Journal of Statistical Physics, Vol. 32, No. 3, 1983

On the Boltzmann Equation for the Lorentz Gas

C. Boldrighini,[1] L. A. Bunimovich,[2] and Ya. G. Sinai[2]

Received March 10, 1983; revised March 31, 1983

We consider the Boltzmann–Grad limit for the Lorentz, or wind-tree, model. We prove that if ω is a fixed configuration of scatterer centers belonging to a set of full measure with respect to the Poisson distribution with parameter $\lambda > 0$, then the evolution of an initial a.c. particle density tends in the Boltzmann–Grad limit to the solution of the Boltzmann equation for the model. As an intermediate step we prove that the process of the free path lengths and impact parameters induced by the Lebesgue measure on a small region tends to a limiting independent process.

KEY WORDS: Boltzmann–Grad limit; Boltzmann equation; Lorentz model.

INTRODUCTION

For the classical Lorentz gas the kinetic Boltzmann equation goes, in the Boltzmann–Grad limit[(1)] into the linear Fokker–Planck–Kolmogorov equation for the corresponding Markov process. What is known up to now on this problem can be found in the paper of Spohn.[(2)]

In the generally accepted approach to the derivation of the Boltzmann equation for the Lorentz gas, which was first proposed by Gallavotti,[(3,4)] one writes down first the equations for the correlation functions averaged over scatterer configurations, and then takes the Boltzmann–Grad limit of such equations. Spohn[(2)] strengthened this result by proving convergence in probability. In this paper, using a different method, we show that the limiting Boltzmann equation holds for typical configurations.

We hope that the development of our approach for more complicated systems may be useful to obtain new results on the existence and unique-

[1] Istituto Matematico dell'Università, Camerino, C.N.R., G.N.F.M., Italy.
[2] Landau Institute for Theoretical Physics, Academy of Sciences of the U.S.S.R., Moscow.

ness problem for the general Boltzmann equation. Moreover, we should like to emphasize that our approach is natural if one wants to compare the theory with results of computer experiments.[3]

1. BASIC NOTATIONS AND FORMULATION OF THE RESULTS

We consider only the two-dimensional case for simplicity. The case of arbitrary finite dimension can be treated in a similar way. Points of the plane $\mathbb{R}^2$ are denoted by $q=(q_1,q_2)$. The scalar product in $\mathbb{R}^2$ is denoted by $\cdot$. All the subsets of $\mathbb{R}^2$ we introduce below are supposed to be measurable.

A particle moving in the plane with velocity of modulus 1 is described by a point in the phase space $M=\mathbb{R}^2\times S^1$. The points of M are denoted by (q,ψ), where $q\in\mathbb{R}^2$ is the particle position and $\psi\in[0,2\pi)$ is the direction of the velocity vector $\boldsymbol{\psi}=(\cos\psi,\sin\psi)$. The Borel σ-algebra of M is denoted by $\mathfrak{B}$.

By Ω we denote the space of the locally finite subsets of $\mathbb{R}^2$. A point $\omega\in\Omega$ identifies a configuration of scatterer centers. For any subset $A\subset\mathbb{R}^2$ and any $\omega\in\Omega$ we denote by ω_A the intersection $\omega\cap A$ and by Ω_A the space $\Omega_A=\{\omega\in\Omega:\omega=\omega_A\}$. By $|\omega|$ we denote the cardinality of ω. Ω endowed with the topology of pointwise convergence is a polish space. The corresponding Borel σ-algebra is denoted by $\mathfrak{M}$.

The Poisson measure with parameter $\lambda>0$ is the probability measure on $(\Omega,\mathfrak{M})$ such that for any integer $k>0$ and any collection of nonintersecting subsets $A_1,A_2,\ldots,A_k$ of $\mathbb{R}^2$, the random variables $|\omega_{A_1}|,|\omega_{A_2}|,\ldots,|\omega_{A_k}|$ are independent and each $|\omega_{A_i}|$ is distributed according to the Poisson law with parameter $\lambda m(A_i)$, $i=1,2,\ldots,k$, where $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^2$. Throughout this paper λ is fixed and the corresponding Poisson measure is denoted by $\mathrm{Prob}(\cdot)$.

Let $q\in\mathbb{R}^2$, $a>0$. By $D_a(q)=\{q'\in\mathbb{R}^2:|q'-q|\leqslant a\}$ we denote the circle of radius a and center q, and by $K_a(q)=\{q'\in\mathbb{R}^2:|q'-q|=a\}$ its boundary.

Let now $\omega\in\Omega$ and $a>0$ be fixed. Consider a particle moving uniformly with velocity of modulus 1 in $R_{a,\omega}=\mathbb{R}^2\backslash[\bigcup_{q\in\omega}D_a(q)]$ and undergoing elastic collisions with the "scatterers" $D_a(q)$, $q\in\omega$. If we prescribe at collision the particle to be in the outgoing configuration, its position and velocity are given by a point in the set $\hat{K}_a(q)=\{q',\psi)\in M:q'\in K_a(q),(q'-q)\cdot\boldsymbol{\psi}\geqslant 0\}$ for some $q\in\omega$.

Let $K_{a,\omega}$ denote the boundary of $R_{a,\omega}$. In general $K_{a,\omega}\neq\bigcup_{q\in\omega}K_a(q)$

[3] We are indebted to H. van Beijren for this remark.

since scatterers may overlap. Let $\overline{K}_{a,\omega} = \bigcup_{q,q' \in \omega, q \neq q'} \{K_a(q) \cap K_a(q')\}$ denote the set of the points which belong to the intersection of different scatterer boundaries (angular points), and set $K'_{a,\omega} = K_{a,\omega} \setminus \overline{K}_{a,\omega}$. We define the phase space of our mechanical system as

$$M_{a,\omega} = \mathring{M}_{a,\omega} \cup \hat{K}_{a,\omega} \tag{1.1}$$

where

$$\mathring{M}_{a,\omega} = R_{a,\omega} \times S^1 \qquad \hat{K}_{a,\omega} = \left\{ (q', \psi) \in \bigcup_{q \in \omega} \hat{K}_a(q) : q' \in K'_{a,\omega} \right\}$$

The evolution of a point $(q, \psi) \in M_{a,\omega}$ is given by $T_t^{(a,\omega)}(q, \psi) = (q_t, \psi_t)$, $t \geqslant 0$, where q_t is the end point of a continuous path which starts at q in the direction ψ, and consists of straight line segments which join on scatterer boundaries. The angle between two consecutive segments is determined according to the law of elastic collision, and ψ_t is the direction of the path at q_t.

Clearly if the path of the particle (q, ψ) hits an angular point it cannot be continued. Therefore we must exclude all the points of $M_{a,\omega}$ the paths of which fall for some time $t \in \mathbb{R}^1$ on an angular point. It is not hard to see that the Lebesgue measure of this set is zero for all $\omega \in \Omega$.

In this way we define a one-parameter group of transformations (a flow) $\{T_t^{(a,\omega)}, t \in \mathbb{R}^1\}$ on a subset $M'_{a,\omega} \subset M_{a,\omega}$ such that its complement has Lebesgue measure 0. We shall call such a flow the Lorentz gas with scatterer configuration ω and scatterer radius a.

For $(q, \psi) \in M_{a,\omega}$ we denote by $\tau_a^\omega(q, \psi) \in [0, \infty)$ the length of the free path of the (q, ψ) particle, and by $b_a^\omega(q, \psi)$ the corresponding impact parameter (see Fig. 1). If we admit the value ∞, τ_a^ω is defined for all points of $M_{a,\omega}$, whereas b_a^ω is defined only on the subset of the points $(q, \psi) \in M_{a,\omega}$ such that $\tau_a^\omega(q, \psi) < \infty$ and the straight line from q in the direction ψ does not hit an angular point. We denote this subset by $M''_{a,\omega}$.

We denote by $T_a^\omega : M''_{a,\omega} \to \hat{K}_{a,\omega}$ the map which associates to each point $(q, \psi) \in M''_{a,\omega}$ the point $T_a^\omega(q, \psi) \in \hat{K}_{a,\omega}$ corresponding to the first reflection of the particle (q, ψ) for positive times.

Let $(q, \psi) \in M''_{a,\omega}$ and consider the impact parameter $b_a^\omega(q + s\psi^\perp, \psi)$, where $\psi^\perp = (-\sin\psi, \cos\psi)$, as a function of $s \in \mathbb{R}^1$. We extend its definition to values of s for which $q + s\psi^\perp \in D_a(q')$ for some $q' \in \omega$ (i.e., for which the starting point is "covered" by some scatterer), simply by omitting the scatterer $D_a(q')$. We obtain a function of s defined on some union of intervals, which is continuous on each such interval; we denote by $-d_1^{(\omega,a)}(q, \psi)$ and $d_2^{(\omega,a)}(q, \psi)$, the left and right end points, respectively, of the interval which contains the point $s = 0$ (see Fig. 2).

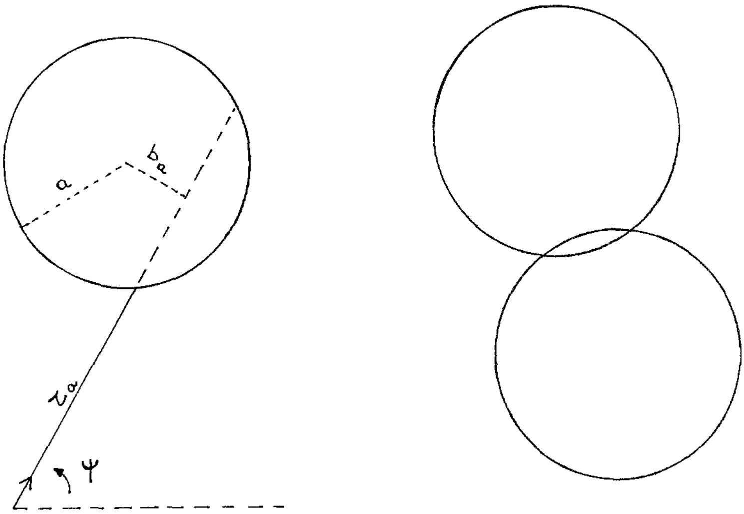

Fig. 1.

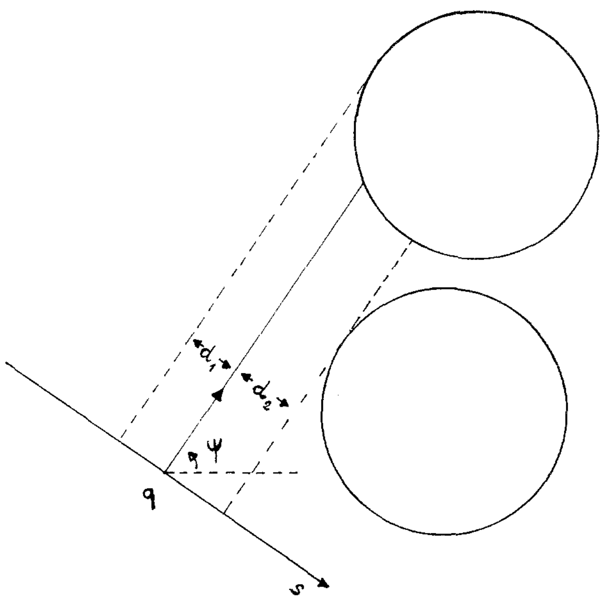

Fig. 2.

For $(q,\psi) \in M_{a,\omega}$ we denote by $\tau_a^{(n)}(q,\psi)$ and $b_a^{(n)}(q,\psi)$, $n = 1,2,\ldots$, the sequences of the free path lengths and of the impact parameters, i.e., $\tau_a^{(1)}(q,\psi) = \tau_a^{\omega}(q,\psi)$, $\tau_a^{(2)}(q,\psi) = \tau_a^{\omega}(T_a^{\omega}(q,\psi)), \ldots,$ $b_a^{(1)}(q,\psi) = b_a^{\omega}(q,\psi)$, $b_a^{(2)}(q,\psi) = b_a^{\omega}(T_a^{\omega}(q,\psi)), \ldots$.

For what follows it is convenient to introduce the rescaled quantities $\hat{\tau}_a^{\omega} = a\tau_a^{\omega}$, $\hat{b}_a^{\omega} = a^{-1}b_a^{\omega}$, $\hat{\tau}_a^{(n)} = a\tau_a^{(n)}$, $\hat{b}_a^{(n)} = a^{-1}b_a^{(n)}$, $n = 1,2,\ldots,$ and $\hat{d}_i^{(\omega,a)} = a^{-1}d_i^{(\omega,a)}$, $i = 1,2$.

We shall sometimes drop the indices ω and a not to overload the notation.

We introduce now a convenient representation of the point $(q_t,\psi_t) = T_t^{(a,\omega)}(q,\psi)$, $t > 0$, $(q,\psi) \in M'_{a,\omega}$, in terms of the free path lengths and of the impact parameters.

When a particle undergoes a collision with normalized impact parameter equal to $y \in [-1,1]$, the direction of its velocity changes by a quantity $z(y) = 2\arcsin y + \pi$. Therefore it is not hard to see that if we set $R_k(y_1,\ldots,y_k) = \sum_{j=1}^{k} z(y_j)$, $k \geqslant 1$, $R_0 = 0$, and

$$\psi^{(k)}(\psi; y_1,\ldots,y_k) = \psi + R_k(y_1,\ldots,y_k) \tag{1.2a}$$

$$\delta_t^{(k)}(\psi; x_1, y_1; \ldots; x_k, y_k) = \sum_{j=1}^{k} x_j \psi^{(j-1)}(\psi; y_1,\ldots,y_{j-1}) + \left(t - \sum_{j=1}^{k} x_j\right)\psi^{(k)}(\psi; y_1,\ldots,y_k) \tag{1.2b}$$

for $k \geqslant 0$, $(y_1,\ldots,y_k) \in [-1,1]^k$, $x_j \in [0,\infty)$, $j = 1,\ldots,k$, and $\sum_{j=1}^{k} x_j \leqslant t$, then

$$q_t = q + \delta_t^{(n)}\big(\psi; \tau_a^{(1)}(q,\psi), \hat{b}_a^{(1)}(q,\psi); \ldots; \tau_a^{(n)}(q,\psi), \hat{b}_a^{(n)}(q,\psi)\big) \tag{1.3a}$$

$$\psi_t = \psi^{(n)}\big(\psi; \hat{b}_a^{(1)}(q,\psi), \ldots, \hat{b}_a^{(n)}(q,\psi)\big) \tag{1.3b}$$

where $n = n_t(q,\psi)$ is the number of collisions which the particle (q,ψ) undergoes in the time interval $(0,t]$.

If a configuration $\omega \in \Omega$ of scatterers of radius $a > 0$ is given, and μ_0 is an absolutely continuous measure on M, we define the "compatible" measure $\mu_0^{(a,\omega)}$ by setting

$$\mu_0^{(a,\omega)}(A) = \mu_0(A \cap M_{a,\omega}) \qquad A \in \mathfrak{B} \tag{1.4}$$

The family of measure $\{\mu_t^{(a,\omega)}, t \in \mathbb{R}^1\}$ given by

$$\mu_t^{(a,\omega)}(A) = \mu_0^{(a,\omega)}\big(T_{-t}^{(a,\omega)}(A \cap M'_{a,\omega})\big), \qquad A \in \mathfrak{B}, \quad t \in \mathbb{R}^1 \tag{1.5}$$

gives the evolution of the initial measure μ_0 under the Lorentz gas's dynamics.

For any $\omega \in \Omega$ and $\rho \in (0, 1)$, we define the ρ-contracted scatterer configuration ω_ρ by setting $\omega_\rho = \{q \in \mathbb{R}^2 : \rho^{-1} q \in \omega\}$. The statistical behavior of the Lorentz gas in the Boltzmann–Grad limit for a fixed (microscopic) scatterer configuration ω is given by the behavior as $\rho \to 0$ of the measures $\mu_t^{(\rho^2 a, \omega_\rho)}$, $t \in \mathbb{R}^1$.[2] In what follows we shall set $a = 1$.

The main result of our paper is the following.

Theorem 1. Let μ_0 be an absolutely continuous measure on M with density $f_0 \in C^1(M)$. Then for Prob-almost all $\omega \in \Omega$ the measures $\mu_t^{(\rho^2, \omega_\rho)}$ given by Eq. (1.5), tend locally weakly, as $\rho \to 0$, to a limiting measure μ_t. Moreover, μ_t is absolutely continuous and its density $f_t(q, \psi)$ is the unique solution of the equation

$$\frac{\partial}{\partial t} f_t(q, \psi) + (\psi \cdot \nabla_q) f_t(q, \psi) = \frac{\lambda}{2} \int_{-\pi}^{\pi} d\psi' \left| \sin \frac{\psi - \psi'}{2} \right| \{ f_t(q, \psi') - f_t(q, \psi) \} \tag{1.6}$$

with initial data f_0.

In Section 3 we will indicate how it is possible to give a "weak" version of the theorem for $f_0 \in L^1_{\text{loc}}$.

For what follows it is convenient to adopt another description of the Boltzmann–Grad limit. Namely, we can keep the scatterer configuration fixed and contract by a factor $\rho \to 0$ the particle trajectories. This fact is based on the following relation, which is easily proved: if we set $(q_t, \psi_t) = T_t^{(\rho^2, \omega_\rho)}(q, \psi)$, then $T_{\rho^{-1}t}^{(\rho, \omega)}(\rho^{-1} q, \psi) = (\rho^{-1} q_t, \psi_t)$.

The plan of the paper is the following. In Section 2 we give the necessary preliminaries on the theory of the Lorentz gas, and some simple probabilistic results. In Section 3 we prove a basic lemma and the main theorem. Section 4 contains concluding remarks.

2. SOME FACTS FROM THE THEORY OF BILLIARDS AND AUXILIARY RESULTS

Throughout this section $\rho \in (0, 1)$ is the scatterer radius.

A point $(q', \psi) \in \hat{K}_\rho(q)$, $q \in \mathbb{R}^2$ can be conveniently identified by two angles: $\theta = \arctan[(q' - q)_2/(q' - q)_1]$ and $\phi = \psi - \theta \pmod{2r}$. θ is the angle of the radial vector $q' - q$ with respect to the q_1 axis, and ϕ is the angle of the velocity vector ψ with respect to the radial vector $q' - q$ (see Fig. 3). In this way to any scatterer center q we associate a copy $S(q)$ of the cylinder $S = S^1 \times [-\pi/2, \pi/2]$.

Let $\gamma \subset S(q)$, $q \in \mathbb{R}^2$, be a curve of class C^2 which is "increasing," i.e.; such that in the (θ, ϕ)-plane it is described by an increasing function $\phi(\theta)$ (vertical segments for which $\theta = \text{const}$ are also admitted). We shall call

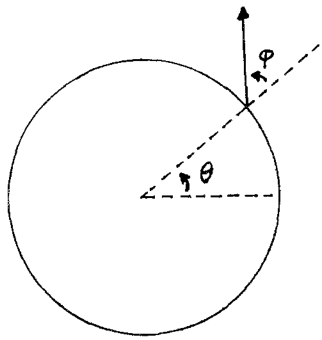

Fig. 3.

such curves simply "increasing curves." Given the scatterer radius ρ, γ identifies a curve in M, which we denote by the same symbol. It is often convenient to take as a parameter describing the curve the outgoing direction $\psi = \theta + \phi$.

The basic facts from the theory of billiards which we need are contained in the following proposition (see Refs. 5–7).

Proposition 2.1. Let $\rho \in (0, 1)$ and $\omega \in \Omega$ be fixed, and suppose that $\gamma \subset S(q)$, $q \in \omega$, is an increasing curve on which T_ρ^ω is continuous. Then its image $\gamma_1 = T_\rho^\omega \gamma$ identifies a curve in $S(\bar{q})$, for some $\bar{q} \in \omega$, which is also increasing, and, if (θ_1, ϕ_1) are the coordinates on $S(\bar{q})$, it is given by a function $\phi_1(\theta_1)$ which satisfies the equation

$$\frac{d\phi_1}{d\theta_1} = 1 + \frac{\cos\phi_1}{\tau/\rho + \cos\phi(d\theta/d\psi)} \tag{2.1}$$

Moreover

$$-\frac{d\psi_1}{d\psi} = 1 + \frac{2}{\cos\phi_1}\left(\frac{\tau}{\rho} + \cos\phi\,\frac{d\theta}{d\psi}\right) \tag{2.2}$$

where $\psi_1 = \theta_1 + \varphi_1\, \phi_1$ denotes the outgoing angle on γ_1.

Remark. Clearly it is not needed that $q \in \omega$, provided that γ, as a curve in M, is contained in $M''_{\rho,\omega}$. In any case when we give a curve $\gamma \subset S(q)$ the identification of the corresponding curve in M has to be done by setting the radius equal to ρ.

The following result can be proved using simple geometric considerations and the properties of the Poisson measure.

Proposition 2.2. Let $(q,\psi) \in M$ be fixed, and set, for $\rho \in (0,1)$, $x \in [0,\infty)$, $y \in [-1,1]$, $u \in [0,2]$

$$F^{(\rho)}(x,y) = \operatorname{Prob}\{\hat{\tau}_\rho^\omega(q,\psi) < x, \hat{b}_\rho^\omega(q,\psi) < y\} \tag{2.3a}$$

$$G_i^{(\rho)}(u) = \operatorname{Prob}\{d_i^{(\omega,\rho)}(q,\psi) < u\}, \qquad i = 1,2 \tag{2.3b}$$

Then the following relations hold:

$$\lim_{\rho\to 0} F^{(\rho)}(x,y) = F(x,y) \tag{2.4a}$$

$$\lim_{\rho\to 0} G_i^{(\rho)}(u) = G(u), \qquad i = 1,2 \tag{2.4b}$$

where $F(x,y) = \frac{1}{2}(1+y)(1-e^{-2\lambda x})$, $G(u) = 2u/(2+u)$, and convergence is uniform in $(x,y) \in [0,\infty) \times [-1,1]$, and $u \in [0,2]$.

If $\gamma \subset S(\bar{q})$, $\bar{q} \in \mathbb{R}^2$, is an increasing curve, as a curve in M it is parametrizable by the angle ψ, i.e., it is given by $\gamma = \{(q(\psi),\psi) : \psi \in \Delta\}$, where $\Delta \subset S^1$ is some interval, and $\sup_{\psi\in\Delta}|dq/d\psi| < \infty$. A curve of class C^2 in M satisfying these conditions will be called "admissible."

We shall denote by $|\cdot|$ the "Lebesgue" measure on S^1 (more precisely the Haar measure on S^1 normalized to 2π), and if $\Delta \subset S^1$ is an interval, by $d\hat{\mu}_\Delta = d\psi/|\Delta|$ the normalized Lebesgue measure on Δ. Furthermore, if $\gamma \subset M$ is an admissible curve, we denote by Δ_γ the corresponding interval of variation of ψ, and for $\omega \in \Omega$ we set

$$\mathscr{M}_\gamma^{\rho,\omega}(x,y) = \{\psi \in \Delta_\gamma : \hat{\tau}_\rho^\omega(q(\psi),\psi) < x, \hat{b}_\rho^\omega(q(\psi),\psi) < y\} \tag{2.5a}$$

$$\mathscr{P}_{i,\gamma}^{\rho,\omega}(u) = \{\psi \in \Delta_\gamma : \hat{d}_i^{(\omega,\rho)}(q(\psi),\psi) < u\}, \qquad i = 1,2 \tag{2.5b}$$

In the following proposition we consider a family of increasing curves $\{\gamma_\rho, \rho \in (0,1)\} \subset S(\bar{q})$, on a scatterer of radius ρ and fixed center $\bar{q}$. We prove that, if the curves do not become too small, the distributions of the quantities $\hat{\tau}, \hat{b}, \hat{d}_i$, $i = 1,2$, induced by the Lebesgue measure on the angles, tend to a limit as $\rho \to 0$. We shall write for simplicity Δ_ρ, $\hat{\mu}$, $\mathscr{M}^\rho$ and $\mathscr{P}_i^\rho$ instead of Δ_{γ_ρ}, $\hat{\mu}_{\Delta_{\gamma_\rho}}$, $\mathscr{M}_{\gamma_\rho}^{\rho,\omega}$ and $\mathscr{P}_{i,\gamma_\rho}^{(\omega,\rho)}$, respectively.

Proposition 2.3. Suppose that $|\Delta_\rho| > \rho^{\alpha_1}$ for some constant $\alpha_1 \in [0,1)$, and let $\rho_n \leqslant n^{-t}$ for some $t > 0$ and $n = 1,2,\ldots$. Then for Prob-almost all $\omega \in \Omega$ the following relations hold, uniformly in $(x,y) \in [0,\infty) \times [-1,1]$ and $u \in [0,2]$:

$$\lim_{n\to\infty} \hat{\mu}(\mathscr{M}^{\rho_n}(x,y)) = F(x,y) \tag{2.6a}$$

$$\lim_{n\to\infty} \hat{\mu}(\mathscr{P}_i^{\rho_n}(u)) = G(u), \qquad i = 1,2 \tag{2.6b}$$

Proof. Let β_1 and β_2 be two numbers such that $0 < \beta_1 < \beta_2 < 1 - \alpha_1$, set

$$\kappa(\rho) = \left[\frac{1}{\rho^{\beta_1} + \rho^{\beta_2}} \right]$$

where $[\cdot]$ denotes the integer part, and consider the following subsets of $\Delta_\rho = [\psi_1, \psi_2]$ (we take Δ_ρ closed for definiteness):

$$I_j = \left[\psi_1 + (j-1)|\Delta_\rho|(\rho^{\beta_1} + \rho^{\beta_2}), \psi_1 + j|\Delta_\rho|\rho^{\beta_1} + (j-1)|\Delta_\rho|\rho^{\beta_2}\right)$$

$$J_j = \left[\psi_1 + j|\Delta_\rho|\rho^{\beta_1} + (j-1)|\Delta_\rho|\rho^{\beta_2}, \psi_1 + j|\Delta_\rho|(\rho^{\beta_1} + \rho^{\beta_2})\right), \quad j = 1, \ldots, \kappa(\rho)$$

Clearly we can write $\Delta_\rho = \bigcup_{j=1}^{\kappa(\rho)}(I_j \cup J_j) \cup I'$, and, setting $\gamma_j = \{(q,\psi) \in \gamma_\rho : \psi \in I_j\}$, $\tilde{\gamma}_j = \{(q,\psi) \in \gamma_\rho : \psi \in J_j\}$ and $\gamma' = \{(q,\psi) \in \gamma_\rho : \psi \in I'\}$, we have

$$\mathcal{M}^\rho(x,y) = \left[\bigcup_{j=1}^{\kappa(\rho)} \mathcal{M}_{\gamma_j}^{\rho,\omega}(x,y)\right] \cup \left[\bigcup_{j=1}^{\kappa(\rho)} \mathcal{M}_{\tilde{\gamma}_j}^{\rho,\omega}(x,y)\right] \cup \mathcal{M}_{\gamma'}^{\rho,\omega}(x,y)$$

Therefore

$$\begin{aligned} 0 &\leqslant \hat{\mu}(\mathcal{M}^\rho(x,y)) - \sum_{j=1}^{\kappa(\rho)} \hat{\mu}(\mathcal{M}_{\gamma_j}^{\rho,\omega}(x,y)) \\ &\leqslant \sum_{j=1}^{\kappa(\rho)} \hat{\mu}(\Delta_{\tilde{\gamma}_j}) + \hat{\mu}(\gamma') \leqslant \epsilon_1(\rho) \end{aligned} \tag{2.7}$$

with $\epsilon_1(\rho) = 2\pi(\rho^{\beta_2 - \beta_1}/2 + 2\rho^{\beta_1})$. Let $r(\rho) = \pi\rho^{1-\beta_2-\alpha_1}$ and consider the set

$$\Omega^{(\rho)} = \{\omega \in \Omega : \mathrm{dist}(\bar{q}, \omega) > r(\rho)\}$$

Let $P^{(\rho)}(\cdot) = \mathrm{Prob}(\cdot \,|\, \Omega^{(\rho)})$ and $\mathbb{E}^{(\rho)}$ denote the corresponding conditional probability and conditional expectation. It is not hard to see that

$$|\mathbb{E}^{(\rho)}\hat{\mu}(\mathcal{M}^\rho(x,y)) - F^{(\rho)}(x,y)| \leqslant \epsilon_2(\rho) \tag{2.8}$$

where $\epsilon_2(\rho)$ is a nondecreasing function (depending on α_1 and β_2) and $\lim_{\rho\to 0}\epsilon_2(\rho) = 0$. Therefore, setting

$$R_j^{\rho,\omega}(x,y) = \hat{\mu}(\mathcal{M}_{\gamma_j}^{\rho,\omega}(x,y)) - \mathbb{E}^{(\rho)}\hat{\mu}(\mathcal{M}_{\gamma_j}^{\rho,\omega}(x,y)),$$

since estimate (2.7) induces a corresponding estimate for the expected values, we find

$$|\hat{\mu}(\mathcal{M}^\rho(x,y)) - F^{(\rho)}(x,y)| \leqslant \left|\sum_{j=1}^{\kappa(\rho)} R_j^{\rho,\omega}(x,y)\right| + \epsilon_3(\rho) \tag{2.9}$$

where $\epsilon_3 = 2\epsilon_1 + \epsilon_2$ is independent of $\omega \in \Omega$ and $(x,y) \in [0,\infty) \times [-1,1]$.

A crucial point in the proof is the observation that for $\rho \in (0,1)$ the random variables $R_j(x,y), j = 1, \ldots, \kappa(\rho)$, are independent with respect to the measure $P^{(\rho)}$. Moreover, since

$$\sup_{\omega \in \Omega} |R_j^{\rho,\omega}| \leqslant 2\rho^{\beta_1}, \qquad j = 1, 2, \ldots, \kappa(\rho) \tag{2.10}$$

the Chebyshev inequality gives the following estimate of the Bernstein type: for any $X > 0$

$$P^{(\rho)}\left[\left\{\omega \in \Omega : \left|\sum_{j=1}^{\kappa(\rho)} R_j^{\rho,\omega}(x,y)\right| > X\right\}\right]$$

$$\leqslant 2\exp(-hX)\exp\left[\frac{h^2}{2}B^2(\rho)\left(1 - \frac{hH}{3}\right)\right] \tag{2.11}$$

where

$$H = 2\rho^{\beta_1}, \qquad B^2(\rho) = \sum_{j=1}^{\kappa(\rho)} \mathbb{E}^{(\rho)}(R_j(x,y))^2$$

and h is a positive number such that $hH < 3$. Now it is not hard to prove, using the properties of the Poisson measure and some simple estimates, that

$$\mathbb{E}^{(\rho)}(R_j(x,y))^2 \leqslant C\frac{\rho^{2+\beta_1}}{|\Delta_\rho|}\log\left(\frac{1}{\rho}\right) \tag{2.12}$$

where C is an absolute constant. Hence, setting $h = \rho^{-\beta_1}$, we find $h^2B^2(\rho)$ $(1 - hH)^{-1} \leqslant C'\log(1/\rho)\rho^{2(1-\beta_1)}|\Delta_\rho| \leqslant C''\log(1/\rho)\rho^{2-2\beta_1-\alpha_1}$ and the last quantity goes to zero as $\rho \to 0$. Therefore, setting $\epsilon_4(\rho) = \rho^{\beta_1 - \delta}$ for some $\delta < \beta_1$, we get from inequalities (2.11), (2.12), for ρ small enough

$$P^{(\rho)}\left[\left\{\omega \in \Omega : \left|\sum_{j=1}^{\kappa(\rho)} R_j^{\rho,\omega}(x,y)\right| > \epsilon_4(\rho)\right\}\right] \leqslant \frac{5}{2}\exp(-\rho^{-\delta}) \tag{2.13}$$

Consider now the points $x_0 = 0$, $y_0 = 0$ and

$$x_i = i(\log(1/\rho))^{-1}, \qquad i = 1, \ldots, N_1(\rho) = \left[(\log(1/\rho))^2\right], \qquad x_{N_1+1} = \infty$$

$$y_l = l(\log(1/\rho))^{-1}, \qquad l = 1, \ldots, N_2(\rho) = \left[2\log(1/\rho)\right], \qquad y_{N_2+1} = 1$$

From inequalities (2.9), (2.13) it follows that for ρ small enough

$$P^{(\rho)}\left[\left\{\omega \in \Omega : \max_{\substack{i=0,\ldots,N_1+1 \\ l=0,\ldots,N_2+1}} |\hat{\mu}(\mathscr{M}^\rho(x_i, y_l)) - F^{(\rho)}(x_i, y_l)| > \epsilon_5(\rho)\right\}\right]$$

$$\leqslant 3\exp(\rho^{-\delta}) \tag{2.14}$$

where $\epsilon_5 = \epsilon_3 + \epsilon_4$. Now, if $x \in (x_i, x_{i+1})$, $y \in (y_l, y_{l+1})$ for some i and l, $\hat{\mu}(\mathscr{M}^\rho(x, y))$ will lie somewhere between $\mu(\mathscr{M}^\rho(x_i, y_l))$ and $\hat{\mu}(\mathscr{M}^\rho(x_{i+1}, y_{l+1}))$, and hence, because of inequality (2.14), between $F^{(\rho)}(x_i, y_l) - \epsilon_5(\rho)$ and $F^{(\rho)}(x_{i+1}, y_{l+1}) + \epsilon_5(\rho)$. Since, as is easy to see, $F^{(\rho)}$ is uniformly continuous, uniformly in $\rho \in (0, 1)$, we find, setting

$$\epsilon_6(\rho) = \epsilon_5(\rho) + \max_{\substack{i=0,\ldots,N_1 \\ l=0,\ldots,N_2}} \left[F^{(\rho)}(x_{i+1}, y_{l+1}) - F^{(\rho)}(x_i, y_l) \right]$$

$$P^{(\rho)}\left(\left\{ \omega \in \Omega : \| \hat{\mu}(\mathscr{M}^\rho(\cdot, \cdot)) - F^{(\rho)}(\cdot, \cdot)\|_\infty > \epsilon_6(\rho) \right\}\right) \leqslant 3\exp(-\rho^{-\delta}) \tag{2.15}$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbb{R}^1 \times [-1, 1]$.

To conclude the proof, observe that, since $\Omega^{(\rho_m)} \supset \Omega^{(\rho_n)}$ for any $m > n$, we have

$$P^{(\rho_n)}(A) \leqslant \frac{\mathrm{Prob}(\Omega^{(\rho_m)})}{\mathrm{Prob}(\Omega^{(\rho_n)})} P^{(\rho_m)}(A), \qquad A \in \mathfrak{M}$$

Setting $A = \{\omega \in \Omega : \| \hat{\mu}(\mathscr{M}^{\rho_m}(\cdot, \cdot)) - F^{(\rho_m)}(\cdot, \cdot)\|_\infty > \epsilon_6(\rho_m)\}$, using inequality (2.15), and letting $m \to \infty$ for fixed n, we find by the Borel–Cantelli lemma that for $P^{(\rho_n)}$-almost all $\omega \in \Omega$ $\lim_{m\to\infty} \| \hat{\mu}(\mathscr{M}^{\rho_m}(\cdot, \cdot)) - F(\cdot, \cdot)\|_\infty = 0$. Since $\bigcup_{n=1}^\infty \Omega^{(\rho_n)} = \Omega \bmod 0$, Eq. (2.4a) is proved. Equations (2.4b) are proved in a similar way. Proposition 2.3 is proved. ■

We shall now show that convergence to the limiting distribution takes place for a sufficiently dense family of curves and for a large class of measures on them.

For $a > 0$ we denote by $\mathbb{Z}_a = \{q \in \mathbb{R}^2 : q = \mathbf{k}a,\ \mathbf{k} \in \mathbb{Z}^2\}$ the square lattice of constant a. If $\Delta \subset S^1$ is an interval and f a nonnegative function on it, we denote by $\mu_{\Delta,f}$ the normalized measure on Δ induced by f (when no confusion arises we write simply μ_f). Furthermore for $\alpha_2 \in [0, 1)$ we denote be $\mathscr{F}_\Delta^\rho(\alpha_2)$ the class of the positive functions f on Δ such that $f \in C^1$ and

$$\sup_{\psi\in\Delta} |f'(\psi)| \Big/ \inf_{\psi\in\Delta} f(\psi) < \rho^{-\alpha_2} \tag{2.16}$$

For fixed $\alpha_2 \in [0, 1)$ and $q \in \mathbb{R}^2$ we consider the curves in M $\gamma_q^{(i)} = \{(q, \psi) : \psi \in [(i-1)(2\pi/\kappa(\rho)), i(2\pi/\kappa(\rho)))\}$ $i = 1, \ldots, \kappa(\rho)$, and $\kappa(\rho) = [2\pi\rho^{-\alpha_2}(1 + \log(1/\rho))]$. We shall call $\Gamma_\rho(q)$ the collection $\{\gamma_q^{(i)} : i = 1, \ldots, \kappa(\rho)\}$.

Proposition 2.4. Let $a(\rho) = \rho^l$ for $l > 1$. Then for any choice of $\delta_1 > 0$, $t > 0$ and $\beta \in (0, 1 - \alpha_2)$, if we set $\mathbb{Z}^{(\rho)}(\omega) = \{q \in \mathbb{Z}_{a(\rho)} \cap D_{\rho^{-\delta_1}}(0) :$ $\mathrm{dist}(q, \omega) > r(\rho)\}$ for $r(\rho) = \pi\rho^\beta$ and $\rho_n \leqslant n^{-t}$, $n = 1, 2, \ldots,$ the following relations hold for Prob-almost all $\omega \in \Omega$:

$$\lim_{n\to\infty} \max_{\substack{q\in\mathbb{Z}^{(\rho_n)}(\omega)\\ \gamma\in\Gamma_{\rho_n}(q)}} \sup_{f\in\mathscr{F}^{\rho_n}_{\Delta_\gamma}(\alpha_2)} \| \mu_f(\mathscr{M}^{\rho_n,\omega}_\gamma(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty = 0 \tag{2.17a}$$

$$\lim_{n\to\infty} \max_{\substack{q\in\mathbb{Z}^{(\rho_n)}(\omega)\\ \gamma\in\Gamma_{\rho_n}(q)}} \sup_{f\in\mathscr{F}^{\rho_n}_{\Delta_\gamma}(\alpha_2)} \| \mu_f(\mathscr{P}^{\rho_n,\omega}_{i,\gamma}(\cdot)) - G(\cdot)\|_\infty = 0, \quad i+1,2 \tag{2.17b}$$

Proof. Clearly a curve $\gamma \in \Gamma_\rho(q)$ is an admissible curve in M, which can be identified with an increasing curve on some scatterer. Choosing $\epsilon \in (0, 1-\beta-\alpha_2)$, and setting $\beta_2 = 1-\beta-\alpha_2-\epsilon$ we have $|\Delta_\gamma| > \rho^{\alpha_2+\epsilon}$ for ρ small enough, so that repeating the steps in the proof of Proposition 2.3 we find that for any $q \in \mathbb{R}^2$, ρ small enough and all $\gamma \in \Gamma_\rho(q)$,

$$P^{(\rho)}_q(\{\omega\in\Omega : \|\hat{\mu}(\mathscr{M}^{\rho,\omega}_\gamma(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty > \epsilon'_6(\rho)\}) \leqslant 3\exp(-\rho^{-\delta}),$$
$$\gamma\in\Gamma_\rho(q) \tag{2.18}$$

where $P^{(\rho)}_q$ is the conditional measure under the condition $\Omega^{(\rho)}_q = \{\omega\in\Omega : \operatorname{dist}(q,\omega) > r(\rho)\}$, $\hat{\mu}$ denotes the normalized Lebesgue measure on Δ_γ, and ϵ'_6 can be computed as in the proof of Proposition 2.3. Now, if $f \in \mathscr{F}^{\rho}_{\Delta_\gamma}(\alpha_2)$ it is easy to see that for any measurable $A \subset \Delta_\gamma$

$$|\hat{\mu}(A) - \mu_f(A)| \leqslant 2|A|\left(\sup_{\psi\in\Delta_\gamma}|f'(\psi)| \Big/ \inf_{\psi\in\Delta_\gamma} f(\psi)\right) \leqslant \frac{4\pi}{\log(1/\rho)} \tag{2.19}$$

Putting together inequalities (2.18) and (2.19) we find

$$P^{(\rho)}_q\left(\left\{\omega\in\Omega : \max_{\gamma\in\Gamma_\rho(q)} \sup_{f\in\mathscr{F}^{\rho}_{\Delta_\gamma}(\alpha_2)} \|\mu_f(\mathscr{M}^{\rho,\omega}_\gamma(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty > \epsilon_7(\rho)\right\}\right)$$
$$\leqslant 3\kappa(\rho)\exp(-\rho^{-\delta})$$

with ϵ_7 nondecreasing and $\lim_{\rho\to 0}\epsilon_7(\rho) = 0$, and consequently

$$\operatorname{Prob}\left[\left\{\omega\in\Omega : \max_{\substack{q\in\mathbb{Z}^{(\rho)}(\omega)\\ \gamma\in\Gamma_\rho(q)}} \sup_{f\in\mathscr{F}^{\rho}_{\Delta_\gamma}(\alpha_2)} \|\mu_f(\mathscr{M}^{\rho,\omega}_\gamma(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty > \epsilon_7\right\}\right]$$
$$\leqslant \sum_{q\in\mathbb{Z}^{(\rho)}(\omega)} P(\Omega^{(\rho)}_q)3\kappa(\rho)\exp(-\rho^{-\delta})$$
$$\leqslant c\rho^{-2\delta_1-2l-\alpha_2}\log(e/\rho)\exp(-\rho^{-\delta})$$

Equation (2.17a) follows now by the Borel–Cantelli lemma. Equations (2.17b) are proved in a similar way. Proposition 2.4 is proved. ■

In what follows $\mathscr{G}_\rho$ will denote a family of increasing curves in $S^1 \times [-\pi/2, \pi/2]$ such that $|\Delta_\gamma| > \rho^{-\alpha_1}$ for some fixed $\alpha_1 \in [0,1)$ and any $\gamma \in \mathscr{G}_\rho$, and the number of the curves does not exceed ρ^{-s}, where s is a positive integer. By $\mathscr{G}_\rho(q)$ we denote a copy of $\mathscr{G}_\rho$ located on the scatterer of center $q \in \mathbb{R}^2$ and radius ρ. Furthermore, if r is a positive number and $\omega \in \Omega$ we denote by $\omega^{(r)} = \{q \in \omega : \omega \cap D_r(q) = \{q\}\}$ the configuration of the points $q \in \omega$ which are at a distance larger than r from the other points of ω. The following proposition holds.

Proposition 2.5. For any choice of $\delta_1 > 0$, $t > 0$, $\alpha_2 \in [0,1)$, $\beta \in (0, 1 - \bar{\alpha})$ where $\bar{\alpha} = \max(\alpha_1, \alpha_2)$, if $\omega_{(\rho)} = \omega^{(r(\rho))} \cap D_{\rho^{-\delta_1}}(0)$, for $r(\rho) = \pi\rho^\beta$ and $\omega \in \Omega$, and $\rho_n \leqslant n^{-t}$, $n = 1, 2, \ldots,$ the following relations hold for Prob-almost all $\omega \in \Omega$:

$$\lim_{n\to\infty} \max_{\substack{q \in \omega_{(\rho_n)} \\ \gamma \in \mathscr{G}_{\rho_n}(q)}} \sup_{f \in \mathscr{F}^{\rho_n}_{\Delta_\gamma}(\alpha_2)} \| \mu_f(\mathscr{M}^{\rho_n,\omega}_\gamma(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty = 0 \tag{2.20a}$$

$$\lim_{n\to\infty} \max_{\substack{q \in \omega_{(\rho_n)} \\ \gamma \in \mathscr{G}_{\rho_n}(q)}} \sup_{f \in \mathscr{F}^{\rho_n}_{\Delta_\gamma}(\alpha_2)} \| \mu_f(\mathscr{P}^{\rho_n,\omega}_{i,\gamma}(\cdot)) - G(\cdot)\|_\infty = 0 \quad i = 1,2 \tag{2.20b}$$

Proof. The situation is similar to that of the previous proposition, except for the fact that the curves have now random positions. To overcome this difficulty we use a simple construction. Let $a(\rho) = \rho^l$, $l > 1$, and $Q_a(q)$, $q \in \mathbb{R}^2$, denote the square of center q and sides of length a parallel to the coordinate axes. Consider the event $\mathscr{E}^\rho_q = \{\omega \in \Omega : \omega_{Q_a(q)} \cap \omega^{(r(\rho))} \neq \emptyset\}$. Clearly if $\omega \in \mathscr{E}^\rho_q$ the intersection $\omega_{Q_a(q)} \cap \omega^{(r(\rho))}$ consists of exactly one point, which we denote by $\hat{q}(q)$. Consider the set

$$E^\rho_q(\epsilon) = \left\{ \omega \in \mathscr{E}^\rho_q : \max_{\gamma \in \mathscr{G}_\rho(\hat{q})} \sup_{f \in \mathscr{F}^\rho_{\Delta_\gamma}(\alpha_2)} \| \mu_f(\mathscr{M}^{\rho,\omega}_\gamma(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty > \epsilon \right\}$$

We have, denoting by P and $\tilde{P}$ the probability measures induced by Prob on $\Omega_{Q_a(q)}$ and $\Omega_{\mathbb{R}^2 \setminus Q_a(q)}$, respectively,

$$\begin{aligned}
&\mathrm{Prob}(E^\rho_q(\epsilon)) \\
&\quad = \int_{\Omega_{Q_a(q)}} P(d\omega) \int_{\Omega_{\mathbb{R}^2\setminus Q_a(q)}} \tilde{P}(d\omega')\, \chi_{E^\rho_q(\epsilon)}(\omega \cup \omega') \\
&\quad \leqslant c \int_{Q_a(q)} d\hat{q}\, P^{(\rho)}_{\hat{q}} \Bigg(\Bigg\{ \omega \in \Omega : \max_{\gamma \in \mathscr{G}_\rho(\hat{q})} \sup_{f \in \mathscr{F}^\rho_{\Delta_\gamma}(\alpha_2)} \| \mu_f(\mathscr{M}^{\rho,\omega}_\gamma(\cdot,\cdot)) \\
&\qquad\qquad - F(\cdot,\cdot)\|_\infty > \epsilon \Bigg\} \Bigg)
\end{aligned} \tag{2.21}$$

For $\gamma \in \mathscr{G}_\rho(\hat{q})$ we set $\gamma_i = \{(q(\psi),\psi) \in \gamma : \psi \in \Delta_\rho^{(i)}\}$ with $\Delta_\rho^{(i)} = [(i-1)|\Delta_\gamma|/\kappa(\rho), i|\Delta_\gamma|/\kappa(\rho))$, $i = 1, \ldots, \kappa(\rho)$ and $\kappa(\rho) = [|\Delta_\gamma|\rho^{-\bar{\alpha}}(1 + \log(1/\rho)]$. As in the previous proof, one finds inequalities analogous to (2.18) and (2.19), which lead to the inequality

$$P_{\hat{q}}^{(\rho)}\left(\left\{\omega \in \Omega : \max_{\gamma \in \mathscr{G}_\rho(\hat{q})} \sup_{f \in \mathscr{F}_{\Delta_\gamma}^\rho(\alpha_2)} \| \mu_f(\mathscr{M}_\gamma^{\rho,\omega}(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty > \epsilon\right\}\right) \leqslant \exp(\rho^{-\delta}/2) \tag{2.22}$$

which holds for some $\delta > 0$ and ρ small enough, and in which we can take $\epsilon = \epsilon_7'(\rho)$, with ϵ_7' nondecreasing and $\lim_{\rho\to 0}\epsilon_7'(\rho) = 0$. Putting together (2.21) and (2.22) we find

$$\mathrm{Prob}\left[\left\{\omega \in \Omega : \max_{\substack{\hat{q} \in \omega_{(\rho)} \\ \gamma \in \mathscr{G}_\rho(\hat{q})}} \sup_{f \in \mathscr{F}_{\Delta_\gamma}^\rho(\alpha_2)} \| \mu_f(\mathscr{M}_\gamma^{\rho,\omega}(\cdot,\cdot)) - F(\cdot,\cdot)\|_\infty > \epsilon_7'(\rho)\right\}\right]$$
$$\leqslant \sum_{q \in \mathbb{Z}^{(\rho)}(\omega)} \mathrm{Prob}(E_q^\rho(\epsilon_7')) \leqslant c\rho^{-2\delta_1}\rho^{-2l}\exp(-\rho^{-\delta}/2)$$

whence, using again the Borel–Cantelli lemma, we get Eq. (2.20a). Equations (2.20b) are proved in a similar way. ■

3. THE MAIN THEOREM

The proof of the main theorem is based on some partial results which we give separately. The idea of the proof is the following. As we have proved (Propositions 2.4 and 2.5) there is a large family of increasing curves on which the distributions of the quantities $\hat{\tau}, \hat{b}, \hat{d}_i$, $i = 1,2$, are close to the corresponding limits, for a large class of density functions, when ω belongs to a set of full measure. The next step is a continuity argument, namely, we show that if a curve γ' is sufficiently close to a curve γ of the above family, then the distributions of $\hat{\tau}, \hat{b}, \hat{d}_i$, $i = 1,2$, on γ' are also close to the limiting values. We are then able to prove a fundamental lemma on the joint distribution of the successive free path lengths and impact parameters, which shows that they constitute, in the limit as $\rho \to 0$, a process with independent values for Prob-almost all $\omega \in \Omega$. This leads easily to the proof of the theorem.

The "continuity property" for the distributions on curves which are close to each other is given by the following result.

Proposition 3.1. Let $\gamma = \{(q(\psi),\psi) : \psi \in \Delta\gamma\}$ and $\gamma' = \{(q'(\psi),\psi) : \psi \in \Delta_{\gamma'}\}$ be two admissible curves in M such that $\Delta_\gamma \supset \Delta_{\gamma'}$ and for some values of $\rho \in (0,1)$, $\alpha_1 \in [0,1)$, $\kappa_1, \kappa_2 > 1$ the following relations hold: (i) $|\Delta_\gamma| > \rho^{\alpha_1}$, (ii) $|\Delta_\gamma \backslash \Delta_{\gamma'}| < \rho^{\kappa_1}$, and (iii) $\sup_{\psi \in \Delta_{\gamma'}}|q(\psi) - q'(\psi)| < \rho^{\kappa_2}$. Suppose furthermore that $\omega \in \Omega$ is such that $\gamma, \gamma' \subset M_{\rho,\omega}$, and moreover for some $\alpha_2 \in [0,1)$

$$\sup_{f \in \mathscr{F}^{\rho}_{\Delta_{\gamma'}}(\alpha_2)} \| \mu_f(\mathscr{M}^{\rho,\omega}_{\gamma'}(\cdot,\cdot)) - F(\cdot,\cdot)\|_{\infty} < \epsilon \tag{3.1a}$$

$$\sup_{f \in \mathscr{F}^{\rho}_{\Delta_{\gamma'}}(\alpha_2)} \| \mu_f(\mathscr{P}^{\rho,\omega}_{i,\gamma'}(\cdot)) - G(\cdot)\|_{\infty} < \epsilon, \qquad i = 1,2 \tag{3.1b}$$

Then there is a constant C such that if

$$\bar{\alpha} = \max(\alpha_1, \alpha_2), \quad \text{and} \quad \alpha = \min(\kappa_1 - \bar{\alpha}, \kappa_2 - 1)$$

the following relations hold:

$$\sup_{f \in \mathscr{F}^{\rho}_{\Delta_{\gamma}}(\alpha_2)} \| \mu_f(\mathscr{M}^{\rho,\omega}_{\gamma}(\cdot,\cdot)) - F(\cdot,\cdot)\|_{\infty} < C(\epsilon + \rho^{\alpha}) \tag{3.2a}$$

$$\sup_{f \in \mathscr{F}^{\rho}_{\Delta_{\gamma}}(\alpha_2)} \| \mu_f(\mathscr{P}^{\rho,\omega}_{i,\gamma}(\cdot)) - G(\cdot)\|_{\infty} < C(\epsilon + \rho^{\alpha}) \tag{3.2b}$$

Proof. Set $\hat{\gamma} = \{(q,\psi) \in \gamma : \psi \in \Delta_{\gamma'}\}$ and let $\hat{f}$ denote the restriction of $f \in \mathscr{F}^{\rho}_{\Delta_{\gamma}}(\alpha_2)$ to $\Delta_{\gamma'}$. We have

$$\| \mu_f(\mathscr{M}^{\rho,\omega}_{\gamma}(\cdot,\cdot)) - \mu_{\hat{f}}(\mathscr{M}^{\rho,\omega}_{\hat{\gamma}}(\cdot,\cdot))\|_{\infty} \leqslant 2\mu_{f,\Delta_{\gamma}}(\Delta_{\gamma} \setminus \Delta_{\gamma'})$$

$$\leqslant 2\rho^{\kappa_1} \sup_{\psi \in \Delta_{\gamma}} \frac{f(\psi)}{\int_{\Delta_{\gamma}} f(\psi')\, d\psi'} \leqslant 4\rho^{\kappa_1 - \bar{\alpha}} \tag{3.3}$$

In order to estimate $\| \mu_{\hat{f}}(\mathscr{M}^{\rho,\omega}_{\hat{\gamma}}(\cdot,\cdot)) - \mu_{\hat{f}}(\mathscr{M}^{\rho,\omega}_{\gamma'}(\cdot,\cdot))\|_{\infty}$ observe that if $(q,\psi) \in \gamma'$ is such that $\min_{i=1,2} d_i^{(\omega,\rho)}(q',\psi) > \rho^{\kappa_2}$ then

$$|\hat{\tau}^{\omega}_{\rho}(q(\psi),\psi) - \hat{\tau}^{\omega}_{\rho}(q'(\psi),\psi)| < \rho^2 + \rho^{1+\kappa_2} < 2\rho^2$$

$$|\hat{b}^{\omega}_{\rho}(q(\psi),\psi) - \hat{b}^{\omega}_{\rho}(q'(\psi),\psi)| < \rho^{\kappa_2 - 1}$$

Therefore, setting $\mathscr{N} = \{\psi \in \Delta_{\gamma'} : \min_{i=1,2} d_i^{(\omega,\rho)}(q'(\psi),\psi) > \rho^{\kappa_2}\}$ we find

$$\mu_{\hat{f}}\left(\mathscr{M}^{\rho,\omega}_{\hat{\gamma}}(\hat{x},\hat{y}_-) \cap \mathscr{N}\right) \leqslant \mu_{\hat{f}}\left(\mathscr{M}^{\rho,\omega}_{\hat{\gamma}}(x,y)\right)$$

$$\leqslant \mu_{\hat{f}}(\Delta_{\gamma'} \setminus \mathscr{N}) + \mu_{\hat{f}}\left(\mathscr{M}^{\rho,\omega}_{\gamma'}(x + 2\rho^2, \hat{y}_+)\right) \tag{3.4}$$

where $\hat{x} = \max(0, x - 2\rho^2)$, $\hat{y}_+ = \min(1, y + \rho^{\kappa_2 - 1})$, $\hat{y}_- = \max(-1, y - \rho^{\kappa_1 - 1})$. Since $\hat{f} \in \mathscr{F}^{\rho}_{\Delta_{\gamma'}}(\alpha_2)$, using equations (3.1b) and (3.4) and the fact that the function $F(x,y)$ is uniformly Lipschitz it follows that

$$\| \mu_{\hat{f}}(\mathscr{M}^{\rho,\omega}_{\hat{\gamma}}(\cdot,\cdot)) - \mu_{\hat{f}}(\mathscr{M}^{\rho,\omega}_{\gamma'}(\cdot,\cdot))\|_{\infty} \leqslant C'(\epsilon + \rho^{\kappa_2 - 1}) \tag{3.5}$$

Using inequalities (3.1a), (3.3), and (3.5) we obtain the result (3.2a). Equation (3.2b) can be proved in a similar way. ■

An increasing curve $\gamma \subset S(q)$, $q \in \mathbb{R}^2$, beaks into a (at most countable) number of pieces, on which T_ρ^ω is continuous, each piece being again, by Proposition 2.1, an increasing curve. The information that we need on how measures are transformed is given by the following result.

Proposition 3.2. Let $\gamma = \{(\theta(\psi), \phi(\psi)) : \psi \in \Delta_\gamma\} \subset S(\bar{q})$, $\bar{q} \in \mathbb{R}^2$, be an increasing curve, and let f be a positive function on Δ_γ of class C^1. If $\omega \in \Omega$ is such that $\gamma \subset M_{\rho,\omega}$, $\gamma_1 = \{(\theta(\psi_1), \phi_1(\psi_1)) : \psi_1 \in \Delta_{\gamma_1}\}$ is a continuous curve in the image $T_\rho^\omega \gamma$ such that $\tau^0 = \inf_{(q,\psi)\in\gamma : T_\rho^\omega(q,\psi)\in\gamma_1} \tau_\rho^\omega(q,\psi) > 2$, and $\inf_{\psi_1\in\Delta_{\gamma_1}} \cos\phi_1(\psi_1) = u > 0$, and f_1 denotes the density of the measure induced on Δ_{γ_1} by f, the following inequalities hold:

$$\sup_{\psi_1\in\Delta_{\gamma_1}} \left| \frac{d^2\theta_1}{d\psi_1^2} \right| \leqslant \frac{\rho}{\tau^0} + \frac{1}{8}\left(\frac{\rho}{\tau^0}\right)^3 \sup_{\psi\in\Delta_\gamma} \left| \frac{d^2\theta}{d\psi^2} \right| \tag{3.6}$$

$$\frac{\sup_{\psi_1\in\Delta_{\gamma_1}} |f_1'(\psi_1)|}{\inf_{\psi_1\in\Delta_{\gamma_1}} f_1(\psi_1)} \leqslant \frac{C}{u}\left[1 + \frac{\sup_{\psi\in\Delta_\gamma} |f'(\psi)|}{\inf_{\psi\in\Delta_\gamma} f(\psi)}\right] \tag{3.7}$$

where

$$C = 18\left[1 + \left(\frac{\rho}{\tau^0}\right)^2 \sup_{\psi\in\Delta_\gamma} \left| \frac{d^2\theta}{d\psi^2} \right|\right]$$

Proof. Using Eqs. (2.1) and (2.2) it is easily seen that, setting

$$H(\psi) = \frac{1}{\cos\phi_1}\left(\frac{\tau_\rho}{\rho} + \cos\phi \frac{d\theta}{d\psi}\right)$$

we have

$$\left| \frac{d^2\theta_1}{d\psi_1^2} \right| = (2H+1)^{-3} \left| \frac{d}{d\psi} H \right|$$

The main point in the proof is the inequality $|d\tau_\rho/d\psi| \leqslant \rho(1+H)$ which can be derived from geometric arguments. Using this, Eqs. (2.1) and (2.2), and the obvious inequality $H \geqslant 2/\cos\phi_1$, inequality (3.6) is easily proved.

The measure induced by f on Δ_{γ_1} has density $f_1 = -f(\psi)(d\psi/d\psi_1)$. Equation (2.2) gives $|d\psi/d\psi_1| \leqslant (2H+1)^{-1}$, and it is easily seen that $\tau^0/\rho \leqslant H \leqslant (1/u)(\tau^0/\rho + 3)$. Combining these results we get

$$\sup_{\psi_1\in\Delta_{\gamma_1}} \left| \frac{d\psi}{d\psi_1} \right|^2 \Big/ \inf_{\psi\in\Delta_{\gamma_1}} \left| \frac{d\psi}{d\psi_1} \right| \leqslant \frac{2\rho}{u\tau^0} \tag{3.8}$$

Moreover, since $\psi_1 = \theta_1 + \phi_1 = 2\theta_1 + \psi - \pi$, we have $|d^2\psi/d\psi_1^2| =$

$2|d^2\theta_1/d\psi_1^2|$, and, making use of the previous results, we find

$$\sup_{\psi_1 \in \Delta_{\gamma_1}} \left| \frac{d^2\psi}{d\psi_1^2} \right| \left(\inf_{\psi_1 \in \Delta_{\gamma_1}} \left| \frac{d\psi}{d\psi_1} \right| \right)^{-1} \leqslant \frac{8}{u} \left(1 + \left(\frac{\rho}{\tau^0} \right)^2 \sup_{\psi \in \Delta_\gamma} \left| \frac{d^2\theta}{d\psi^2} \right| \right) \tag{3.9}$$

Inequalities (3.8) and (3.9) give inequality (3.7). Proposition 3.2 is proved. ∎

Corollary 3.3. Let $\gamma \subset S(q)$, $\omega \in \Omega$, and f be as in Proposition 3.4. Suppose furthermore that $\gamma_k = \{(\theta_k(\psi_k), \phi_k(\psi_k)) : \psi_k \in \Delta_{\gamma_k}\}$, $k = 1, \ldots, n$, is a sequence of increasing curves such that $\gamma_k \subset T_\rho^\omega \gamma_{k-1}$ $(\gamma_0 = \gamma)$, $k = 1, \ldots, n$, and (i) $\inf_{(q,\psi) \in \gamma_k} \tau_\rho(q, \psi) \geqslant \tau^0 > 2$, $k = 0, \ldots, n-1$, (ii) $\inf_{\psi_k \in \Delta_{\gamma_k}} \cos\phi_k(\psi_k) \geqslant u > 0$, $k = 1, \ldots, n$. Then, if f_n denotes the density of the measure induced by f on Δ_{γ_n}, the following inequality holds for ρ small enough and all $n = 1, 2, \ldots$:

$$\frac{\sup_{\psi_n \in \Delta_{\gamma_n}} |f_n'(\psi_n)|}{\inf_{\psi_n \in \Delta_{\gamma_n}} f_n(\psi_n)} \leqslant \left(\frac{C}{u} \right)^n \frac{\sup_{\psi \in \Delta_\gamma} |f'(\psi)|}{\inf_{\psi \in \Delta_\gamma} f(\psi)} + C \frac{(C/u)^n - 1}{C - u}$$

Proof. The proof follows immediately from Proposition 3.2. ∎

Proposition 3.4. Let $\eta_1 \in [0, 1/2)$, $N_\rho(\omega) = |\omega \cap D_{\rho^{-1-\eta_1}}(0)|$, $\hat{N}_\rho(\omega) = |(\omega/\omega^{(r(\rho))}) \cap D_{\rho^{-1-\eta_1}}(0)|$ for $r(\rho) = \pi\rho^\beta$ and $\beta \in (1/2 + \eta_1, 1)$. Then there is a set $\tilde{\Omega} \subset \Omega$, $\text{Prob}(\tilde{\Omega}) = 1$, such that for $\omega \in \tilde{\Omega}$ and ρ small enough (i) $N_\rho(\omega) \leqslant c_1 \rho^{-2(1+\eta_1)}$ for some constant $c_1 > 0$, and (ii) $\hat{N}_\rho(\omega) \leqslant \rho^{-\eta_2}$ for any $\eta_2 > 3/2 - \beta + \eta_1$.

Proof. We have $\mathbb{E} N_\rho = \mathbb{D} N_\rho = \lambda\pi\rho^{-2(1+\eta_1)}$ ($\mathbb{D}$ denotes the dispersion), whence by the Chebyshev inequality, for any $s \in (\eta_1, 2\eta_1)$ we get

$$\text{Prob}\Big(\big\{ \omega \in \Omega : N_\rho(\omega) > \mathbb{E} N_\rho - (\lambda\pi)^{1/2} \rho^{-(2+s)} \big\} \Big) \leqslant \rho^2$$

from which assertion (i) follows from the Borel–Cantelli lemma and obvious geometric considerations. Consider now $\hat{N}_\rho$. We have

$$\begin{aligned} \mathbb{E} \hat{N}_\rho &\leqslant \lambda^2 \int_{D_{\rho^{-1-\eta_1}}(0)} dq \int_{D_{r(\rho)}(q)} dq' = 4\pi\lambda^2 (r(\rho))^2 \rho^{-2(1+\eta_1)} \\ &\leqslant \bar{c}\rho^{-2(1-\beta+\eta_1)} \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mathbb{D} \hat{N}_\rho &\leqslant \lambda \int_{D_{\rho^{-1-\eta_1}}(0)} dq\, \mathbb{D} |\omega \cap D_{r(\rho)}(q)| \leqslant 4\pi\lambda^2 (r(\rho))^2 \rho^{-2(1+\eta_1)} \\ &\leqslant \bar{c}\rho^{-2(1-\beta+\eta_1)} \end{aligned} \tag{3.11}$$

For $\hat{\eta} > (1 - \beta + \eta_1)$ inequalities (3.10) and (3.11) imply, via Chebyshev inequality, that

$$\operatorname{Prob}\left(\left\{\omega \in \Omega : \hat{N}_\rho(\omega) \geqslant \mathbb{E}\hat{N}_\rho + \rho^{-\hat{\eta}}\right\}\right) \leqslant \bar{c}\rho^{1+\eta}$$

for some $\eta > 0$. Assertion (ii) follows in a standard way for all $\eta_2 > \hat{\eta}$ since $\hat{\eta} > 2(1 - \beta + \eta_1)$. Proposition 3.4 is proved. ■

We are now ready for the proof of our basic lemma on the joint distribution of the free path lengths and impact parameters. The notation is the same as in Proposition 2.4, only we require the following conditions on the parameters: $\alpha_2 \in [0, 1/2)$, $\delta_1 \in (1, 3/2 - \alpha_2)$ and $\beta \in (\delta_1 - 1/2, 1 - \alpha_2)$. Furthermore we denote by $\mathscr{G}_\rho$ the class of the positive functions on M which are the class C^1 and for which

$$\sup_{|q| \leqslant \rho^{-\delta_1}} \left\{ \max_{\psi \in S^1} \left| \frac{\partial}{\partial \psi} f(q, \psi) \right| \Big/ \min_{\psi \in S^1} f(q, \psi) \right\} < \rho^{-\alpha_2} \tag{3.12}$$

and, if γ is an admissible curve in M and $(x_1, y_1), \ldots, (x_n, y_n) \in [0, \infty) \times [-1, 1]$ we set

$$\begin{aligned}
&\mathscr{M}_{\rho,\gamma}^{(n)}(x_1, y_1; \ldots; x_n, y_n) \\
&\quad = \left\{\psi \in \Delta_\gamma : \hat{\tau}_\rho^{(1)}(q(\psi), \psi) < x_1, \hat{b}_\rho^{(1)}(q(\psi), \psi) < y_1, \ldots, \right. \\
&\qquad \left. \hat{\tau}_\rho^{(n)}(q(\psi), \psi) < x_n, \hat{b}_\rho^{(n)}(q(\psi), \psi) < y_n\right\}, \qquad n = 1, 2, \ldots
\end{aligned}$$

Lemma 3.5. The following relations hold for Prob-almost all $\omega \in r$

$$\lim_{\rho \to 0} \sup_{f \in \mathscr{G}_\rho} \sup_{\substack{|q| < \rho^{-\delta_1} \\ \operatorname{dist}(q, \omega) > d(\rho)}} \max_{\gamma \in \Gamma_\rho(q)} \left\| \mu_f\left(\mathscr{M}_{\rho,\gamma}^{(n)}(x_1, y_1; \ldots; x_n, y_n)\right) - \prod_{i=1}^n F(x_i, y_i) \right\|_\infty = 0, \qquad n = 1, 2, \ldots \tag{3.13}$$

where $d(\rho) = r(\rho) + \sqrt{2}\, a(\rho)$ and μ_f denotes the normalized measure induced by $f(q, \cdot)$ on Δ_γ.

Proof. We shall prove Eq. (3.13) for $n = 1, 2$. The extension to the case $n \geqslant 3$ can be made along the same lines.

Set $D_\rho^\omega = \{q \in \mathbb{R}^2 : |q| < \rho^{-\delta_1}, \operatorname{dist}(q, \omega) > d(\rho)\}$. If $q \in D_\rho^\omega$ there is at least one point $\bar{q} \in \mathbb{Z}^{(\rho)}(\omega)$ such that $|q - \bar{q}| \leqslant \sqrt{2}\, a(\rho)$. We can therefore apply Proposition 3.1 to the pairs $\gamma_q^{(i)}, \gamma_{\bar{q}}^{(i)}$, $i = 1, \ldots, \kappa(\rho)$, and Proposition

2.4, and conclude that the Prob-a.a. ω and $\rho_m \leqslant m^{-t}$, $t > 0$, $m = 1, 2, \ldots$

$$\lim_{m\to\infty} \sup_{\substack{q\in D^{\omega}_{\rho_m}\\ f\in\mathscr{G}_{\rho_m}}} \max_{\gamma\in\Gamma_{\rho_m}(q)} \| \mu_f\big(\mathscr{M}^{(1)}_{\rho_m,\gamma}(\cdot,\cdot)\big) - F(\cdot,\cdot)\|_\infty = 0 \tag{3.14}$$

Using again Propositions 2.4 and 3.1 it is not hard to see that if we set

$$\mathscr{N}_\gamma = \left\{\psi \in \Delta_\gamma : \min_{i=1,2} d_i^{(\omega,\rho)}(q,\psi) > (\log(e/\rho))^{-1}, \tau^{\omega}_\rho(q,\psi) \in (\rho^{-1+\eta_3}, \rho^{-\delta_1})\right\}$$

for some $\eta_3 \in (0, 1/2)$, we have

$$\lim_{m\to\infty} \sup_{\substack{q\in D^{\omega}_{\rho_m}\\ f\in\mathscr{G}_{\rho_m}}} \max_{\gamma\in\Gamma_{\rho_m}(q)} \mu_f(\Delta_\gamma \backslash \mathscr{N}_\gamma) = 0 \tag{3.15}$$

This means that we can neglect for each $\gamma \in \Gamma_\rho(q)$ the angles for which the free path is either too long or too short, or which are too close to a discontinuity of the map T^{ω}_ρ. Consider now the sets

$$\mathscr{S}_\gamma = \left\{\psi \in \mathscr{N}_\gamma : T^{\omega}_\rho(q,\psi) \in \hat{K}_\rho(q') \text{ for } q' \in \omega\backslash\omega^{(r(\rho))}\right\} \qquad \gamma \in \Gamma_\rho(q)$$

If we choose $\eta_1 \in (\delta_1 - 1, 1/2 - \alpha_2)$ such that moreover $\eta_1 + 1/2 < \beta$ and $\hat{N}_\rho(\omega)$ is defined as in Proposition 3.4, it is easy to see by simple geometric considerations that for ρ small enough

$$\sup_{\substack{|q|\leqslant\rho^{-\delta_1}\\ f\in\mathscr{G}_\rho}} \max_{\gamma\in\Gamma_\rho(q)} \mu_f(\mathscr{S}_\gamma) \leqslant c\kappa(\rho)\frac{\rho}{\rho^{-1+\eta_3}}\hat{N}_\rho(\omega) \tag{3.16}$$

Therefore, according to Proposition 3.4, assertion (ii), we find that Prob-a.e. the right-hand side of Eq. (3.16) is less than $c'\rho^{2-\eta_3-\alpha_2-\eta_2}\log(e/\rho)$, and we can take $\eta_2 \in (3/2 + \eta_1 - \beta, 3/2 - \alpha_2)$, so that it goes to 0. Therefore if we set $\tilde{\mathscr{N}}_\gamma = \mathscr{N}_\gamma \backslash \mathscr{S}_\gamma$ and $\tilde{\mathscr{M}}^{(n)}_{\rho,\gamma}(\cdot) = \mathscr{M}^{(n)}_{\rho,\gamma}(\cdot) \cap \tilde{\mathscr{N}}_\gamma$, $\gamma \in \Gamma_\rho(q)$, $q \in D^{\omega}_\rho$, we find that

$$\lim_{m\to\infty} \sup_{\substack{q\in D^{\omega}_{\rho_m}\\ f\in\mathscr{G}_{\rho_m}}} \max_{\gamma\in\Gamma_{\rho_m}(q)} \| \mu_f\big(\mathscr{M}^{(n)}_{\rho_m,\gamma}(\cdot)\backslash\tilde{\mathscr{M}}^{(n)}_{\rho_m,\gamma}(\cdot)\big)\|_\infty = 0, \qquad n = 1, 2, \ldots \tag{3.17}$$

Consider now on $S^1 \times [-\pi/2, \pi/2]$ the lattice $\hat{\mathbb{Z}}_b$ of the points $z_{\mathbf{k}} = b\mathbf{k}$, where $b = b(\rho) = (\pi/2)[\pi/\rho^2]^{-1}$ and $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ for $k_1 = -2[\pi/\rho^2], \ldots, 2[\pi/\rho^2] - 1$, $k_2 = -[\pi/\rho^2], \ldots, [\pi/\rho^2]$. Let $\mathscr{Q}_\rho$ denote the family of the straight segments of slope 1 on $S^1 \times [-\pi/2, \pi/2]$ which have initial and end points in $\hat{\mathbb{Z}}_b$ and length larger than $\frac{1}{2}[\log(e/\rho)]^{-1}$. For any choice of $s > 4$ and $\alpha_1 \in (0, 1)$ the number of such segments does not exceed ρ^{-s} and their length is larger than ρ^{α_1}, for ρ small enough. We can therefore apply Proposition 2.5 to find that for any $\alpha \in (0, 1)$ and Prob-a.a.

$\omega \in \Omega$

$$\lim_{m\to\infty} \max_{\substack{|q| \leqslant \rho^{-1-\eta_1} \\ q \in \omega^{(r(\rho_m))}}} \max_{\gamma \in \mathscr{F}_{\rho_m}(q)} \sup_{f \in \mathscr{F}_{\Delta_\gamma}^{\rho_m}(\alpha)} \| \mu_f(\mathscr{M}_\gamma^{\rho_m,\omega}(\cdot,\cdot)) - F(\cdot,\cdot) \|_\infty = 0 \tag{3.18}$$

Now if $\gamma \in \Gamma_\rho(q)$, $q \in D_{\rho_m}^\omega$, the image of $\tilde{\mathscr{M}}_{\rho,\gamma}^{(1)}(x, y)$ under T_ρ^ω consists, for ρ small enough, of a finite number of increasing curves of length larger than $(\log(e/\rho))^{-1}$, on which $\cos\phi_1 \geqslant (\log(e/\rho))^{-1/2}$, and which are on scatterers with centers in $\omega^{(r(\rho))} \cap D_{\rho^{-1-\eta_1}}(0)$. Let us denote by $\Gamma_1(\gamma)$ the collection of such curves. If $\gamma_1 \in \Gamma_1(\gamma)$ the density f_1 of the measure induced by μ_f, via T_ρ^ω, on Δ_{γ_1}, belongs, according to Proposition 3.2 to the class $\mathscr{F}_{\Delta_{\gamma_1}}^\rho(\alpha)$ for any $\alpha > \alpha_2$ and ρ small enough. Moreover, let (θ_0, ϕ_0) and $(\bar{\theta}, \bar{\phi})$ be the end points on γ_1 ($\bar{\phi} > \phi_0$), and let $(\hat{\theta}_0, \hat{\phi}_0)$ denote the point of the lattice $\hat{\mathbb{Z}}_b$ which is closest to (θ_0, ϕ_0) under the condition $\hat{\theta}_0 + \hat{\phi}_0 \geqslant \theta_0 + \phi_0$. Let $(\hat{\theta}, \hat{\phi})$ be the last point of $\hat{\mathbb{Z}}_b$ which is met moving away from $(\hat{\theta}_0, \hat{\phi}_0)$ on the straight line of slope 1, in the direction of increasing θ, for which $\hat{\theta} + \hat{\phi} \leqslant \bar{\theta} + \bar{\phi}$, and let $\hat{\gamma}_1$ be the segment of $\mathscr{Q}_\rho$ with end points $(\hat{\theta}_0, \hat{\phi}_0)$ and $(\hat{\theta}, \hat{\phi})$. Since $|d\phi_1/d\theta_1 - 1| = (\rho/\tau)\cos\phi_1 \leqslant \rho^{2-\eta_3}$ [see Eq. (2.1)], it is easy to see that we can apply Proposition 3.1 taking γ_1 as γ, $\hat{\gamma}_1$ as γ', $\kappa_1 = 2$ and $\kappa_2 = 3 - \eta_3$. Therefore the Prob-a.a. $\omega \in \Omega$

$$\lim_{m\to\infty} \sup_{\substack{q \in D_{\rho_m}^\omega \\ f \in \mathscr{G}_{\rho_m}}} \max_{\gamma \in \Gamma_{\rho_m}(q)} \max_{\gamma_1 \in \Gamma_1(\gamma)} \| \mu_{f_1}(\mathscr{M}_{\gamma_1}^{\rho_m,\omega}(\cdot,\cdot)) - F(\cdot,\cdot) \|_\infty = 0 \tag{3.19}$$

Now, for $\gamma \in \Gamma_\rho(q)$, denoting by $\hat{T}_\gamma$ the restriction of T_ρ^ω to γ, and setting, for $\gamma_1 \in \Gamma_1(\gamma)$ $\tilde{\gamma}_1 = \hat{T}_\gamma^{-1}\gamma_1$ we have

$$\frac{\mu_f(\tilde{\mathscr{M}}_{\rho,\gamma}^{(2)}(x_1, y_1; x_2, y_2))}{\mu_f(\tilde{\mathscr{M}}_{\rho,\gamma}^{(1)}(x_1, y_1))} = \sum_{\gamma_1 \in \Gamma_1(\gamma)} \frac{\mu_f(\Delta_{\tilde{\gamma}_1})}{\mu_f(\tilde{\mathscr{M}}_{\rho,\gamma}^{(1)}(x_1, y_1))} \mu_{f_1}(\mathscr{M}_{\gamma_1}^{\rho,\omega}(x_2, y_2)) \tag{3.20}$$

Equations (3.14), (3.17), (3.19), and (3.20) imply that there is a set $\hat{\Omega}$, $\text{Prob}(\hat{\Omega}) = 1$, such that for $\omega \in \hat{\Omega}$

$$\lim_{m\to\infty} \sup_{\substack{q \in D_{\rho_m}^\omega \\ f \in \mathscr{G}_{\rho_m}}} \max_{\gamma \in \Gamma_{\rho_m}(q)} \| \mu_f(\mathscr{M}_{\rho_m,\gamma}^{(2)}(x_1, y_1; x_2, y_2)) - F(x_1, y_1)F(x_2, y_2) \|_\infty = 0 \tag{3.21}$$

By Propositions 2.4, 2.5, and 3.1, one can choose $\hat{\Omega}$ in such a way that for

$\omega \in \hat{\Omega}$

$$\lim_{m\to\infty} \sup_{\substack{q \in D^{\omega}_{\rho_m} \\ f \in \mathscr{G}_{\rho_m}}} \max_{\gamma \in \Gamma_{\rho_m}(q)} \| \mu_f(\mathscr{P}^{\rho_m,\omega}_{i,\gamma}(\cdot)) - G(\cdot)\|_\infty = 0, \quad i = 1,2 \quad (3.22)$$

$$\lim_{m\to\infty} \sup_{\substack{q \in D^{\omega}_{\rho_m} \\ f \in \mathscr{G}_{\rho_m}}} \max_{\gamma \in \Gamma_{\rho_m}(q)} \max_{\gamma_1 \in \Gamma_1(\gamma)} \| \mu_{f_1}(\mathscr{P}^{\rho_m,\omega}_{i,\gamma}(\cdot)) - G(\cdot)\|_\infty = 0, \quad i = 1,2 \quad (3.23)$$

Now it is easy to check that for fixed $q \in \mathbb{R}^2$ the system of curves $\Gamma_\rho(q)$ changes only when ρ assumes some discrete values ρ_m, $m = 1, 2, \ldots,$ and it is easy to see that $\rho_m^{-1} > m^t$ for m large enough and any $t < 1/\alpha_2$. Therefore we have only to prove, in addition to Eq. (3.21), that for $\omega \in \hat{\Omega}$

$$\lim_{m\to\infty} \sup_{\substack{q \in D^{\omega}_{\rho_{m+1}} \\ f \in \mathscr{G}_{\rho_{m+1}}}} \max_{\gamma \in \Gamma_{\rho_m}(q)} \sup_{\rho \in (\rho_{m+1}, \rho_m]} \| \mu_f\big(\mathscr{M}^{(2)}_{\rho,\gamma}(x_1, y_1; x_2, y_2)\big)$$

$$- F(x_1, y_1)F(x_2, y_2)\|_\infty = 0 \quad (3.24)$$

Equation (3.24) can be proved making use of Eqs. (3.21), (3.22), (3.23) and simple geometric considerations. Therefore, Eq. (3.13) is proved for $n = 2$. For $n = 1$ it follows from Eqs. (3.14) and (3.22). For $n > 2$ the proof follows if one establishes for the increasing curves in the iterated images of the curves $\gamma \in \Gamma_\rho(q)$ a relation analogous to relation (3.19). This can be done by using Corollary 3.3 and again Propositions 2.5 and 3.1. Lemma 3.5 is proved. ∎

Proof of Theorem 1.1. We assume for the moment that $\inf_{(q,\psi)\in M} f_0(q,\psi) = c_1 > 0$. The theorem will be proved if we show that, if ω belongs to a set of full measure, for any square Q and any interval $I = [\psi_1, \psi_2] \subset S^1$, and for all $t \in \mathbb{R}^1$ we have

$$\lim_{\rho\to 0} \mu_t^{(\rho^2,\omega_\rho)}(Q \times I) = \int_{Q\times I} f_t(q,\psi)\, dq\, d\psi \quad (3.25)$$

where f_t, $t \in \mathbb{R}^1$, is the unique solution of Eq. (1.6) with initial data f_0. By the observation made at the end of Section 1

$$\mu_t^{(\rho^2,\omega_\rho)}(Q \times I) = \int_{M'_{\rho,\omega}} dq\, d\psi\, \hat{f}_\rho(q,\psi)\chi^t_{Q,I}(q,\psi) \quad (3.26)$$

where $\chi^t_{Q,I}(q,\psi)$ is the indicator function of the set $T^{(\rho,\omega)}_{-\rho^{-1}t}((Q_\rho \times I) \cap M'_{\rho,\omega})$, $Q_\rho = \rho^{-1}Q$,[4] and $\hat{f}_\rho(q,\psi) = \rho^2 f_0(\rho q, \psi)$. The function $\hat{f}_\rho$ does not

[4] We use here the notation $\rho^{-1}Q = \{q' \in \mathbb{R}^2 : \rho q' \in Q\}$.

necessarily belong to the class $\mathscr{G}_\rho$ for any $\alpha_2 \in [0, 1/2)$ and $\delta_1 \in (1, 3/2 - \alpha_2)$. However, we can fix $\alpha_2 \in (0, 1/2)$, choose $\delta_1 \in (1, 3/2 - \alpha_2)$ and take the set

$$\hat{D}_\rho^\omega = D_\rho^\omega \cap \left\{ q \in \mathbb{R}^2 : \frac{1}{c_1\rho^2} \max_{\psi \in S^1} \left| \frac{\partial}{\partial\psi} \hat{f}_\rho(q,\psi) \right| < \rho^{-\alpha_2} \right\}$$

Clearly Lemma 3.5 implies that

$$\lim_{\rho\to 0} \sup_{q \in \hat{D}_\rho^\omega} \max_{\gamma \in \Gamma_\rho(q)} \left\| \mu_{\hat{f}_\rho}\left(\mathscr{M}_{\rho,\gamma}^{(n)}(x_1, y_1; \ldots; x_n, y_n)\right) - \prod_{i=1}^{n} F(x_i, y_i) \right\|_\infty = 0,$$
$$n = 1, 2, \ldots \tag{3.27}$$

If we set $B_t^\rho = \{ q \in \mathbb{R}^2 : \mathrm{dist}(q, Q_\rho) \leqslant \rho^{-1} t \}$, it is clear that $T_{-\rho^{-1}t}^{(\rho,\omega)}((Q_\rho \times I) \cap M'_{\rho,\omega}) \subset B_t^\rho \times S^1$. Moreover, setting $\bar{D}_\rho^\omega = \mathbb{R}^2 \backslash \hat{D}_\rho^\omega$, it is easily seen by Proposition 3.4, assertion (i), that there is a set $\tilde{\Omega}$, $\mathrm{Prob}(\tilde{\Omega}) = 1$, such that for $\omega \in \tilde{\Omega}$ and all $t \in \mathbb{R}^1$

$$\lim_{\rho\to 0} \int_{\left(B_t^\rho \cap \bar{D}_\rho^\omega\right) \times S'} dq\, d\psi\, \hat{f}_\rho(q,\psi) = 0 \tag{3.28}$$

Therefore the limit as $\rho \to 0$ of expression on the right-hand side of Eq. (3.26) is equal to that of the expression

$$\int_{\hat{B}_t^{\rho,\omega} \times S^1} dq\, d\psi\, \hat{f}_\rho(q,\psi) \chi_{Q,I}^t(q,\psi) = \int_{\hat{B}_t^{\rho,\omega}} dq \sum_{\gamma \in \Gamma_\rho(q)} \mu_q(\Delta_\gamma) \sum_{n=0}^{\infty} \mu_{\hat{f}_\rho}\left(E_\gamma^{(n)}(q,t)\right) \tag{3.29}$$

where $\hat{B}_t^{\rho,\omega} = B_t^\rho \cap \hat{D}_\rho^\omega$, μ_q is the measure on S^1 with density $\hat{f}_\rho(q, \cdot)$, and $E_\gamma^{(n)}(q,t) = \{ \psi \in \Delta_\gamma : n_t(q,\psi) = n$ and $T_{\rho^{-1}t}^{(\rho,\omega)}(q,\psi) \in Q_\rho \times I \}$. From Eqs. (1.2.a, b) and (1.3a, b) we see that

$$\begin{aligned} &E_\gamma^{(n)}(q,t) \\ &= \Bigg\{ \psi \in \Delta : \sum_{j=1}^{n} \hat{\tau}^{(j)}(q,\psi) \leqslant t < \sum_{j=1}^{n+1} \hat{\tau}^{(j)}(q,\psi), \\ &\qquad q + \delta_{\rho^{-1}t}^{(n)}\left(\psi; \tau^{(1)}(q,\psi), \hat{b}^{(1)}(q,\psi); \ldots; \tau^{(n)}(q,\psi), \hat{b}^{(n)}(q,\psi)\right) \in Q_\rho, \\ &\qquad \psi + R_n\left(\hat{b}^{(1)}(q,\psi), \ldots, \hat{b}^{(n)}(q,\psi)\right) \in I \Bigg\} \end{aligned}$$

Let $\bar{\psi}_\gamma$ denote the middle point of the interval Δ_γ and consider the sets

$$\hat{E}_\gamma^{(n)}(q,t)$$
$$= \Big\{ \psi \in \Delta_\gamma : \sum_{j=1}^{n} \hat{\tau}^{(j)}(q,\psi) \leqslant t \leqslant \sum_{j=1}^{n+1} \hat{\tau}^{(j)}(q,\psi),$$
$$q + \delta_{\rho^{-1}t}^{(n)}\big(\bar{\psi}_\gamma ; \tau^{(1)}(q,\psi), \hat{b}^{(1)}(q,\psi), \ldots, \tau^{(n)}(q,\psi), \hat{b}^{(n)}(q,\psi)\big) \in \hat{Q}_\rho,$$
$$\bar{\psi} + R_n\big(\hat{b}^{(1)}(q,\psi), \ldots, \hat{b}^{(n)}(q,\psi)\big) \in \hat{I}_\rho \Big\}, \qquad \gamma \in \Gamma_\rho(q)$$

where $\hat{Q}_\rho = \{q \in \mathbb{R}^2 : \mathrm{dist}(q, Q_\rho) \leqslant 2\pi t(\rho\kappa(\rho))^{-1}\}$, $\hat{I}_\rho = \{\psi \in \Delta_\gamma : \mathrm{dist}(\psi, I) \leqslant 2\pi(\kappa(\rho))^{-1}\}$. It is easy to see that $E_\gamma^{(n)}(q,t) \subset \hat{E}_\gamma^{(n)}(q,t)$. Moreover from the weak convergence, uniform in $q \in \hat{D}_\rho^\omega$, of the distributions of free path lengths and impact parameters, given by Eq. (3.27), and taking into account Eq. (3.28), we find

$$\lim_{\rho\to 0} \int_{\hat{B}_t^{\rho,\omega}} dq \sum_{\gamma \in \Gamma_\rho(q)} \mu_q(\Delta_\gamma)\mu_{\hat{f}_\rho}\big(\hat{E}_\gamma^{(n)}(q,t)\big)$$
$$= \exp(-2\lambda t)\lambda^n \int_M dq\, d\psi\, f_0(q,\psi) \int_{x_1 + \cdots + x_n \leqslant t} dx_1 \ldots dx_n$$
$$\times \int_{[-1,1]^n} dy_1 \ldots dy_n\, \chi_I\big(\psi + R_n(y_1, \ldots, y_n)\big)$$
$$\chi_Q\big(q + \delta_t^{(n)}(\psi; x_1, y_1; \ldots x_n, y_n)\big)$$

which implies (see Ref. 3)

$$\limsup_{\rho\to 0} \mu_t^{(\rho^2,\omega_\rho)}(Q \times I) \leqslant \int_{Q\times I} d\bar{q}\, d\bar{\psi} \int_M dq\, d\psi\; g_t(\bar{q},\bar{\psi}; q,\psi) f_0(q,\psi)$$
$$= \int_{Q\times I} dq\, d\psi\, f_t(q,\psi) \tag{3.30}$$

where g_t denotes the distribution (Green's function)

$$g_t(\bar{q},\bar{\psi}; q,\psi) = \sum_{n=0}^{\infty} \lambda^n \int_{x_1 + \cdots + x_n \leqslant t} dx_1 \ldots dx_n \int_{[-1,1]^n} dy_1 \ldots dy_n$$
$$\times\ \delta\big((\bar{q},\bar{\psi}) - \big(\psi + R_n(y_1, \ldots, y_n),$$
$$q + \delta_t^{(n)}(\psi; x_1, y_1; \ldots; x_n, y_n)\big)\big)$$

In a similar way we prove that

$$\liminf_{\rho\to 0} \mu_t^{(\rho^2,\omega_\rho)}(Q \times I) \geqslant \int_{Q\times I} dq\, d\psi\, f_t(q,\psi) \tag{3.31}$$

Equations (3.30) and (3.31) imply Eq. (3.25).

If the condition $\inf_{(q,\psi)\in M} f_0(q,\psi) = c_1 > 0$ does not hold one can modify the proof by excluding from the set $\hat{B}_t^{\rho,\omega}$ the points in which $\hat{f}_\rho(q,\psi) < \rho^2\eta(\rho)$ for some function $\eta(\rho)$ going to zero slowly enough as $\rho \to 0$. Theorem 1.1 is proved. ■

We now show how the result can be extended to nonsmooth initial data $f_0 \in L^1_{\mathrm{loc}}(M)$. For each $N = 1, 2, \ldots,$ we take a function $f_0^{(N)} \in C^1(M)$ such that

$$\int_{D_N(0)\times S^1} |f_0 - f_0^{(N)}|\, dq\, d\psi < \epsilon_N$$

where $\lim_{N\to\infty} \epsilon_N = 0$. Correspondingly, by the above result, one can find a set $\Omega^{(N)}$, $\mathrm{Prob}(\Omega^{(N)}) = 1$, such that for $\omega \in \Omega^{(N)}$ the limiting behavior of the Lorentz gas is given by the solution $f_t^{(N)}$, $t \in \mathbb{R}^1$, of Eq. (1.6) with initial data $f_0^{(N)}$. Now, for any fixed $t \in \mathbb{R}^1$, and any bounded $A \in \mathfrak{B}$, we have, for $N' > N > R = t + \sup_{(q,\psi)\in A} |q|$,

$$\int_A dq\, d\psi\, |f_t^{(N)}(q,\psi) - f_t^{(N')}(q,\psi)| \leqslant \int_{D_R(0)\times S^1} dq\, d\psi\, |f_0^{(N)}(q,\psi) - f_0^{(N')}(q,\psi)|$$
$$< 2\epsilon_N$$

which shows that the sequence $\{f_t^{(N)}\}_{N=1}^{\infty}$ is a Cauchy sequence in $L^1(A)$. If we denote by f_t the corresponding limit, clearly $f_t \in L^1_{\mathrm{loc}}(M)$, and it is easily seen, setting $\hat{\Omega} = \bigcap_{N=1}^{\infty} \Omega^{(N)}$, that $\mathrm{Prob}(\hat{\Omega}) = 1$, and for $\omega \in \hat{\Omega}$

$$\lim_{\rho\to 0} \mu_t^{(\rho^2,\omega_\rho)}(A) = \lim_{\rho\to 0} \int_{T_{-t}^{(\rho^2,\omega_\rho)}(A \cap M'_{\rho^2,\omega_\rho})} dq\, d\psi\, f_0(q,\psi) = \int_A dq\, d\psi\, f_t(q,\psi)$$

for any bounded $A \in \mathfrak{B} \cdot f_t$, $t \in \mathbb{R}^1$, gives a weak solution of Eq. (1.6).

4. CONCLUDING REMARKS

As we said in the Introduction, the kinetic equation of Boltzmann for the classical Lorentz gas, goes, when we apply the limiting procedure of Boltzmann and Grad, into the Fokker–Planck–Kolmogorov equation for the limiting Markov process. From Lemma 3.7 it follows that in the limit we have a process with independent values. This explains why the Boltzmann equation for the Lorentz gas which we obtain is of the form

$$\left(\frac{\partial}{\partial t} + \psi \cdot \nabla_{\mathbf{q}}\right) f_t(q,\psi) = l^{-1}(\hat{P} - \mathbb{1}) f_t(q,\psi)$$

where l is the mean free path, $\hat{P}$ an averaging over directions

$$\hat{P}f_t(q,\psi) = \int_{S^{d-1}} \sigma(\psi \mid \psi') f_t(q,\psi')\, d\psi'$$

S^{d-1} is the surface of the d-dimensional unit sphere, $d\psi'$ is the usual measure over solid angles and $\sigma(\psi \mid \psi')$ is the scattering cross section.

Apparently it is possible to consider other distributions of the scatterers centers, not necessarily Poisson. However, such a generalization is not a direct consequence of our paper.

In conclusion we want to say something about the difference between the problems which arise in ergodic theory and in kinetic theory (kinetics). Ergodic theory is interested in the asymptotic properties, as t goes to infinity, of dynamical systems, whereas in kinetic theory we look at the behavior of the system for finite times (of the order of the mean free flight time). This is the reason for the difference in the methods which are used. The basic object of investigation in the ergodic theory of Hamiltonian systems are the so-called horospheres, or, more generally, the stable and unstable foliations. On the other hand the present paper shows that to derive kinetic equations it is sufficient to study the evolution for finite times of the spheres which are given by a set of velocity vectors corresponding to one point of the configuration space of the system under consideration. One can introduce such objects for a large class of potentials, including potentials for which horospheres have not been constructed.

REFERENCES

1. H. Grad, Principles of the Kinetic Theory of Gases, in *Handbunch der Physik*, Vol. 12, S. Flügge, ed. (Springer, Berlin, 1958).
2. H. Spohn, *Rev. Mod. Phys.* **52**:569–615 (1980).
3. G. Gallavotti, Rigorous Theory of the Boltzmann equation in the Lorentz Gas, Nota interna $n°$ 358, Istituto di Fisica, Università di Roma, 1972.
4. G. Gallavotti, *Phys. Rev.* **185**:308 (1969).
5. Ya. G. Sinai, *Russ. Math. Surv.* **25**:132 (1970).
6. G. Gallavotti, Lectures on the Billiard, in *Dynamical Systems, Theory and Application*, J. Moser, ed. (Springer, Berlin, 1975).
7. L. A. Bunimovich and Ya. G. Sinai, *Commun. Math. Phys.* **78**:247–280 (1980).