found expressions for n_{s0} and n_{d0}, we now substitute them into Eq. (2.9) to obtain an exact self-consistent equation for Z. Due to the property $n_{d0}(\psi) = n_{d0}(\psi + \pi)$, the oscillators of the D group do not contribute to the order parameter value. Our self-consistent equation thus becomes

$$\begin{aligned} Z &= S(Z) \\ &= \int_0^{2\pi} n_{s0}(\psi; Z) e^{i\psi}\, d\psi \\ &= 2\int_0^1 dy\, KZg(K|Z|\, y)(1-y^2)^{1/2} \end{aligned} \tag{2.32}$$

Since $S(Z)$ is an odd function of Z, it is expanded for small Z as

$$S(Z) = (1+\varepsilon)Z - \beta|Z|^2 Z + O(|Z|^5) \tag{2.33}$$

where

$$\varepsilon = (K - K_c)/K_c \tag{2.34a}$$

$$K_c = 2/\pi g(0) \tag{2.34b}$$

$$\beta = -\tfrac{1}{16}\pi K_c^3\, g''(0) \tag{2.34c}$$

Suppose that $g(\omega)$ is convex at $\omega = 0$, i.e., $\beta > 0$, and consider the situation where K is increased continuously. Up to K_c, the only possible solution of Eq. (2.33) is the zero solution. At K_c a new solution branch bifurcates from the zero branch. Near K_c the new solution is given by

$$Z = (\varepsilon/\beta)^{1/2} e^{i\Theta} \tag{2.35}$$

where Θ is an arbitrary constant. One would expect that a transfer of stability occurs at K_c from the trivial zero solution to the nontrivial one, since this actually happens in a supercritical pitchfork bifurcation; if β is negative, in contrast, the nontrivial branch that appears on the side of negative ε is naturally expected to be unstable, analogously to a subcritical pitchfork bifurcation. At this stage of the present theory, however, nothing can yet be claimed about the stability of these macroscopic steady states. It would be only when an evolution equation for Z has been derived that something can be stated about their stability. The derivation of such an equation is the subject of the next section.

As additional information gained from our steady state theory, we derive here some formulas for the distribution of the coupling-modified frequencies. Such formulas will become relevant to the study in Section 4.

The normalized distribution of $\tilde{\omega}$, which will be denoted as $G(\tilde{\omega})$, can be expressed as a sum of two parts:

$$G(\tilde{\omega}) = G_s(\tilde{\omega}) + G_d(\tilde{\omega}) \tag{2.36}$$

Since every oscillator in the S group has vanishing $\tilde{\omega}$, the quantity $G_s(\tilde{\omega})$ is proportional to $\delta(\tilde{\omega})$, and its total intensity equals r, so that

$$G_s(\tilde{\omega}) = r\,\delta(\tilde{\omega}) \tag{2.37}$$

For obtaining $G_d(\tilde{\omega})$, it is convenient to use the identity

$$G_d(\tilde{\omega})\,d\tilde{\omega} = g(\omega)\,d\omega, \qquad |\omega/KZ| > 1 \tag{2.38}$$

and the $\omega - \tilde{\omega}$ relation given by Eq. (2.26). Their combination immediately leads to

$$G_d(\tilde{\omega}) = g((\tilde{\omega}^2 + |KZ|^2)^{1/2}) \frac{|\tilde{\omega}|}{(\tilde{\omega}^2 + |KZ|^2)^{1/2}} \tag{2.39}$$

It is seen that the distribution G almost coincides with the bare distribution g in the high-frequency region, $|\tilde{\omega}/KZ| \gg 1$, while its deviation from g becomes pronounced for $|\tilde{\omega}/KZ| \lesssim 1$. The delta peak is obviously the consequence of a finite fraction of the population being pulled into a single frequency. This in turn causes a marked intensity drop around the delta peak. Near the transition point, $G_d(\tilde{\omega})$ at extremely low frequencies behaves as

$$G_d(\tilde{\omega}) \simeq \frac{g(0)}{K|Z|} |\tilde{\omega}| \tag{2.40}$$

Finally, it is remarked that the equilibrium measure $\rho_0(\psi)$ where $\psi \equiv (\psi_s, \psi_d)$, is simply given by the product

$$\rho_0(\psi) = \rho_{s0}(\psi_s)\,\rho_{d0}(\psi_d) \tag{2.41}$$

It is important to realize that the above ρ_0 is not exact for finite N, because the parameter Z on which ρ_{s0} and ρ_{d0} depend has been supposed to be constant, whereas Z is actually not strictly constant, but always involves some fluctuations. Of course, our approximate ρ_0 may be accurate enough for calculating averages of macrovariables such as Z, and may also be useful for obtaining order parameter fluctuations in some situations (see Section 4). The insufficiency of ρ_0 will become apparent in Section 5, where the fluctuations are more carefully examined near the critical point K_c.

3. EVOLUTION EQUATION FOR THE ORDER PARAMETER

The self-consistent equation (2.32), which is exact in the limit $N \to \infty$, will now be extended to a dynamic form, and some of its consequences will also be discussed. One may anticipate that such dynamical extension will be possible at least near the critical point due to the expected slowing down of the order parameter motion there. Slowing down of Z allows the microscopic degrees of freedom, i.e., individual ψ_i, to follow Z almost adiabatically. Consequently, these rapid variables will be completely eliminated from the expression, resulting in a great reduction of the dynamics. In this way, a universal evolution equation for Z will be obtained. In the theory presented below, it is always assumed that K is close to K_c, so that Z is sufficiently slow, and also that $|Z|$ is much smaller than 1, which means that the system is near a steady state.

If Z evolves extremely slowly, the phase distribution $n(\psi, t)$ at each moment will establish its steady state almost completely, that is,

$$n(\psi, t) \simeq n_0(\psi; Z(t)) = n_{s0}(\psi; Z(t)) + n_{d0}(\psi; Z(t))$$

Unfortunately, this simple approximation for $n(\psi, t)$ merely leads to $Z(t) = S(Z(t))$, which is identical to Eq. (2.32) and no evolution of Z is produced at all. The deviation of $n(\psi, t)$ from its time-local steady state form is therefore essential in generating the dynamics of Z. Slow variation of Z at least enables us a fairly clear-cut separation of the population into S and D subsystems at each moment according to the criterion $|\omega/KZ(t)| \gtrless 1$. Actually, however, the threshold condition $\omega = \pm K|Z|$ will be slightly obscured due to the motion of Z, and there should be a small number of vague oscillators lying near the borderline between the two subpopulations. A crude picture of what makes the borderline obscure is the following. The oscillators near the borderline have extremely long time scales, say T_1, which could be even longer than that of Z, say T_2. On the other hand, the minimum time interval needed for deciding upon the group to which these oscillators should belong is comparable to T_1. While such a decision would only be possible if the variation of Z over the period T_1 were negligible, the inequality $T_1 > T_2$ clearly contradicts this requirement.

The present theory completely ignores the effects of these marginal oscillators, and its theoretical justification would require much more elaborate study. A few minor technical difficulties arising from such an unnaturally clear division using the condition $|\omega/KZ| \gtrless 1$ will be eliminated by a suitable prescription. We now concentrate on the investigation of non-adiabatic effects on $n(\psi, t)$, and treat groups S and D separately again.

S Group. In the zeroth approximation in slowness of Z, the solution of Eq. (2.11) takes the equilibrium form

$$\psi_i(t) = \psi_{i0}(Z(t)) \tag{3.1}$$

In order to include nonadiabatic effects, which should necessarily be small, we put

$$\psi_i(t) = \psi_{i0}(Z(t)) + \delta\psi_i(t) \tag{3.2}$$

and linearize Eq. (2.11) in $\delta\psi_i$ as

$$\frac{d\delta\psi_i}{dt} = -(|KZ|^2 - \omega_i^2)^{1/2}\,\delta\psi_i - \frac{d\psi_{i0}}{dt} \tag{3.3}$$

The solution of Eq. (3.3) as $t \to \infty$ is of the form

$$\delta\psi_i(t) = -\int_{-\infty}^{t} dt' \frac{d\psi_{i0}(Z(t'))}{dt'} \left\{ -\int_{t'}^{t} dt'' \, [|KZ(t'')|^2 - \omega_i^2]^{1/2} \right\} \tag{3.4}$$

Partial integration of Eq. (3.4) and its substitution into Eq. (3.2) lead to

$$\psi_i(t) = \int_0^{\infty} dt' \, \psi_{i0}(Z(t-t')\, P(t, t'; \omega_i) \tag{3.5}$$

where P is a normalized weighting function and is defined by

$$P(t, t'; \omega_i) = -\frac{d}{dt'} \exp\left\{ -\int_{t-t'}^{t} dt'' [|KZ(t'')|^2 - \omega_i^2]^{1/2} \right\} \tag{3.6}$$

By comparison of Eq. (3.5) with Eq. (3.1), it is clear that the nonadiabatic effects are the consequence of the fact that the decay time τ_i of P as a function of t' is nonvanishing. The τ_i is nothing but the time scale of the ith oscillator and is typically of the order of $|KZ(t)|^{-1}$. This is a large quantity because of the assumed smallness of $|Z(t)|$; still, the nonadiabaticity remains small because by assumption the time scale of Z is even longer than τ_i or, equivalently, Z changes only a little in relative magnitude over the time interval τ_i. Of course, the dynamical equation for $Z(t)$, whose derivation is the main purpose in this section, should be consistent with our slowness assumption for Z, and this will be checked at the end of this section.

The fact that $Z(t)$ changes much more slowly than P (as a function of t') leads to the following simplifications in two ways. First, the t'' dependence of Z in the integral in Eq. (3.6) can be neglected and replaced by $Z(t)$. This leads to

$$\begin{aligned} P(t, t'; \omega) &\simeq [|KZ(t)|^2 - \omega^2]^{1/2} \exp\{-[|KZ(t)|^2 - \omega^2]^{1/2} t'\} \\ &\equiv P(t'; Z(t), \omega) \end{aligned} \tag{3.7}$$

Second, $\psi_i(t)$ in Eq. (3.5) is somewhat simplified. We note that the range of integration in that equation is practically restricted to $(0, \tau_i)$. Since in this range ψ_{i0} may be regarded as a linear function of the small deviation of Z from some standard value of it, say $Z(t)$, the time-averaging operation in Eq. (3.5) acts practically on Z, that is,

$$\psi_i(t) \simeq \psi_{i0}(\overline{Z(t)}^i) \tag{3.8}$$

where the i-dependent bar is defined by

$$\overline{Z(t)}^i = \int_0^\infty dt'\, Z(t-t')\, P(t'; Z(t), \omega_i) \tag{3.9}$$

The expression for $\psi_i(t)$ permits us a simple interpretation, that is, an S-group oscillator, even though it might be unable to follow immediately the motion of Z, will nevertheless rest on a time-local steady state under a *virtual* mean field $\overline{Z(t)}^i$ rather than the true field $Z(t)$.

Although the virtual mean field is oscillator-dependent, this dependence is so weak that it would be instructive to ignore it first and observe what follows. The result is that

$$n_s(\psi, t) = n_{s0}(\psi; \overline{Z(t)}) \tag{3.10}$$

which leads to

$$Z_s(t) = S(\overline{Z(t)}) \tag{3.11}$$

Furthermore, if Z_d vanishes, which is the case for steady states, then Eq. (3.11) becomes

$$Z(t) = S(\overline{Z(t)}) \tag{3.12}$$

This equation describes the evolution of the order parameter, though not in a differential form.

It would probably be too crude an approximation to neglect the oscillator dependence of the virtual field. We shall find, however, that Eq. (3.12) still holds if a slightly different interpretation is given to $\overline{Z(t)}$. Its correct definition is

$$\overline{Z(t)} = \int_0^\infty dt'\, Z(t-t')\, \Pi_s(t'; Z(t)) \tag{3.13}$$

where Π_s is a new weighting function given by

$$\Pi_s(t'; Z(t)) = \frac{4}{\pi}\int_0^1 dy\,(1-y^2)^{1/2}\, P(t'; Z(t), K|Z|\,y)$$

$$= \frac{4}{\pi} K|Z(t)| \int_0^1 dy\,(1-y^2)\exp\{-K|Z(t)|\,(1-y^2)^{1/2}\,t'\} \quad (3.14)$$

Formulas (3.11), (3.13), and (3.14) are derived in Appendix A. It is easy to see that Π_s is normalized, i.e.,

$$\int_0^\infty \Pi_s(t'; Z)\,dt' = 1 \quad (3.15)$$

and that it behaves for large t' as

$$\Pi_s(t'; Z) \simeq (\pi K^3|Z|^3 t'^4)^{-1}, \qquad t' \gg |KZ|^{-1} \quad (3.16)$$

Equation (3.13) is now converted into a more familiar differential form. Partial integration of that equation gives

$$\overline{Z(t)} = Z(t) + \int_0^\infty dt'\,\frac{dZ(t-t')}{dt'}\,\Theta_s(t'; Z(t)) \quad (3.17)$$

Here the function Θ_s is related to Π_s through

$$\Theta_s(t'; Z) = \int_{t'}^\infty \Pi_s(t''; Z)\,dt'' \quad (3.18)$$

and satisfies

$$\Theta_s(0; Z) = 1 \quad (3.19)$$

due to Eq. (3.15). The characteristic time of $\Theta_s(t'; Z)$ is the same as that of $\Pi_s(t'; Z)$ and is given by $|KZ|^{-1}$, and the t' dependence of Θ_s is only through $K|Z|\,t'$. From this and the fact that the initial amplitude of Θ_s is independent of Z [see Eq. (3.19)], we see that

$$\int_0^\infty \Theta_s(t'; Z)\,dt' = \xi_s |KZ|^{-1} \quad (3.20)$$

where ξ_s is a constant of ordinary magnitude. By assumption, the time interval $|KZ(t)|^{-1}$, i.e., the time scale of Θ_s, is shorter than the time scale of Z, which enables us to approximate Eq. (3.17) by the Markovian form

$$\overline{Z(t)} = Z(t) - \frac{dZ(t)}{dt}\,\xi_s\,|KZ(t)|^{-1} \quad (3.21)$$

D Group. It was argued at the beginning of this section that $n_d(\psi, t)$ roughly equals $n_d(\psi; Z(t))$, but that a small difference between them could be important. Analogous to the finding about $n_s(\psi, t)$, one would expect that $n_d(\psi, t)$ might be approximated by something like $n_{d0}(\psi; \overline{Z(t)})$, where $\overline{Z(t)}$ means some time average of Z, but may generally differ from the previous quantity under the same notation. Such an anticipation turns out to be basically valid, but actually the situation is slightly more complicated. Let us introduce the normalized phase distribution $\hat{n}_d(\psi, t)$, which we need to consider for some technical reasons. In Appendix B it is shown that $\hat{n}_d(\psi, t)$ and $n_d(\psi, t)$ are approximately given by

$$\hat{n}_d(\psi, t) = \hat{n}_{d0}(\psi; \overline{Z(t)}^{\psi}) \tag{3.22}$$

and

$$\begin{aligned} n_d(\psi, t) &= \int_0^{2\pi} n_d(\psi', t)\, d\psi' \cdot \hat{n}_{d0}(\psi; \overline{Z(t)}^{\psi}) \\ &\simeq \int_0^{2\pi} n_{d0}(\psi'; Z(t))\, d\psi' \cdot \hat{n}_{d0}(\psi; \overline{Z(t)}^{\psi}) \end{aligned} \tag{3.23}$$

where the ψ dependent bar is the time average defined by

$$\begin{aligned} \overline{Z(t)}^{\psi} &= \int_0^{\infty} dt'\, Z(t-t')\, \Pi_d(\psi, t'; Z(t)) \\ &= Z(t) + \int_0^{\infty} dt' \frac{dZ(t-t')}{dt'} \Theta_d(\psi, t'; Z(t)) \end{aligned} \tag{3.24}$$

The functions Π_d and Θ_d are quantities analogous to Π_s and Θ_s, respectively, and satisfy

$$\frac{d\Theta_d(\psi, t; Z)}{dt} = -\Pi_d(\psi, t; Z) \tag{3.25}$$

and

$$\int_0^{\infty} \Pi_d(\psi, t'; Z)\, dt' = \Theta_d(\psi, 0; Z) = 1 \tag{3.26}$$

Explicit forms of Π_d and Θ_d are unknown, but their physical implication is rather simple. Appendix B shows that Θ_d describes the relaxation of $\hat{n}_d(\psi, t)$ from one equilibrium to another when the field Z (supposed to be external) makes a sudden and sufficiently small-amplitude jump. To be more precise, suppose that the distribution $\hat{n}_d$ is in equilibrium under constant $Z \equiv Z_0$ up to $t=0$. Then let the field value be switched to a slightly different value Z_1, and observe how $\hat{n}_{\rm d}(\psi, t)$ evolves. Appendix B shows that the evolution is described by

$$\hat{n}_{\rm d}(\psi, t) = \hat{n}_{\rm d0}(\psi; Z_0) + (Z_1 - Z_0)\{1 - \Theta_{\rm d}(\psi, t; Z_0)\} \frac{d\hat{n}_{\rm d0}(\psi; Z_0)}{dZ_0} \tag{3.27}$$

The order parameter component $Z_{\rm d}$ is now given by

$$Z_{\rm d}(t) = \int_0^{2\pi} n_{\rm d0}(\psi'; Z(t))\, d\psi' \int_0^{2\pi} \hat{n}_{\rm d0}(\psi; \overline{Z(t)}^{\psi}) e^{i\psi}\, d\psi \tag{3.28}$$

which may further be simplified as

$$Z_{\rm d}(t) = \int_0^{2\pi} \hat{n}_{\rm d0}(\psi; \overline{Z(t)}^{\psi}) e^{i\psi}\, d\psi \tag{3.29}$$

because the first integral on the right-hand side of Eq. (3.28) is close to 1 near $K_{\rm c}$. This does not mean, however, that $\hat{n}_{\rm d0}$ can be replaced by $n_{\rm d0}$ in Eq. (3.29), because the estimation of $Z_{\rm d}(t)$ involves the differentiation of $\hat{n}_{\rm d0}(\psi; Z)$ with respect to Z, as can be seen below. It should be noted that the difference between $\overline{Z(t)}^{\psi}$ and $Z(t)$, which is equal to the last term in Eq. (3.24), measures the degree of nonadiabaticity. To the first order in nonadiabaticity, the integrand in Eq. (3.29) becomes

$$\hat{n}_{\rm d0}(\psi; \overline{Z(t)}^{\psi}) = \hat{n}_{\rm d0}(\psi; Z(t)) + \frac{d\hat{n}_{\rm d0}(\psi; Z(t))}{dZ(t)} \int_0^{\infty} dt' \frac{dZ(t-t')}{dt'} \Theta_{\rm d}(\psi, t'; Z(t)) \tag{3.30}$$

which immediately leads to

$$Z_{\rm d}(t) = \int_0^{2\pi} d\psi \int_0^{\infty} dt' \frac{d\hat{n}_{\rm d0}(\psi; Z(t))}{dZ(t)} \frac{dZ(t-t')}{dt'} \Theta_{\rm d}(\psi, t'; Z(t)) e^{i\psi} \tag{3.31}$$

Recalling that $\Theta_{\rm d}(\psi, t'; Z)$ describes the relaxation of $\hat{n}_{\rm d}(\psi, t')$ when the Z value makes a small jump, we expect that the time scale of $\Theta_{\rm d}$ as a function of t' will be of the order of $|KZ|^{-1}$. The reason is that the oscillators responsible for the relaxation of $n_{\rm d}$ are practically restricted to those with natural frequencies not much larger than $|KZ|$; the other oscillators, even though they may constitute the majority of the group D, will virtually form a uniform phase distribution and remain unaffected by the variation of Z. Thus, $\Theta_{\rm d}$ will satisfy, similarly to $\Theta_{\rm s}$, the equation

$$\int_0^{\infty} \Theta_{\rm d}(\psi, t'; Z)\, dt' = \xi_{\rm d}^{\psi}\, |KZ|^{-1} \tag{3.32}$$

where ξ_{d}^{ψ}, still a function ψ, is of ordinary magnitude. With the use of Eq. (3.32), the Markovian approximation on Eq. (3.31) leads to

$$Z_{\mathrm{d}}(t) = -\frac{dZ(t)}{dt} |KZ(t)|^{-1} \int_0^{2\pi} d\psi \, \xi_{\mathrm{d}}^{\psi} \frac{d\hat{n}_{\mathrm{d}0}(\psi; Z(t))}{dZ(t)} e^{i\psi} \tag{3.33}$$

For small Z the integrand in the last equation gives rise to a small factor proportional to $|KZ|$, as is easily confirmed from Eq. (2.31). Thus,

$$Z_{\mathrm{d}}(t) \sim O\left(\frac{dZ(t)}{dt}\right) \tag{3.34}$$

Putting Eqs. (3.11), (3.21), and (3.34) together, we obtain

$$\begin{aligned} Z &= S\left(Z - \xi_{\mathrm{s}} \frac{dZ}{dt} |KZ|^{-1}\right) + O\left(\frac{dZ}{dt}\right) \\ &\simeq (1+\varepsilon)\left(Z - \xi_{\mathrm{s}} \frac{dZ}{dt} |KZ|^{-1}\right) - \beta |Z|^2 Z + O\left(\frac{dZ}{dt}\right) \end{aligned} \tag{3.35}$$

This may further be reduced to

$$\xi_{\mathrm{s}} \frac{dZ}{dt} |KZ|^{-1} \simeq \varepsilon Z - \beta |Z|^2 Z \tag{3.36}$$

It should be noted that the D-group oscillators are completely irrelevant to the order parameter dynamcs. We have thus succeeded in generalizing our self-consistent equation (2.32) into a dynamic form. When the Markovian approximation is not permitted (for reasons stated later), one should use the equation

$$Z(t) = Z_{\mathrm{s}}(t) = S(\overline{Z(t)}) \tag{3.37}$$

Equation (3.36) allows us to study the stability of steady states and the relaxation to stable ones. It is obvious that the phase factor of Z cancels between the two sides. The phase Θ is thus an arbitrary constant and preserves its initial value. We choose Θ to be zero and regard Z as a real number. In the weak coupling region ($\varepsilon < 0$), the zero solution is seen to be stable. Retaining only the lowest order term in Z, we get

$$dZ/dt = -|\varepsilon| \, \xi_{\mathrm{s}}^{-1} K Z^2 \tag{3.38}$$

whose solution behaves like

$$Z(t) \sim 1/|\varepsilon| \, t \tag{3.39}$$

In the strong coupling region ($\varepsilon > 0$), the zero solution loses stability and gives way to the nontrivial solution $(\varepsilon/\beta)^{1/2} \equiv Z_{st}$ if β is positive. The equation linearized in the deviation η, where

$$\eta(t) = Z(t) - Z_{st} \tag{3.40}$$

then becomes

$$d\eta/dt = -\varepsilon^{3/2}\gamma_0\eta \tag{3.41}$$

where

$$\gamma_0 = 2K_c/\xi_s\beta^{1/2} \tag{3.42}$$

Thus,

$$\eta(t) \sim \exp(-\gamma_0\varepsilon^{3/2}t) \tag{3.43}$$

If β is negative, the nontrivial solution that appears in the weak coupling region is easily seen to be unstable. In the rest of the paper, we shall always assume the normal case, i.e., the case of positive β.

Finally, we remark that our starting assumption that the time scale $|KZ|^{-1}$ is shorter than that of Z is consistent. This is seen from Eq. (3.36), showing that the time scale of Z is $O(|\varepsilon KZ|^{-1})$, which is in fact much longer than $|KZ|^{-1}$ near K_c.

4. NORMAL FLUCTUATIONS OF THE ORDER PARAMETER

Let us come back to the steady state problem and study order parameter fluctuations around the steady state value found previously. The order parameter fluctuations here are of deterministic origin and are related to the ergodic motion of the D subsystem on T^{N_d}. We begin with some preliminary remarks on the statistical equilibrium of our system. We found an equilibrium measure $\rho_0(\psi)$ in section 2 [see Eq. (2.41)]. Our ρ_0 contains a constant parameter Z, which should now be interpreted as the statistical average of the dynamical variable $Z(\psi)$ by means of $\rho_0(\psi)$. Let the statistical average of some variable $f(\psi)$ be denoted as $\langle f(\psi)\rangle$, i.e.,

$$\langle f(\psi)\rangle = \int_0^{2\pi} d\psi\, f(\psi)\,\rho(\psi) \tag{4.1}$$

Then the self-consistent equation derived in Section 2 should more correctly be written as

$$\langle Z\rangle = \langle Z_s\rangle + \langle Z_d\rangle, \qquad \langle Z_s\rangle = S(\langle Z\rangle), \qquad \langle Z_d\rangle = 0 \tag{4.2}$$

Although the average of Z_d is zero, it exhibits irregular temporal variation in the course of the motion of $\boldsymbol{\psi}_d$ on T^{N_d}. The fact that Z_d (and hence Z) fluctuates gives the very reason that our equilibrium measure $\rho_0(\boldsymbol{\psi})$ obtained from the assumption of constant Z is only approximate. In this section we shall still use this approximate ρ_0 to study fluctuations, whereas an improved fluctuation theory will be developed in the next section.

As long as we use $\rho_0(\boldsymbol{\psi})$, the order parameter component Z_s exhibits no fluctuation, but tends to the definite value $S(\langle Z\rangle)$. Thus, we have

$$Z(t) = S(\langle Z\rangle) + Z_d(t) \qquad \text{for} \quad t \to \infty \tag{4.3}$$

We will now investigate statistical properties of the stochastic process of $Z_d(t)$, or, equivalently, $Z(t)$. This can be achieved by means of $\rho_{d0}(\boldsymbol{\psi}_d)$ and the equations of motion

$$d\boldsymbol{\psi}_d/dt = \mathbf{v}_d(\boldsymbol{\psi}_d; \langle Z\rangle) \tag{4.4}$$

The quantities of basic importance are some time correlations of Z. In Appendix C the following formulas are proved:

$$\begin{aligned}
\langle Z_d(t_0)\, Z_d^*(t_0+t)\rangle &= \langle Z_d(0)\, Z_d^*(t)\rangle && (4.5\text{a})\\
&= \langle Z_d(0)\, Z_d^*(-t)\rangle && (4.5\text{b})\\
&= \frac{1}{N}\int_{-\infty}^{\infty} d\omega\, G_d(\omega) \sum_{l=1}^{\infty} |D_l(\omega)|^2\, e^{-il\omega t} && (4.5\text{c})\\
&\equiv F(t)\\
\langle \operatorname{Re} Z_d(0) \operatorname{Re} Z_d(t)\rangle &= \langle \operatorname{Im} Z_d(0) \operatorname{Im} Z_d(t)\rangle\\
&= \frac{1}{2}\langle Z_d(0)\, Z_d^*(t)\rangle && (4.5\text{d})\\
\langle \operatorname{Re} Z_d(0) \operatorname{Im} Z_d(t)\rangle &= \langle \operatorname{Im} Z_d(0) \operatorname{Re} Z_d(t)\rangle = 0 && (4.5\text{e})
\end{aligned}$$

where $G_d(\omega)$ is the distribution of the coupling-modified frequencies for the D group [see Eq. (2.39)], and $D_l(\omega)$ are defined by

$$e^{i\psi_j(t)} = \sum_{l=1}^{\infty} D_l(\tilde{\omega}_j)\, e^{il\tilde{\omega}_j t} \tag{4.6}$$

Let the frequency spectrum of the order parameter fluctuation be denoted as $F(\omega)$, or

$$F(\omega) = \frac{1}{\pi}\int_0^{\infty} F(t) e^{i\omega t}\, dt = \langle |Z_d(\omega)|^2\rangle \tag{4.7}$$

where

$$Z_{\rm d}(\omega) = \lim_{T\to\infty} \frac{1}{(\pi T)^{1/2}} \int_0^T Z_{\rm d}(t) e^{-i\omega t}\, dt \tag{4.8}$$

Then the Fourier transform of Eq. (4.5c) is expressed as

$$F(\omega) = \frac{1}{N} \sum_{l=1}^{\infty} G_d\left(\frac{\omega}{l}\right) \left| D_l\left(\frac{\omega}{l}\right)\right|^2 l^{-1} \tag{4.9}$$

In the disordered phase ($K < K_{\rm c}$), where $G_{\rm d}(\omega) = g(\omega)$, we have $D_1(\omega) = 1$ and $D_l(\omega) = 0$ ($l \neq 1$), because the motion of $\exp(i\psi_j)$ is perfectly sinusoidal like $\exp[i(\omega_j t + \text{const})]$. This means that

$$F(\omega) = \frac{1}{N} g(\omega), \qquad (K < K_{\rm c}) \tag{4.10}$$

In the ordered phase ($K > K_{\rm c}$), in contrast, $F(\omega)$ deviates from $G_{\rm d}(\omega)$ especially at low frequencies ($\omega \lesssim |K\langle Z\rangle|$) because the corresponding oscillations are highly nonsinusoidal. It is shown in Appendix C that

$$\begin{aligned} F(\omega) &\simeq \frac{1}{N} G_{\rm d}(\omega) \simeq \frac{1}{N} g(\omega), && |\omega/K\langle Z\rangle| \gg 1 \\ &\simeq \frac{b}{N} \omega^2, && |\omega/K\langle Z\rangle| \ll 1 \end{aligned} \tag{4.11}$$

where

$$b = \frac{g(0)}{|K\langle Z\rangle|^2} \sum_{l=1}^{\infty} l^{-3} \tag{4.12}$$

while $G_{\rm d}(\omega)$ was found to be linear in ω in the low-frequency region, $F(\omega)$ in the same region is quadratic in ω, thus exhibiting even an stronger intensity drop there.

The total intensity of the order parameter fluctuations is given by

$$\langle |Z - \langle Z\rangle|^2\rangle = \langle |Z_{\rm d}|^2\rangle = \frac{1}{N} \int_{-\infty}^{\infty} d\omega\, G_{\rm d}(\omega) \sum_{l=1}^{\infty} |D_l(\omega)|^2 = \frac{1-r}{N} \tag{4.13}$$

where identity (C.8) has been used and r is defined by Eq. (2.18). The total intensity of fluctuations is thus constant in the disordered phase because r vanishes there, and as K increases beyond $K_{\rm c}$ it decreases with the development of the long-range order. So far, the order parameter fluctuations have entirely been attributed to D-group oscillators, and as a result there has been no chance for the fluctuations to be enhanced near K_c. Such a theory is insufficient and needs to be improved so as to include fluctuations from S-group oscillators. This is what we do in the next section.

5. STOCHASTIC EVOLUTION EQUATION AND ANOMALOUS FLUCTUATIONS OF THE ORDER PARAMETER

The analysis of the order parameter fluctuations carried out in the preceding section was based on the approximate invariant measure $\rho_0(\psi)$. In obtaining this measure, we neglected the effect of the order parameter fluctuations on the motion of the individual oscillators. Since the resulting individual motions themselves make Z fluctuate, we must admit that our previous treatment was not consistent enough. According to the previous treatment, the fluctuations in Z come entirely from the D subsystem. Actually, however, the oscillators in the S group, being agitated by the fluctuating Z, can also fluctuate, and such an effect could be fed back to Z. A more specific description of this kind of secondary process, which could be important near the critical point, is the following. Suppose that a fluctuation in the order parameter occurred at some time. This may capture a few oscillators of relatively low frequencies and hold them in a transient phase-locked state. As a result, the fluctuation will be enhanced, which may force more oscillators into locking states, and so forth. In this way, the order parameter fluctuations may develop cooperatively starting from an initial seed of normal fluctuation into an anomalous level and even into a macroscopic level. In the present section we try to incorporate this kind of secondary (but possibly important) effect into our theory, and discuss how the previous results on fluctuations should be modified near the critical point. In trying to formulate our statistical mechanical theory, any attempt to find a correct invariant measure beyond the previous one would be hopelessly difficult. Instead, we take a different approach, trying to find a stochastic evolution equations for Z with a noise term of known statistical properties.

Our starting assumption is again that the population can unambiguously be divided into S and D subgroups as

$$Z(t) = Z_s(t) + Z_d(t) \tag{5.1}$$

Although every oscillator should experience fluctuating Z, the effect of order parameter fluctuation seems to be more important to the S group than to the D group, as is expected from the above qualitative argument on the secondary effect of fluctuations. We may, assume, therefore that the statistical properties of $Z_d(t)$ remain the same as those clarified in Section 4, where Z was supposed to be constant. The problem now is to express $Z_s(t)$ in terms of the solutions for $\psi_s(t)$ under fluctuating Z. We saw in Section 3 that Z_s behaves as in Eq. (3.11) provided that the temporal variation of Z is sufficiently slow. Although Z now involves a rapidly fluctuating part superimposed on the slowly varying part, we still assume that Eq. (3.11) remains valid and substitute it for the systematic part of the stochastic evolution equation we are trying to find. This does not seem unreasonable, and what should rather be noticed is that the condition we thought necessary for the derivation of Eq. (3.11) was a little too restrictive. As is easily confirmed, the fact is that the condition we actually needed for deriving Eq. (3.11) was that the *net* variation of Z over the typical time scale of S-group oscillators is much smaller than Z itself, and by no means that Z is free from any ripples of rapidly varying components. Assuming that Eq. (3.11) can be used as the systematic part of our stochastic evolution equation, we have

$$Z(t) = S(\overline{Z(t)}) + Z_d(t) \tag{5.2}$$

This equation may be looked upon as a generalization of Eq. (3.37) and also of Eq. (4.3); recall that those equations are generalizations of our self-consistent equation (2.32) in different ways.

We now reduce Eq. (5.2) to a form more convenient for our present purposes. Since the difference between $Z(t)$ and $\overline{Z(t)}$ is supposed to be small, as mentioned repeatedly, Eq. (5.2) may be approximated near K_c as

$$Z(t) = (1+\varepsilon)\,\overline{Z(t)} - \beta\,|\overline{Z(t)}|^2\,\overline{Z(t)} + Z_d(t) \tag{5.3}$$

or even by

$$Z(t) - \overline{Z(t)} = \varepsilon\overline{Z(t)} - \beta\,|Z(t)|^2\,Z(t) + Z_d(t) \tag{5.4}$$

It is interesting to observe that the stochastic term $Z_d(t)$ in Eq. (5.4) plays qualitatively different roles, depending on the frequency range of the order parameter fluctuations in which we are interested. If we focus on sufficiently slow components of $Z(t)$, the left-hand side becomes proportional to $dZ(t)/dt$, as was shown in Section 3. Then the stochastic term acts as fluctuating *forces*. On the contrary, if we are concerned with rapidly fluctuating components of $Z(t)$, we see that Eq. (5.4) practically reduces to $Z(t) \simeq Z_d(t)$ because $Z(t)$ contains no such components and the other two terms on the right-hand side are much smaller than $Z(t)$. Thus, the stochastic term no longer acts as forces, but contributes directly to the order parameter fluctuations.

In the disordered phase ($\varepsilon < 0$), one may neglect the cubic term in Eq. (5.4) to obtain

$$Z(t) - \overline{Z(t)} = -|\varepsilon| \, \overline{Z(t)} + Z_d(t) \tag{5.5}$$

In the ordered phase ($\varepsilon > 0$), it is more suitable to transform Eq. (5.4) into a set of equations for the amplitude and phase of Z. We define the amplitude deviation η by

$$\eta(t) = |Z(t)| - \langle |Z| \rangle \simeq |Z(t)| - |\langle Z \rangle| \tag{5.6}$$

In the last near-equality the smallness of the phase fluctuation of Z is assumed, the validity of which will be confirmed later. Linearization of Eq. (5.4) in η then leads to

$$\eta(t) = \overline{\eta(t)} = -2\varepsilon \overline{\eta(t)} + f(t, \Theta(t)) \tag{5.7}$$

where

$$f(t, \Theta(t)) = \mathrm{Re}[Z_d(t) \, e^{-i\Theta(t)}] \tag{5.8}$$

In deriving an equation for Θ, one may neglect amplitude fluctuation. Then we get

$$\Theta(t) - \overline{\Theta(t)} = h(t, \Theta(t)) \tag{5.9}$$

where

$$h(t, \Theta(t)) = |\langle Z \rangle|^{-1} \, \mathrm{Im}[Z_d(t) \, e^{-i\Theta(t)}] \tag{5.10}$$

The quantities with bars in Eqs. (5.5), (5.7), and (5.9) need to be expressed more explicitly. In order to do this, we employ the approximation in which the quantity $|Z(t)|$ appearing in the definition of Π_d [see Eq. (3.14)] is replaced by its statistical average $\langle |Z| \rangle$. Note that $\langle |Z| \rangle$ is positive definite and should not be approximated by $\langle |Z| \rangle$ in the disordered phase. It is expected that in the disordered phase $\langle |Z| \rangle$ is equal to a typical amplitude of order parameter fluctuation, or

$$\langle |Z| \rangle = \alpha_1 \langle |Z|^2 \rangle^{1/2} \qquad (\varepsilon < 0) \tag{5.11}$$

where α_1 is some constant of order 1. One may now express $\Pi_s(t)$ in the scaling form

$$\Pi_s(t) = K \langle |Z| \rangle \, \tilde{\Pi}_s(K \langle |Z| \rangle t) \tag{5.12}$$

where

$$\tilde{\Pi}_s(\lambda) = \frac{4}{\pi} \int_0^1 dy \, (1 - y^2) \exp[-(1 - y^2)^{1/2} \lambda] \tag{5.13}$$

Property (5.12) together with the normalization condition (3.15) implies that the Fourier components defined by

$$\Pi_s(\omega) \equiv \int_0^\infty \Pi_s(t) e^{-i\omega t}\, dt \tag{5.14}$$

have the following asymptotic properties:

$$\begin{aligned} \Pi_s(\omega) &\simeq 0, && |\omega|/K\langle |Z| \rangle \gg 1 \\ &\simeq 1 - \alpha_2\, i\omega/K\langle |Z| \rangle, && |\omega|/K\langle |Z| \rangle \ll 1 \end{aligned} \tag{5.15}$$

where α_2 is a constant of ordinary magnitude. In order to calculate the order parameter fluctuations in the disordered phase, we rewrite Eq. (5.5) in terms of Fourier components:

$$Z(\omega) = \frac{Z_d(\omega)}{1 - (1+\varepsilon)\, \Pi_s(\omega)} \tag{5.16}$$

or

$$\langle |Z(\omega)|^2 \rangle = \frac{F(\omega)}{|1 - (1+\varepsilon)\, \Pi_s(\omega)|^2} = \frac{g(\omega)}{N|1 - (1+\varepsilon)\, \Pi_s(\omega)|^2} \tag{5.17}$$

From Eqs. (5.15) and (5.17), the asymptotic forms of the fluctuation spectrum become

$$\begin{aligned} \langle |Z(\omega)|^2 \rangle &\simeq \frac{g(\omega)}{N}, && \frac{|\omega|}{K\langle |Z| \rangle} \gg 1 \\ &\simeq \frac{1}{N} \frac{g(\omega)}{\varepsilon^2 + (\alpha_2 \omega/K\langle |Z| \rangle)^2}, && \frac{|\omega|}{K\langle |Z| \rangle} \ll 1 \end{aligned} \tag{5.18}$$

The total fluctuation intensity is then roughly estimated as

$$\begin{aligned} \langle |Z|^2 \rangle &= \int_0^\infty \langle |Z(\omega)|^2 \rangle\, d\omega \\ &\simeq \frac{1}{N} \int_{K\langle |Z| \rangle}^\infty g(\omega)\, d\omega + \frac{1}{N} \int_0^{K\langle |Z| \rangle} \frac{g(\omega)}{\varepsilon^2 + (\alpha_2 \omega/K\langle |Z| \rangle)^2}\, d\omega \\ &\simeq \frac{1}{N} + O\left(\frac{K\langle |Z|^2 \rangle^{1/2}}{\varepsilon N} \right) \end{aligned} \tag{5.19}$$

The last equation represents a self-consistent equation for fluctuations, and its solution behaves as follows. If $\varepsilon \gtrsim N^{-1/2}$, the normal part of fluctuations is dominant, i.e., $\langle |Z|^2 \rangle \simeq 1/N$, and we have the same result as in Section 4. By including the first correction, we have

$$\langle |Z|^2 \rangle \simeq 1/N + O(1/\varepsilon N^{3/2}) \tag{5.20}$$

If $\varepsilon \lesssim N^{-1/2}$, the anomalous part is dominant, and we have

$$\langle |Z|^2 \rangle \simeq O(1/\varepsilon^2 N^2) \tag{5.21}$$

In any case, critical fluctuations on the order of $1/N$ are absent, and this result is in sharp contrast to ordinary thermodynamic phase transitions.

We now proceed to the ordered phase. The stochastic term in Eqs. (5.7) and (5.9) are state-dependent (i.e., Θ-dependent), but the variable Θ may safely be replaced there by its statistical average (which actually exists). Thus, $f(t, \Theta(t)) \simeq f(t, \langle \Theta \rangle)$, and similarly for $h(t, \Theta(t))$. The Fourier components of $f(t)$ and $h(t)$ are now denoted as $f(\omega)$ and $h(\omega)$, respectively. One may easily check the properties

$$\langle |f(\omega)|^2 \rangle = \tfrac{1}{2} F(\omega) \tag{5.22}$$

$$\langle |h(\omega)|^2 \rangle = \tfrac{1}{2} |\langle Z \rangle|^{-2} F(\omega) \tag{5.23}$$

From Eq. (5.7) the Fourier components of the amplitude fluctuation become

$$\eta(\omega) = \frac{f(\omega)}{1 - (1 - 2\varepsilon)\, \Pi_s(\omega)} \tag{5.24}$$

whose mean square behaves asymptotically as

$$\begin{aligned} \langle |\eta(\omega)|^2 \rangle &\simeq \frac{G_d(\omega)}{N}, && \left| \frac{\omega}{K\langle Z \rangle} \right| \gg 1 \\ &\simeq \frac{1}{N} \frac{b\omega^2}{4\varepsilon^2 + |\alpha_2 \omega / K\langle Z \rangle|^2}, && \left| \frac{\omega}{K\langle Z \rangle} \right| \ll 1 \end{aligned} \tag{5.25}$$

Here we have used the asymptotic form of $F(\omega)$ [see Eq. (4.11)]. Equation (5.25) enables us to make a rough estimation of the total intensity, and we find

$$\begin{aligned} \langle |\eta|^2 \rangle &\simeq \frac{1}{N} \int_{|K\langle Z \rangle|}^{\infty} G_d(\omega)\, d\omega + \frac{b}{N} \int_0^{|K\langle Z \rangle|} \frac{\omega^2}{4\varepsilon^3 + |\alpha_2 \omega / K\langle Z \rangle|^2}\, d\omega \\ &\simeq \frac{1}{N} \int_0^{\infty} G_d(\omega)\, d\omega + O\left(\frac{\varepsilon^{-1}}{N} \int_0^{\varepsilon^{1/2}} \frac{\omega^2}{\varepsilon^2 + \varepsilon^{-1} \omega^2}\, d\omega \right) \\ &\simeq \frac{1 - r}{N} + O\left(\frac{\varepsilon^{1/2}}{N} \right) \end{aligned} \tag{5.26}$$

It is seen that the low-frequency part gives only a negligible contribution, so that there are no critical fluctuations at all.

The phase fluctuation of Z can be estimated from Eq. (5.9). Its Fourier components satisfy

$$\Theta(\omega)=h(\omega)/[1-\Pi_s(\omega)] \tag{5.27}$$

Thus,

$$\begin{aligned}\langle|\Theta(\omega)|^2\rangle &= \frac{1}{2}\frac{|\langle Z\rangle|^{-2}F(\omega)}{|1-\Pi_s(\omega)|^2} \\ &\simeq \frac{\beta g(\omega)}{2\varepsilon N}, \qquad \left|\frac{\omega}{K\langle Z\rangle}\right| \gg 1 \\ &\simeq \frac{K^2 b}{2\alpha_2^2 N}, \qquad \left|\frac{\omega}{K\langle Z\rangle}\right| \ll 1 \end{aligned} \tag{5.28}$$

It is remarkable that $\langle|\Theta(\omega)|^2\rangle$ remains finite in the limit $\omega \to 0$ (under fixed ε). As a result, the total intensity $\langle|\Theta|^2\rangle$ is also nondivergent. This property should be contrasted with ordinary systems of broken continuous symmetry under stochastic driving forces for which the order parameter phase exhibits diffusion, thus eventually restoring the original symmetry.

Finally, we make a few comments on the above peculiar fluctuation characteristics, i.e., unexpectedly weak critical fluctuations and non-divergent phase fluctuation. A qualitative reason for small low-frequency fluctuations in the disordered phase can be better appreciated if one approximates Eq. (5.5) by the Markovian form

$$\frac{dZ}{dt}|\langle Z\rangle|^{-1} \simeq -|\varepsilon| Z + Z_d \tag{5.29}$$

which is valid for time scales longer than $|K\langle Z\rangle|^{-1}$. Because of the large factor $|\langle Z\rangle|^{-1}$ multiplying dZ/dt, the above equation resembles the Langevin equation

$$m\, dv/dt = -\gamma v + \text{fluctuating forces} \tag{5.30}$$

for a Brownian particle with large mass m. The smallness in critical fluctuation is thus interpreted as originating from the large inertia of the order parameter, in much the same way as a heavy Brownian particle exhibits small velocity fluctuations. In the presence of macroscopic order, the order parameter fluctuations are further suppressed by the additional effect that the intensity of the low-frequency components of the stochastic term becomes extremely small. The same effect is responsible for the absence of phase diffusion. It is easily seen from Eq. (5.28) that the low-frequency singularity of the phase fluctuation, which would appear if the noise intensity were constant in the low-frequency region, is perfectly canceled by $F(\omega)$, which is proportional to $|\omega|^2$ in the same region.

No computer studies on critical fluctuations and order parameter relaxation exist except Daido's brief reports on a computer simulations for a time-discrete version of the present model.[11] It is hoped that extensive numerical studies and comparison with the present theory and related ones will be undertaken in the near future.

6. CONCLUDING REMARKS

Dissipative dynamical systems of infinitely many degrees of freedom are formidable objects for scientific research. Any effort at finding new methods and concepts with potential universal applicability would therefore be welcome. In this final section we direct our attention to a certain unique feature of the present approach, hoping that it might possibly enjoy wider applicability in the future. The system treated in the present theory is rather special in some respects. In particular, we assumed an externally given distribution in frequency and also a special form of coupling (i.e., mean field coupling). These assumptions made it possible to represent almost exactly the high-dimensional attractor by a high-dimensional torus. As a generalization of our system, one may imagine a system of short-range interaction. Our high-dimensional attractor would then be a genuine strange attractor, and what the present approach suggests is that this attractor should also be replaced by a torus. From the viewpoint of dynamical system theory, such an approximation may be absurd. From the viewpoint of statistical mechanics of many-body systems, however, the same kind of approximation is by no means unreasonable. Quite on the contrary, it is of daily use and is known as the mean field approximation. In mechanical language, the mean field approximation (or, more generally, one-particle approximation) in usual thermodynamic systems with a Hamiltonian implies replacement of an ergodic orbit by a set of tori, and essentially the same step was also taken in the present approach. The only difference is that in thermodynamic systems each of those tori has its own statistical weight determined from the uniform measure on the surface of constant energy, whereas in our particular dissipative system only one torus is sufficient because it covers the entire measure. Since we know nothing about the general form of the invariant measure in large dissipative systems, some approximation made at the mechanical level (typically, replacement of a strange attractor by tori) should inevitably precede the construction of an invariant measure. For many-body Hamiltonian systems, in contrast, the introduction of one-particle pictures, possibly in various sophisticated forms far beyond the mean field theory, comes after the general form for the invariant measure has been found.

In any case, our proposal of reversing the order of traditional steps in approaching the statistical mechanics of large dynamical systems seems to deserve further serious consideration.

Appendix A

Formulas (3.11), (3.13), and (3.14) are derived below. We begin with the definition of Z_s:

$$Z_s(t) = \int_0^{2\pi} n_s(\psi, t) e^{i\psi} \, d\psi \tag{A.1}$$

For each oscillator in the S group, the natural frequency ω and the phase value as $t \to \infty$ have a one-to-one correspondence:

$$\psi(t) = \overline{\Theta(t)}^{\omega} + \sin^{-1} \frac{\omega}{K|\overline{Z(t)}^{\omega}|} \tag{A.2}$$

which is nothing but an explicit form of Eq. (3.8). With the help of Eq. (A.2), the phase distribution n_s is most easily calculated from the identity

$$n_s(\psi) \, d\psi = g(\omega) \, d\omega \tag{A.3}$$

the insertion of which into Eq. (A.1) yields

$$Z_s(t) = \int_{\text{S group}} d\omega \, g(\omega) \exp\left\{ i \left[\overline{\Theta(t)}^{\omega} + \sin^{-1} \frac{\omega}{K|\overline{Z(t)}^{\omega}|} \right] \right\} \tag{A.4}$$

The new variable y defined by

$$y = \omega / K|\overline{Z(t)}^{\omega}| \tag{A.5}$$

is now used instead of ω in Eq. (A.4). Because the subpopulation boundaries existing at about $\omega = \pm K|Z(t)|$ are somewhat obscured, and may as well be regarded as existing at $\omega = \pm K|\overline{Z(t)}^{\omega}|$, one can fix the range of the ω integration in Eq. (A.4) according to

$$\int_{\text{S group}} d\omega = \int_{-1}^{1} dy \, K|\overline{Z(t)}^{\omega}| \tag{A.6}$$

In this way, Eq. (A.4) becomes

$$Z_s(t) = \int_{-1}^{1} dy\, K|\overline{Z(t)}^y|\, g(K|\overline{Z(t)}^y|\, y)$$
$$\times \exp[i(\sin^{-1} y + \overline{\Theta(t)}^y)]$$
$$= \int_{-1}^{1} dy\, K|\overline{Z(t)}^y|\, e^{i\overline{\Theta(t)}^y} g(K|\overline{Z(t)}^y|\, y)(1-y^2)^{1/2}$$
$$= 2\int_{0}^{1} dy\;\; K\overline{Z(t)}^y\, g(K|\overline{Z(t)}^y|\, y)(1-y^2)^{1/2} \tag{A.7}$$

where we have ignored the possibility of a correlated motion between the amplitude and phase of Z; this is consistent with the conclusion in Section 3, where we find that Θ is a constant of motion. Note that the last expression in Eq. (A.7) coincides with $S(\overline{Z(t)})$ if the y dependence of the bar is ignored [see Eq. (2.32)]. A small-amplitude expansion of Eq. (A.7) gives

$$Z_s(t) = 2Kg(0) \int_0^1 dy\, \overline{Z(t)}^y (1-y^2)^{1/2}$$
$$+\; K^3 g''(0) \int_0^1 dy\, y^2 (1-y^2)^{1/2} |\overline{Z(t)}^y|^2\, \overline{Z(t)}^y + \cdots \tag{A.8}$$

The first term in the above expansion is expressed as

$$2Kg(0) \int_0^1 dy\, \overline{Z(t)}^y (1-y^2)^{1/2} = (1+\varepsilon)\, \overline{Z(t)} \tag{A.9}$$

where the bar without oscillator dependence is defined by

$$\overline{Z(t)} = \frac{4}{\pi} \int_0^1 dy\, \overline{Z(t)}^y (1-y^2)^{1/2} \tag{A.10}$$

Equation (A.10) is identical to Eq. (3.13), as is easily seen from the definition of the oscillator-dependent bar given by Eq. (3.9). The cubic term in Eq. (A.8) is not like $|\overline{Z(t)}|^2\, \overline{Z(t)}$. However, the quantity $\overline{Z(t)}^y$ is only slightly different from $\overline{Z(t)}$, so that this small difference can safely be neglected in the cubic term, which itself is much smaller than the linear term. The same reasoning applies to terms higher than the third, and in this way we arrive at Eq. (3.11).

APPENDIX B

Formulas (3.22)–(3.27) are derived here. Before trying to find a general form of $n_{\mathrm{d}}(\psi, t)$ under slowly varying Z, let us consider a simpler process in which Z varies in small amplitude about some value Z_0 like

$$Z(t) = Z_0 + \delta Z(t) \tag{B.1}$$

The distribution $n_{\mathrm{d}}(\psi, t)$ should then be not much different from $n_{\mathrm{d}0}(\psi; Z_0)$, but we are concerned with this small difference as a functional of $\delta Z(t)$. It should be remembered, however, that the boundary of the D group is somewhat blurred due to the temporal variation of Z. As is stated in the text, we always ignore such a blurring effect or the effect from the small number of borderline oscillators. In the linear process under consideration, this means that we ignore the inflow and outflow across the boundaries. Thus, we confine our observation to those oscillators that are in some sense deep in the D group over the entire linear process (B.1), so that the total number of such well-defined D-group oscillators is supposed to be constant (but of course dependent on Z_0). For this reason, it is more appropriate to work mainly with the normalized distribution

$$\hat{n}_{\mathrm{d}}(\psi, t) = n_{\mathrm{d}}(\psi, t) \Big/ \int_0^{2\pi} n_{\mathrm{d}}(\psi, t)\, d\psi \tag{B.2}$$

rather than $n_{\mathrm{d}}(\psi, t)$. Because $\hat{n}_{\mathrm{d}}(\psi, t)$ can be regarded as a linear functional of $\delta Z(t)$, it should generally take the form

$$\hat{n}_{\mathrm{d}}(\psi, t) = \hat{n}_{\mathrm{d}0}(\psi; Z_0) + \int_0^{\infty} dt'\, \delta Z(t - t')\, M(\psi, t'; Z_0) \tag{B.3}$$

If δZ is constant in time, $n_{\mathrm{d}}(\psi, t)$ must coincide with $\hat{n}_{\mathrm{d}0}(\psi; Z_0 + \delta Z)$, so that in the linear approximation

$$\hat{n}_{\mathrm{d}}(\psi, t) = \hat{n}_{\mathrm{d}0}(\psi; Z_0) + \delta Z \frac{d\hat{n}_{\mathrm{d}0}(\psi; Z_0)}{dZ_0} \tag{B.4}$$

By comparison of Eqs. (B.4) and (B.3) for the special case $\delta Z = \text{const}$, we obtain the identity

$$\int_0^{\infty} M(\psi, t'; Z)\, dt' = \frac{d\hat{n}_{\mathrm{d}0}(\psi; Z)}{dZ} \tag{B.5}$$

Introduce a real function $\Pi_{\mathrm{d}}(\psi, t; Z)$ by

$$M(\psi, t; Z) = \frac{d\hat{n}_{\mathrm{d}0}(\psi; Z)}{dZ} \Pi_{\mathrm{d}}(\psi, t; Z) \tag{B.6}$$

Obviously,

$$\int_0^\infty \Pi_{\mathrm{d}}(\psi, t'; Z)\, dt' = 1 \tag{B.7}$$

Equation (B.3) now becomes

$$\hat{n}_{\mathrm{d}}(\psi, t) = \hat{n}_{\mathrm{d}0}(\psi; Z_0) + \int_0^\infty dt' \frac{d\hat{n}_{\mathrm{d}0}(\psi; Z_0)}{dZ_0} \delta Z(t-t')\, \Pi_{\mathrm{d}}(\psi, t'; Z_0) \tag{B.8}$$

which may be represented compactly in the linear approximation as

$$\begin{aligned} \hat{n}_{\mathrm{d}}(\psi, t) &= \int_0^\infty dt'\, \hat{n}_{\mathrm{d}0}(\psi; Z(t-t'))\, \Pi_{\mathrm{d}}(\psi, t'; Z_0) \\ &\simeq \int_0^\infty dt'\, \hat{n}_{\mathrm{d}0}(\psi; Z(t-t'))\, \Pi_{\mathrm{d}}(\psi, t'; Z(t)) \end{aligned} \tag{B.9}$$

Thus, the unnormalized distribution is given by

$$\begin{aligned} n_{\mathrm{d}}(\psi, t) &\simeq \int_0^{2\pi} n_{\mathrm{d}0}(\psi', t; Z_0)\, d\psi' \cdot \hat{n}_{\mathrm{d}}(\psi, t) \\ &\simeq \int_0^{2\pi} n_{\mathrm{d}0}(\psi', t; Z(t))\, d\psi' \int_0^{2\pi} dt'\, \hat{n}_{\mathrm{d}0}(\psi; Z(t-t'))\, \Pi_{\mathrm{d}}(\psi, t'; Z(t)) \end{aligned} \tag{B.10}$$

again within linear approximation.

So far the study has been on a linear process with respect to the amplitude variation of Z. We now proceed to the case where the variation of Z is nonlinear. But the evolution of Z is so slow that its variation can still be regarded as linear over a finite period. This period can be much longer than the decay time of Π_{d}, which is $O(|KZ|^{-1})$. Let the time axis be divided into segments the length of which about time t is chosen to be longer than $|KZ(t)|^{-1}$ but shorter than the characteristic time of Z. Then in each such interval Eqs. (B.9) and (B.10) hold, which is equivalent to saying that the same equations hold over the entire time domain. Moreover, Eqs. (B.9) and (B.10) reduce, respectively, to Eqs. (3.22) and (3.23) because of the assumed local linearity in the variation of Z.

One can understand the physical implications of the function Π_{d} by applying formula (B.8) to the special process

$$Z(t) = \begin{cases} Z_0, & t \leqslant 0 \\ Z_1, & t > 0 \end{cases} \tag{B.11}$$

where $Z_1 - Z_0 \equiv \delta Z$ is sufficiently small. It is clear that

$$\hat{n}_{\mathrm{d}}(\psi, t) = \begin{cases} \hat{n}_{\mathrm{d}0}(\psi; Z_0), & t \leqslant 0 \\ \hat{n}_{\mathrm{d}0}(\psi; Z_0) + \delta Z \dfrac{d\hat{n}_{\mathrm{d}0}(\psi; Z_0)}{dZ_0} \displaystyle\int_0^t \Pi_{\mathrm{d}}(\psi, t'; Z_0)\, dt', & t > 0 \end{cases} \tag{B.12}$$

Since we have

$$\begin{aligned} \int_0^t \Pi_{\mathrm{d}}(\psi, t'; Z_0)\, dt' &= -\int_t^\infty \Pi_{\mathrm{d}}(\psi, t'; Z_0)\, dt' + \int_0^\infty \Pi_{\mathrm{d}}(\psi, t'; Z_0)\, dt' \\ &= -\Theta_{\mathrm{d}}(\psi, t; Z_0) + 1 \end{aligned} \tag{B.13}$$

where Θ_{d} is defined by Eq. (3.25), we finally obtain Eq. (3.27).

APPENDIX C

Formulas (4.5a)–(4.5e) will be proved first. By definition,

$$\begin{aligned} &\langle Z_{\mathrm{d}}(t_0)\, Z_{\mathrm{d}}^*(t_0 + t) \rangle \\ &= \int_0^{2\pi} d\boldsymbol{\psi}_{\mathrm{d}}\, \rho_{\mathrm{d}0}(\boldsymbol{\psi}_{\mathrm{d}}) \frac{1}{N^2} \sum_{j, j' \in D}\sum \exp\{ i[\psi_j(t_0) - \psi_j(t_0 + t)] \} \end{aligned} \tag{C.1}$$

where the notation $\boldsymbol{\psi}_{\mathrm{d}}$ without time argument should be understood as $\boldsymbol{\psi}_{\mathrm{d}}(0)$. Because the oscillators are mutually uncorrelated, the terms to be retained in the summation in Eq. (C.1) are only those with $j = j'$. In this way, we get

$$\begin{aligned} &\langle Z_{\mathrm{d}}(t_0)\, Z_{\mathrm{d}}^*(t_0 + t) \rangle \\ &= \frac{1}{N^2} \sum_{j \in D} \int_0^{2\pi} d\psi_j \frac{\tilde{\omega}_j}{2\pi} v_j(\psi_j)^{-1} \exp\{ i[\psi_j(t_0) - \psi_j(t_0 + t)] \} \\ &= \frac{1}{N^2} \sum_{j \in D} \frac{\tilde{\omega}_j}{2\pi} \int_0^{2\pi/\tilde{\omega}_j} d\tau \exp\{ i[\psi_j(t_0 + \tau) - \psi_j(t_0 + \tau + t)] \} \\ &= \frac{1}{N^2} \sum_{j \in D} \frac{\tilde{\omega}_j}{2\pi} \int_0^{2\pi/\tilde{\omega}_j} d\tau \exp\{ i[\psi_j(\tau) - \psi_j(\tau + t)] \} \\ &= \langle Z_{\mathrm{d}}(0)\, Z_{\mathrm{d}}^*(t) \rangle = F(t) \end{aligned} \tag{C.2}$$

where the second equation results from the obvious equation

$$\frac{d\psi_j(\tau)}{d\tau} = v_j(\psi_j(\tau)) \tag{C.3}$$

Equation (4.5a) has thus been proved.

The definition (4.6) of $D_l(\omega)$ immediately leads to the equality

$$\frac{\tilde{\omega}_j}{2\pi}\int_0^{2\pi/\tilde{\omega}_j} d\tau \exp\{i[\psi_j(\tau) - \psi_j(\tau + t)]\} = \sum_{l=1}^{\infty} |D_l(\tilde{\omega}_j)|^2 \exp(-il\tilde{\omega}_j t) \tag{C.4}$$

Substitution of Eq. (C.4) into Eq. (C.2) gives

$$F(t) = \frac{1}{N^2} \sum_{j\in D} \sum_{l=1}^{\infty} |D_l(\tilde{\omega}_j)|^2 \exp(-il\tilde{\omega}_j t) \tag{C.5}$$

Equation (C.5) can be expressed in terms of the distribution $G_d(\tilde{\omega})$ of $\tilde{\omega}_0$ in the form of Eq. (4.5c). The time-reversal symmetry in Eq. (4.5b) can be proved as follows. First we notice that the oscillators with real frequencies $\tilde{\omega}$ and $-\tilde{\omega}$ are equally populated:

$$G_d(\tilde{\omega}) = G_d(-\tilde{\omega}) \tag{C.6}$$

Second, $\psi_j(t) = -\psi_j(t) + \text{const}$ for any t, provided that $\tilde{\omega}_j = -\tilde{\omega}_j$. This second property is applied to the definition of $D_l(\tilde{\omega})$ in Eq. (4.6) to give

$$|D_l(\tilde{\omega})|^2 = |D_l(-\tilde{\omega})|^2 \tag{C.7}$$

From Eqs. (C.6), (C.7), and (4.5c), property (4.5b) is clear.

Calculations of $\langle \operatorname{Re} Z_d(0) \operatorname{Re} Z_d(t)\rangle$ and $\langle \operatorname{Im} Z_d(0) \operatorname{Im} Z_d(t)\rangle$ go essentially the same way as that of $F(t)$; the proofs of Eqs. (4.5d) and (4.5e) are straightforward and omitted here.

To find the main features of the power spectrum $F(\omega)$, we need some knowledge about $D_l(\tilde{\omega})$. First, the property

$$\sum_{l=1}^{\infty} |D_l(\tilde{\omega})|^2 = 1 \tag{C.8}$$

is obtained from Eq. (C.4). The l dependence of $D_l(\tilde{\omega})$ is entirely different in high- and low-frequency regions. If the real frequency $\tilde{\omega}$ satisfies $|\tilde{\omega}/K\langle Z\rangle| \gg 1$, the corresponding oscillator undergoes an almost smooth oscillation. This means that practically only the fundamental component is nonvanishing:

$$\begin{aligned} &|D_1(\tilde{\omega})|^2 \simeq 1 \\ &D_l(\tilde{\omega}) \simeq 0, \qquad l \geqslant 2 \end{aligned} \tag{C.9}$$

In contrast, if $|\tilde{\omega}/K\langle Z\rangle| \ll 1$, the oscillation is strongly nonsmooth; for most of the period the oscillator phase sits near $\pi/2$ or $-\pi/2$ (depending on the sign of $\tilde{\omega}$), which is followed abruptly by a departure and relatively quick circuit back to the original position. As a consequence, the quantity $\exp[i\psi(t)]$ behaves as periodic pulses with width $O(|K\langle Z\rangle|)$. This means that $|D_l(\tilde{\omega})|^2$ are nonvanishing and almost l-independent for $l \lesssim |K\langle Z\rangle/\tilde{\omega}|$, while they are virtually zero for $l \gtrsim |K\langle Z\rangle/\tilde{\omega}|$. Taking account of the normalization condition (C.8), we thus obtain

$$\begin{aligned} |D_l(\tilde{\omega})|^2 &\simeq |\tilde{\omega}/K\langle Z\rangle|, \qquad && l \lesssim |K\langle Z\rangle/\tilde{\omega}| \\ &\simeq 0, && l \gtrsim |K\langle Z\rangle/\tilde{\omega}| \end{aligned} \tag{C.10}$$

Properties (C.9) and (C.10) are sufficient for studying the characteristics of the fluctuation spectrum $F(\omega)$ through Eq. (4.9). At high frequencies ($|\tilde{\omega}/K\langle Z\rangle| \gtrsim 1$), only the term with $l=1$ contributes to $F(\omega)$, and we get Eq. (4.10), while at low frequencies ($|\tilde{\omega}/K\langle Z\rangle| \lesssim 1$) we have

$$F(\omega) \simeq \frac{1}{N} \sum_{l=1}^{\infty} G_{\mathrm{d}}\left(\frac{\omega}{l}\right) \left|\frac{\omega}{lK\langle Z\rangle}\right| l^{-1} \tag{C.11}$$

Substitution of the low-frequency form of $G_{\mathrm{d}}(\omega)$ given by Eq. (2.40) into Eq. (C.11) leads to formula (4.11).

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