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On the Time It Takes a State Vector to Reduce

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It is pointed out, in theories which give a dynamical description of the reduction of the state vector, that the reduction should take place in a finite time. It is shown that the reduction time is infinite in the Bohm–Bub theory, and finite in the author's theory.

KEY WORDS: State Vector reduction; Dynkin's equation; first passage time; gambler's ruin problem; diffusion; Bohm–Bub's reduction theory.

1. INTRODUCTION

When quantum theory describes the measurement process, the state vector after the measurement has the form

$$|\psi, t\rangle = \sum_n a_n \, |\phi_n, t\rangle \tag{1.1}$$

Here $|\phi_n, t\rangle$ describes the complete physical system (including apparatus) corresponding to the nth outcome of the experiment. The squared amplitude $|a_n|^2$ is the probability of the nth outcome, and does not change with time once the measurement is completed, as long as $|\psi, t\rangle$ evolves according to the usual Schrödinger equation.

The so-called "reduction of the state vector" is the replacement of (1.1) by

$$|\psi, t\rangle = |\phi_m, t\rangle \tag{1.2}$$

If the state vector is interpreted as describing an ensemble of identical experiments, then the transition from (1.1) to (1.2) does not *need* a

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dynamical description.[1–3] This *nondynamical reduction of the state vector* is an act performed on a piece of paper by a theoretical physicist who wishes to describe the future behavior of a subensemble of experiments, those which resulted in the mth outcome. To do this, the theoretician must be confident that the physical situation is such that the individual terms in (1.1) will not interfere in the future,[2] so he is justified in considering the future evolution of (1.2) by itself. The reduction of the state vector is "a convenience, not a necessity,"[1] because the theoretician would obtain the same future predictions by considering the more cumbersome equation (1.1), and finally projecting out the subensemble of interest.

However, if the state vector is to be interpreted as describing (as being in one-to-one correspondence with) a single experiment, then as the physical system dynamically evolves, the transition from (1.1) to (1.2) must take place dynamically. The result (1.1) given by the Schrödinger equation is an incorrect description of the outcome of the single measurement, so the Schrödinger equation must be modified. In a *dynamical reduction theory* one shows how the amplitude a_m grows to unity while all the other amplitudes decrease to zero.[3]

If a dynamical reduction theory is to make physical sense, *the reduction must take place in a finite time* because an experiment reveals its outcome in a finite time. It is the purpose of this note to discuss the reduction time for the dynamical reduction theories of Bohm and Bub[9] and the author,[4] the first two theories of this type. It will be emphasized that the reduction time for the former theory is infinite, and for the latter theory is finite.

[2] From our point of view, the "problem of the theory of measurement," whose solution is undertaken by Daneri, Loinger, and Prosperi,[4] by Zeh[5] and by Zurek,[6] is simply how to delineate the circumstances under which such a nondynamical reduction is justified. The argument typically consists in showing how, *for practical purposes*, the density matrix of the pure state, $|\psi, t\rangle\langle\psi, t|$, can be replaced after a measurement by the density matrix of the mixture $\sum_n |a_n|^2 |\phi_n, t\rangle\langle\phi_n, t|$.

[3] Machida and Namiki[7] and Wigner[8] alter quantum theory and construct dynamical reduction theories in which a pure density matrix evolves into a mixture (the former by introducing a family of Hilbert spaces to describe the apparatus and averaging over them, the latter by modifying the Schrödinger equation for the density matrix). However, these authors do not show how a single system ends up described by one or another diagonal element of the mixture density matrix. Their work might be regarded as preliminary, to be followed in the future by a more detailed dynamics of each individual system, in the ensemble they are describing.

[4] See Ref. 10. Reference 10c discusses the gambler's ruin analogy. Ref. 10d contains a review of the preceding work. Reference 10e discusses experimental tests which can determine the magnitude of the reduction time.

2. BOHM–BUB'S DYNAMICAL REDUCTION THEORY

In the Bohm–Bub theory, the dynamical reduction is described by

$$\frac{da_n(t)}{dt} = \frac{\gamma}{2} a_n(t) \sum_{m=1}^{N} \left(\frac{|a_n(t)|^2}{|\zeta_n|^2} - \frac{|a_m(t)|^2}{|\zeta_m|^2} \right) |a_m(t)|^2 \tag{2.1}$$

where γ is a constant characterizing the reduction rate. The ζ_n are complex constants, the so-called "hidden variables" of Wiener and Siegal.(11) They are fixed for any single system, and are the components of a vector on the unit sphere in an N-dimensional complex vector space. Each different physical system will have a different vector ζ, and the vectors for a complete ensemble of physical systems are assumed to be uniformly distributed over the unit sphere.

It is assumed that the ordinary Schrödinger equation dominates the description of the measurement until time $t = 0$ (γ is negligibly small for $t < 0$), and that the dynamics described by Eq. (2.1) dominates for $t > 0$ (γ is large for $t > 0$). Therefore, at $t = 0$ the state vector is given by Eq. (1.1) (thus $|a_m(0)|^2$ is the probability of the mth outcome of the measurement according to quantum theory), which provides the initial conditions for Eq. (2.1).

It can be shown(9,12) that, according to Eq. (2.1), the state to which the reduction takes place is the one for which $|a_m(0)|^2/|\zeta_m|^2$ is the largest. It also can be shown that the fraction of state vectors in the ensemble which reduce to $|\phi_m, t\rangle$ is $|a_m(0)|^2$, thereby giving the same predictions as quantum theory once the reduction is completed. The trouble is that the amplitudes approach their asymptotic values of 0 and 1 exponentially with time,[5] so the reduction is not completed in a finite time.

To see this, let us first define $x_n(t) \equiv |a_n(t)|^2$, $z_n \equiv |\zeta_n|^2$; upon multiplying (2.1) by a_n^* and adding the complex conjugate equation we obtain

$$\frac{dx_n}{dt} = \gamma x_n \sum_{m=1}^{N} \left(\frac{x_n}{z_n} - \frac{x_m}{z_m} \right) x_m \tag{2.2}$$

First, consider the special case of a two-state system. Then (2.2) can be integrated exactly, with the result

$$\frac{x_1(t)^{z_2}[1 - x_1(t)]^{z_1}}{x_1(t) - z_1} = \frac{x_1(0)^{z_2}\, x_2(0)^{z_1}}{x_1(0) - z_1} e^{-\gamma t} \tag{2.3}$$

[5] J. J. Tutsch(13) calculates the "slowest collapse time" for the Bohm–Bub theory. By this he means the longest time it takes the largest squared amplitude to rise to 0.99. The exponential behavior is implicit in his result, but he does not comment on the attendant meaninglessness of a superposition of macroscopically distinguishable states that lasts forever.

where $x_2(t) = 1 - x_1(t)$ and $z_2 = 1 - z_1$. If $x_1(0) > z_1$, then it can be seen from (2.3) that $x_1 \to 1$ and $x_2 \to 0$ as $t \to \infty$; indeed,

$$x_1(t) \xrightarrow[t \to \infty]{} 1 - Ae^{-(\gamma/z_1)t}, \qquad x_2(t) \xrightarrow[t \to \infty]{} Ae^{-(\gamma/z_1)t} \tag{2.4a,b}$$

where A is a constant. This behavior holds in the general case. If $x_1/z_1 > x_m/z_m$ ($m \neq 1$), then it follows from (2.2) that $x_1 \to 1$ and $x_m \to 0$ ($m \neq 1$) as $t \to \infty$. Therefore for large times (2.2) becomes

$$\frac{dx_1}{dt} \approx \gamma x_1 \sum_{m=2}^{N} \left(\frac{x_1}{z_1}\right) x_m \approx \frac{\gamma}{z_1}(1 - x_1) \tag{2.5a}$$

[which obtains because $\sum_{m=1}^{N} x_m \equiv 1$ follows from (2.2), and $x_1 \to 1$], and

$$\frac{dx_m}{dt} \approx -\gamma x_m \left(\frac{x_1}{z_1}\right) x_1 \approx -\gamma \frac{x_m}{z_1} \tag{2.5b}$$

Equations (2.5a,b) can be integrated immediately with the results (2.4a,b) [with x_2 replaced by x_m in (2.4b)].

Thus the reduction in the Bohm–Bub theory takes infinite time to be completed, if $\gamma \neq \infty$. In fact, in proposing an experimental test of their theory, Bohm and Bub did take $\gamma = \infty$, and suggested a test of possible dynamics of the hidden variables ζ_n for which they had *not* given a dynamical equation!

A dynamical reduction model has recently been given by Gisin.[14] It also has infinite reduction time.[15]

3. PEARLE'S DYNAMICAL REDUCTION THEORY

In the author's theory, the dynamical reduction is described by

$$i\frac{da_n(t)}{dt} = \sum_{m=1}^{N} \alpha_{nm}(t)\, a_m^*(t) \frac{a_n(t)}{a_n^*(t)} \tag{3.1}$$

(a_m^* is the complex conjugate of a_m). Here $\alpha_{nm} = A_{nm}\dot{B}_{nm}(t)$, where A_{nm} are the constant elements of a Hermitian matrix, and $\dot{B}_{nm}(t)$ are rapidly fluctuating functions of time (a Hermitian matrix of complex white noise). Equation (3.1) is interpreted as a stochastic differential equation in the sense of Stratonovich.[16] It can be shown[10] that, as a consequence of Eq. (3.1), the squared amplitudes $x_n(t) \equiv |a_n(t)|^2$ "play" a modified "gambler's ruin" game.

In the "simple gambler's ruin" game, two gamblers (designated "heads" and "tails") each start with a given amount of dollars and repeatedly toss a fair coin. If the outcome of a toss is heads, then the gambler heads receives a dollar from the gambler tails, and vice versa. The game continues until one gambler loses all his money, and so can play no more.

Let the game be modified so that (i) N gamblers play in pairs (they lose one by one, until a single gambler possesses all the money in the game); (ii) each toss takes place in a short time Δt; (iii) the money exchanged per toss is likewise made small ($\sim \Delta t^{1/2}$); (iv) the rate of play of the nth and mth gamblers is proportional to $[x_n(t)\, x_m(t)]^{1/2}$, where $x_n(t) \equiv$ gambler n's money at time t/total money in the game. Then it can be shown that[10] in the limit $\Delta t \to 0$ (where the game is "played continuously"), the dynamics of the $x_n(t)$ of the gamblers, and the dynamics of the $x_n(t) \equiv |a_n(t)|^2$ described by Eq. (3.1) are identical. Indeed, the *ensemble* of gambler's ruin games and the *ensemble* of solutions of Eq. (3.1) are both described by the same diffusion equation

$$\frac{\partial \rho}{\partial t}(x_1,..., x_n; t) = \frac{1}{2} \sum_{n,m=1}^{N} \sigma_{nm}^2 \left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m}\right)^2 x_n x_m \rho \tag{3.2}$$

$\rho(\mathbf{x}; t)$ is the probability density of the $\mathbf{x}$'s at time t, and σ_{nm}^2 are positive constants.

Because such gambler's ruin games are fair games, or "martingales" ($d\langle x_n \rangle/dt = 0$), it can be shown that the probability the mth gambler (the mth squared amplitude x_m) wins all the money (reaches the value 1) is $x_m(0) =$ the fraction of money with which he started ($=$the probability predicted by conventional quantum theory).[10] Thus the dynamical reduction theory gives the same predictions as quantum theory, once the reduction is completed.

It is also well known[17] that the "simple gambler's ruin" game has a finite mean duration, so that the probability is zero that a particular game will have an infinite duration. However, the modified gambler's ruin game described above is more complicated. Of special concern is modification (iv) above, which states that the rate of play between the nth and mth gamblers slows down as either gambler gets close to losing all his money. It is conceivable that the rate slows down so much that the game never ends. [Indeed, this turns out to be the case if the rate of play is proportional to $(x_n x_m)^{r/2}$ with $r \geqslant 2$.] Therefore, it is worthwhile examining the mean duration of the "games" described by Eq. (3.2), to prove that the reduction time is finite in this theory.

4. MEAN REDUCTION TIME

Equation (3.2) describes a stationary diffusion of the Fokker–Planck type[18]

$$\frac{\partial}{\partial t}\rho(\mathbf{x};\mathbf{x}_0;t)=-\sum_{n=1}^{N}\frac{\partial}{\partial x_n}v_n\rho+\frac{1}{2}\sum_{n,m}^{N}\frac{\partial^2}{\partial x_n\,\partial x_m}D_{nm}\rho \tag{4.1a}$$

$$v_n=\frac{\langle\Delta x_n\rangle}{\Delta t}\bigg|_{\Delta t\to 0},\qquad D_{nm}=\frac{\langle\Delta x_n\,\Delta x_m\rangle}{\Delta t}\bigg|_{\Delta t\to 0} \tag{4.1b}$$

with vanishing drift $v_n=0$. (The vanishing drift is a consequence of the "fairness" of the coin tossed by the gamblers.) One can imagine a single point [representing a member of the ensemble of systems described by Eq. (4.1) or (3.2)] starting out at $\mathbf{x}$ at time $t=0$, and randomly walking over the $\mathbf{x}$ space until it reaches some specified boundary. The amount of time this takes is called the "first passage time."

One can write a differential equation for the mean first passage time $m(x)$ called "Dynkin's equation"[19]:

$$\sum_{n=1}^{N}v_n\frac{\partial}{\partial x_n}m(\mathbf{x})+\frac{1}{2}\sum_{n,m=1}^{N}D_{nm}\frac{\partial^2 m(\mathbf{x})}{\partial x_n\,\partial x_m}=-1 \tag{4.2}$$

This is supplemented by the condition that m must vanish on the boundary, as it takes no time to reach the boundary from the boundary. One way to arrive at Eq. (4.2) is to realize that

$$m(\mathbf{x})=\Delta t+\int d\,\Delta\mathbf{x}\;\rho(\mathbf{x}+\Delta\mathbf{x};\mathbf{x};\Delta t)\,m(\mathbf{x}+\Delta\mathbf{x}) \tag{4.3}$$

which says that the mean first passage time from $\mathbf{x}$ can be found by letting the system evolve for Δt sec to $\mathbf{x}+\Delta\mathbf{x}$, and adding to Δt the mean of the first passage times from the new locations. Equations (4.2) follows from Eq. (4.3) upon expanding $m(\mathbf{x}+\Delta\mathbf{x})$ in a Taylor series in $\Delta\mathbf{x}$, and utilizing (4.1b).

For the diffusion equation (3.2), the mean first passage time is therefore the solution of

$$\frac{1}{2}\sum_{n,m=1}^{N}\sigma_{nm}^2 x_n x_m\left(\frac{\partial}{\partial x_n}-\frac{\partial}{\partial x_m}\right)^2 m(\mathbf{x})=-1 \tag{4.4}$$

However, the boundary condition that must be applied in order that the mean first passage time be identified as the mean reduction time is somewhat unusual, and needs explanation.

The configuration space of the point $\mathbf{x}$ in the modified gambler's ruin problem described by Eq. (3.2) is the hyperplane $\sum_{n=1}^{N} x_n = 1$ bounded by the hyperplanes $x_m = 0$ $(m = 1,..., N)$. For a two-person game this is a line segment, for a three-person game it is an equilateral triangle, for a four-person game it is a regular tetrahedron, etc. For example, a point representing a single four-person game diffuses from the inside of the tetrahedron to one of the triangular boundary faces, *but the diffusion does not stop there.* When the point reaches, e.g., the face $x_4 = 0$, this simply means that the fourth gambler has lost all his money and is out of the game, but the remaining three gamblers continue to play. The phase point diffuses over the triangular face until it reaches an edge of the triangle (another gambler knocked out of the game), and diffuses over the edge until it reaches a vertex (where it stops—the game is over). All this is described by Eq. (3.2)![10] Thus the boundary condition on $m(\mathbf{x})$ requires that it only vanish at the N "vertices" $\{x_m = 1,\ x_k = 0\ (k \neq m);\ \text{for } m = 1 \text{ to } N\}$. On the geometric boundary of the configuration space region for the N person game (e.g., on the triangular faces of the tetrahedron), $m(\mathbf{x})$ must reduce to the expression for the mean reduction time for the $(N-1)$-person game, i.e., to the solution of Eq. (4.4) with N replaced by $N-1$, with similar boundary conditions.

We first specialize to the simplest case, in which $\sigma_{nm}^2 = \sigma^2$ (symmetry of the rate of play of all pairs of gamblers). The solution of Eq. (4.4)

$$\frac{\sigma^2}{2} \sum_{n,m=1}^{N} x_n x_m \left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m} \right)^2 m(\mathbf{x}) = -1 \tag{4.5}$$

which satisfies the correct boundary conditions can be written in closed form:

$$m(\mathbf{x}) = -\frac{1}{\sigma^2} \sum_{n=1}^{N} (1 - x_n) \ln(1 - x_n) \tag{4.6}$$

[Substituting (4.6) in the left side of (4.5) and differentiating yields $-\sum_n \sum_{m \neq n} x_n x_m/(1 - x_n) = -1$.]

It is clear from the entropylike expression (4.6) for the mean reduction time that it is finite, having its largest value

$$m(\mathbf{x}) \leqslant \frac{1}{\sigma^2} (N-1) \ln[1 + (N-1)^{-1}] \tag{4.7}$$

where $x_n = 1/N$ $(n = 1,..., N)$. Since the mean reduction time is finite, those state vector reductions (modified gambler's ruin games) in the ensemble which take an infinite time to be completed comprise only a set of measure zero.

In passing, it is worth noting that, for large N, the maximum mean reduction time on the right-hand side of Eq. (4.7) becomes $1/\sigma^2$, independent of N. This is a desirable feature in a dynamic reduction theory. For example, one would not wish the reduction time to grow without bound or to vanish just because an apparatus is designed with a fine resolution to measure one among many possible eigenvalues.

Finally we treat the general case (4.4), for which we show that the solution of the simpler case (4.5) is a majorant or minorant with appropriate choices of σ^2. That is, if we denote the solution of Eq. (4.5) by $m(\mathbf{x}, \sigma^2)$, retaining $m(\mathbf{x})$ as the solution of Eq. (4.4), and choose $\bar{\sigma}^2 > \max(\sigma^2_{nm})$, $\underline{\sigma}^2 < \min(\sigma^2_{nm})$, then

$$m(\mathbf{x}, \underline{\sigma}^2) \geqslant m(\mathbf{x}) \geqslant m(\mathbf{x}, \bar{\sigma}^2) \tag{4.8}$$

The argument is a standard one.[20] One easily shows that

$$\frac{1}{2} \sum_{n,m} \sigma^2_{nm} x_n x_m \left(\frac{\partial}{\partial x_n} - \frac{\partial}{\partial x_m}\right)^2 [m(\mathbf{x}, \underline{\sigma}^2) - m(\mathbf{x})]$$
$$= \frac{-1}{\underline{\sigma}^2} \sum_n \sum_{m \neq n} \sigma^2_{nm} \frac{x_m x_n}{1 - x_n} + 1 < 0 \tag{4.9}$$

except at the boundary vertices. It follows that $m(\mathbf{x}, \underline{\sigma}^2) - m(\mathbf{x})$ cannot achieve a local minimum (for which the left-hand side of Eq. (4.9) is positive semidefinite, in violation of the inequality). The minimum is thus achieved only at the boundary vertices, where it equals zero, since at those points $m(\mathbf{x}, \underline{\sigma}^2) = m(\mathbf{x}) = 0$. Thus the left-hand inequality in (4.8) is obtained. The right-hand inequality is similarly obtained by considering $m(\mathbf{x}) - m(\mathbf{x}, \bar{\sigma}^2)$.

This completes our demonstration that the reduction time is finite in this theory.

Added Note. After this work was completed, it was brought to my attention by the referee that the diffusion equation (3.2) in the simplest case $\sigma^2_{nm} = \sigma^2$ is known to mathematical population geneticists as the diffusion approximation to the Wright–Fisher model of genetic drift.[21] The model considers a large but finite genetic population composed of a number of alleles (genetic variants such as eye colors). The next generation, of equal size, evolves by random selection with replacement from the previous generation. The model is called "neutral" because no selective differences or mutations are involved. Geneticists are interested in the statistics associated with "fixation" of an allele, by which they mean that the population becomes composed solely of that allele—what we call "reduction"! Known results include that the probability of fixation of an allele equals its initial frequency (in our context, what ensures that the predictions of quantum theory are obtained following reduction) and that the mean time of fixation is finite [and given by Eq. (4.6)].

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