Acta Applicandae Mathematicae **36**: 49–55, 1994.

Monodromy of Boussinesq Elliptic Operators

To J.-L. Verdier, in memoriam

EMMA PREVIATO
Department of Mathematics, Boston University, 111 Cummington Street, Boston, MA 02215, U.S.A.

(Received: 10 July 1993)

Abstract. Verdier's program for classifying elliptic operators with a nontrivial centralizer is outlined. Examples of Boussinesq operators are developed.

Mathematics Subject Classifications (1991). Primary 33E05; secondary 35Q53, 32C38, 32S40.

Key words. Elliptic functions, Kadomtsev–Petviashvili hierarchy, local systems.

A Boussinesq elliptic operator is an ordinary differential operator

$$L = \partial^3 + u(x)\partial + v(x)$$

(we write ∂ for d/dx, as is customary in the theory), such that (i) its coefficients are algebraic functions over an elliptic curve $X = \mathbb{C}/\Lambda$, which is to say u and v are doubly-periodic meromorphic functions of x with the period lattice Λ; (ii) L is a Boussinesq (or 3rd KdV) solution in the sense of [1]: there exists a $W \in \mathrm{Gr}^{(3)}$ such that $L^{1/3}$ is the KP solution associated to W and evaluated at zero higher-times.

In [2] (Lemma A3.6 and Theorem 7.6), a [Boussinesq] elliptic operator is proved to be equivalent to the data: $(\Gamma, k, \xi) \in W(n, X)$, namely the isomorphism class of a 'minimal [Boussinesq] tangential cover' of degree n (cf. Section 1 below) π: $(\Gamma, p) \to (X, q)$, a tangential function $k[\in H^0(3P) \setminus H^0(2P)]$ and an element ξ of the generalized Jacobian of Γ, Pic^{g-1} [if, moreover, $g(\Gamma) \neq 1$, we call these BTC (n, X)]. The Treibich–Verdier theory (cf. [2–5]) provides us with a moduli space $W(n, X)$ of minimal tangential covers; it also answers the KdV question (analogous to Boussinesq) by classifying the minimal hyperelliptic tangential covers: there is a finite number of these for all given X and n, and none of them belong to $W(n, X)$ unless $n = 1$, for they are not minimal among all tangential covers. But for Boussinesq (and higher 'Nth KdV') the situation is quite different basically because the general elliptic curve does not have an automorphism of order N. In [6], the search was confined to the Galois–Boussinesq tangent covers, for which such an automorphism exists; X is essentially unique and the answer is much the same as for KdV, except for the fact that some may belong to $W(n, X)$, i.e. be minimal as covers, a smooth example was constructed in [7].

But if we omit the Galois property, the family of BTC may have dimension greater than zero, as we observe below. The goal of this paper is to state Verdier's program for classifying Boussinesq (and Nth KdV) tangential covers in terms of the monodromy of the corresponding elliptic operators.

In Section 1, we recall the terminology that we need from the Treibich–Verdier theory; we give a Grassmannian interpretation of tangential covers and ask some questions natural to this context. In Section 2, we explain Verdier's idea. In Section 3 we give examples.

The Nth KdV elliptic solitons make up a locus whose geometry is interesting in its own right (in the vein of a Schottky question), has number-theoretic significance (cf. the tantalizing arithmetic of those KdV elliptic solitons whose existence depends on Λ, [4] 6.7), and involves varieties of dimension higher than one. As J.-L. Verdier said in his Colloquium talk at Boston University (October 1988), the KP equation still holds riches unexplored.

1. Grassmannian Formulation

We use the notation Gr as in [1] for the (connected component of spaces W of virtual dimension 0 of the) Grassmannian of $H = L^2(S^1, \mathbb{C})$.

1.1. DEFINITION. Let Gr_2 be the subspace Gr consisting of those W such that the corresponding KP solution $\mathcal{L}_W$ is periodic in $x = t_1$, with respect to some lattice Λ.

1.2. DEFINITION (cf. [2], 2.2). Let Γ be a (projective, integral) curve of (arithmetic) genus > 0, $p \in \Gamma$ a smooth point and π: $(\Gamma, p) \to (X, q)$ a finite, pointed morphism to an elliptic curve. π is said to be a *tangential cover* if $\pi^*(X)$ is tangent to $A_\Gamma(\Gamma)$ at the origin of $\mathrm{Jac}\,\Gamma$, where A_Γ is the Abel map.

1.3. PROPOSITION. *An element $W \in \mathrm{Gr}_2$ gives rise to an algebro-geometric KP solution; in fact, to a tangential cover $(\Gamma, p) \to (X, q)$, where $X = \mathbb{C}/\Lambda$.*

This is a consequence of Proposition 5.1 in [1] and Appendix 3 in [2].

The subscript 2 refers to the two periods; in [1], subspaces $\mathrm{Gr}_0 \subset \mathrm{Gr}_1$ are defined loop-theoretically and shown to correspond to solutions of the rational type (no periods) or exponential type (one period). That Gr_0 is dense in Gr is easily seen by interpreting its elements as graphs. I think it likely that for any fixed lattice Λ the elements of Gr_2 that correspond to it also give a dense subspace of Gr, but I have no evidence for that.* What we can do is to let Λ approach a singular limit within $\mathrm{Gr}^{(2)} = \{W \in \mathrm{GR} \text{ s.t. } z^2 W \subset W\}$:

* This has been proved, cf. E. Colombo, G. P. Pirola and E. Previato, *J. reine angew. Math.* 1994.

1.4. PROPOSITION. $\mathrm{Gr}_2^{(2)} = \mathrm{Gr}_2 \cap \mathrm{Gr}^{(2)}$ *is dense in* $\mathrm{Gr}^{(2)}$.

Proof. $\mathrm{Gr}_0^{(2)}$ has a cell decomposition such that there is exactly one cell C_k of each dimension k; it corresponds to the generalized Jacobian of the curve $y^2 = x^{2k+1}$ and its closure contains all the cells of smaller dimension. Thus, it is enough to remark that the solutions corresponding to C_k are limits of solutions corresponding to $\mathrm{Gr}_2^{(2)}$. This follows from [4] (6.4), where it is proved that Ince's potentials $k(k+1)\wp_\Lambda(x)$ are indeed initial conditions for a solution belonging to $\mathrm{Gr}_2^{(2)}$ (in fact, the unique solution whose corresponding tangential cover $\Gamma \to X$ has degree $k(k+1)/2$).

However, for higher $N \geqslant 3$, the question of the existence of elements in $\mathrm{Gr}_2^{(N)}$ (corresponding to curves of genus > 1) is wide open. For $N = 3$, the following was proved [6]:

(i) If the Galois–Boussinesq tangential covers (GBTC) are defined to be those tangential covers for which X has an automorphism of order 3 which lifts to Γ, then X is unique up to automorphism and for any integer n there exist a finite number of GBTC of X that have degree n; their (arithmetic) genus g is such that $\sqrt{n+1} \leqslant g \leqslant 2\sqrt{n+1} + 1$. It is not known whether there are GBTC of all genera.

(ii) The minimal GBTC of degree n are in 1 : 1 correspondence with rational curves belonging to the linear system $|\lambda(n, a, b)|$ on a certain rational surface $\widetilde{S}$, which is defined by six numbers a_i, b_i, $0 \leqslant i \leqslant 2$, satisfying:
   (1) $2b_i \equiv a_i (\mathrm{mod}\, 3)$,
   (2) $b_i + n \equiv \varepsilon_i (\mathrm{mod}\, 3)$, where $\varepsilon_0 = 1$, $\varepsilon_1 = \varepsilon_2 = 0$,
   (3) $a_i \leqslant 2b_i$ and $b_i \leqslant 2a_i$,
   (4) $a_i \leqslant n - \varepsilon_i$.

(iii) The 'exceptional' GBTC are (up to isomorphism) exceptional divisors on a certain surface T_2 (birational to the ruled surface over X whose ε-invariant is 0). Equivalently, they satisfy
   (5) $\sum_{i=0}^{2}(a_i^2 + b_i^2 - a_i b_i) = 3(n+1)$.

1.5. EXAMPLE. As proved in [7], the hyperelliptic curve Γ_1 with affine equation $\omega_1^2 = \eta_1^6 - 1$ is a degree 2 GBTC of the elliptic curve $\omega^2 = \eta^3 - 1$. It is in fact an exceptional GBTC.

Proof. Γ_1 dominates some minimal GBTC of degree 2. The only solution of conditions (1)–(4) for $n = 2$ is $(a; b) = (1, 2, 2; 2, 1, 1)$. A corresponding GBTC Γ has

$$g(\Gamma) \leqslant \sum_{i=0}^{2} \left(\frac{a_i + b_i}{3}\right) - 1 = 2$$

and Γ is minimal if and only if $g(\Gamma) = 2$ ([6], 2.4). Thus, Γ_1 must be minimal and it is exceptional because (5) holds.

1.6. *Remark.* If we remove the Galois condition, then the moduli space of GBTC of a given elliptic curve X may have dimension > 0. In fact, for degree $n = 2$ and fixed (X, q) the moduli space of BTC has dimension 1.

Proof. The moduli space $W(2, X)$ of minimal tangential covers together with a choice of tangential function has dimension 2, while the isomorphism classes of minimal tangential covers have dimension 1 ([3] and [4]). By Riemann–Roch, $h^0(3p) = 3 - 2 + 1 + h^1(3p) = 2$, thus the tangential cover is Boussinesq unless it is KdV, namely $h^0(2p) = 2$. But there is only a finite number of isomorphism classes of these by [4].

2. Boussinesq Operators

For general N, the moduli space of tangential covers that have a function with Nth order pole at p and are regular elsewhere correspond to the following problem.

PROBLEM. Classify the algebraic differential operators

$$L = \partial^N + u_{N-2}(x)\partial^{N-2} + \cdots + u_0(x),$$

whose coefficients are elliptic functions defined on a given curve X, which have regular singular points ([8] II; [1] 6.10) and which commute with a differential operator B of order relatively prime with N (we refer to this last property by saying that L is algebraic).

Verdier's plan was the following. Given L as above, algebraic over X with regular singular points, the kernel of $L - \lambda$ gives a rank N local system on X for any λ ([8], I.4.5), hence we have a morphism ϕ: $\mathbf{A}^1 \to$ (moduli space of rank N local systems over X). The goal would be that of splitting the local system into N subsystems of rank 1 because the space $\mathcal{D}_1^{(N)}$ is geometrically easier to study, where $\mathcal{D}_1$ is the moduli space of local systems of rank 1 over X, and embed Γ into a compactification of $\mathcal{D}_1^{(N)}$, where

$$\begin{array}{lcc} & \Gamma & \longrightarrow \mathcal{D}_1^{(N)} \\ N:1 & \downarrow & \nearrow \\ & \mathbf{P}^1 & \end{array}$$

and the map ϕ can be continued across ∞ if and only if L is algebraic. The details haven't been worked out.

Remarks. (1) The condition is necessary: if L is algebraic, then there exists a differential operator B that commutes with L, has order prime with N and is also defined over X; the common solutions of $L - \lambda$ and $B - \mu$, give the required rank 1 local systems over X, for $\mu_1, \ldots, \mu_N$ eigenvalues for the action of B on the kernel of $L - \lambda$. By a result of Novikov [9], the sufficiency is ensured if the Floquet solution of L, defined for generic L, is defined on an algebraic curve: in which case the monodromy curve of L with respect to Λ coincides with the algebraic curve of the pair L, B.

(2) The analogous situation holds for the 'usual' algebro-geometric KP solutions, as follows: an operator

$$L = \partial^N + u_{N-2}(x)\partial^{N-2} + \cdots + u_0(x)$$

with meromorphic coefficients and regular singular points, corresponds to an algebro-geometric KP solution if and only if the N formal series

$$\psi(0,z),\ \partial\psi(0,z),\ \ldots,\ \partial^{N-1}\psi(0,z),$$

converge for large z, where ψ is a (normalized) solution of $L - z^N$ ([1], 5.22) which amounts to saying that the map from $z^N \in \mathbf{A}^1$ to the local system of solutions can be extended to $\mathbf{P}^1$.

(3) In the Nth KdV TC situation

$$\begin{array}{lcc} & \Gamma \xrightarrow{\pi} X \\ N:1 & \downarrow \ \nearrow \\ & \mathbf{P}^1 \end{array}$$

the image under π_* of the linear series $\lambda^*(\infty)$ is constant.

Proof. Taking the sum in the group X of the points in the π-image of any divisor in $|\lambda^*(\infty)|$ gives a map $\mathbf{P}^1 \to X$ which must be constant.

3. Examples

To conclude, we look at the moduli spaces of BTC from a different angle, namely the Calogero–Moser–Krichever system (CMK), whose definition is recalled in [2]. In [2], an open dense set of the phase space of this integrable system is identified with $T\mathrm{Sym}^n X \setminus \Delta$, the tangent space to the symmetric product of X minus all diagonals. As recalled in [2], Section 7, if we write a point in the phase space as (x_i, p_i) and denote by y, t the motions under two specific CMK Hamiltonians, then

$$u(x,y,t) = 2\sum_1^n \wp(x - x_i(y,t))$$

is an elliptic soliton if and only if for all $i = 1, \ldots, n$

$$\frac{\partial^2 x_i}{\partial y^2} = 4\sum_{k \neq i} \wp'(x_i - x_k)$$

and

$$\frac{\partial x_i}{\partial t} = \frac{3}{4}\left\{\left(\frac{\partial x_i}{\partial y}\right)^2 - 3\sum_{k\neq i}\wp(x_i - x_k)\right\}. \tag{6}$$

Case $n = 2$. As a consequence of (6), we can determine the KdV/Boussinesq solutions. The tangential polynomial ([10] and [3])

$$F(\mu,\alpha) = 4\mu^2 + 2\mu(p_1 + p_2) + p_1p_2 + 4\wp(x_1 - x_2) - 4\wp(\alpha) \tag{7}$$

is an invariant of the motion and the evolution takes place on the Jacobian of the corresponding curve Γ.

3.1. PROPOSITION. (i) *The set of KdV solutions in the CMK phase space consists of curves isogenous to* X, *where the generalized Ince potentials of* [11] *evolve linearly.*

(ii) *The set of Boussinesq solutions is a two-dimensional subvariety, a Jacobian of genus* 2 *where the flows evolve. One initial condition gives rise to a Boussinesq operator* $L = \partial^3 - 3\times 2\wp(x)\partial - 3\wp'(x) + c$ *where* c *does not depend on* x.

Proof. (i) By (6)

$$p_i = \frac{\partial x_i}{\partial y_i} = 0 \quad \text{and} \quad \wp'(x_1 - x_2) = 0;$$

thus $x_1 - x_2$ is a point of order 2 and the curve (7) $\mu^2 + \wp(x_1 - x_2) - \wp(\alpha) = 0$ is a singular $2:1$ cover of X.

(ii) Choosing $x_1 - x_2$ arbitrary gives a curve (7) which has genus 2 by the Hurwitz formula (covers X with 2 branch points), since the Boussinesq condition (6) implies

$$\frac{\partial}{\partial t} \equiv 0, \qquad p_2 = -p_1, \qquad p_1^2 = 3\wp(x_1 - x_2).$$

A Boussinesq operator $L = \partial^3 + u(x,y)\partial + v(x,y)$ is such that $v_x = \frac{1}{2}(u_y + u_{xx})$ (a straightforward consequence of the KP equation) and for a Boussinesq solution

$$u(x,y) = -3\sum_{i=1}^{n}\wp(x - x_i(y))$$

this implies

$$v_x = \frac{3}{2}\sum_{i=1}^{n}\left(\frac{\partial}{\partial y}x_i\right)\wp'(x - x_i(y)) - \frac{3}{2}\sum_{i=1}^{n}\wp''(x - x_i(y)).$$

For the CMK system, $\mathrm{d}x_i/\mathrm{d}y = p_i$ are independent of x, so

$$v = \frac{3}{2}\sum_{i=1}^{n}(p_i\wp(x - x_i) - \wp'(x - x_i)) + f(y),$$

as claimed.

It should be possible to extend Ince's result (cf. [4]) and prove that the Boussinesq operators $L = \partial^3 - 3n\wp(x)\partial - 3n\wp'(x)$ are algebraic for every positive integer n. As for the generalized Ince potentials of [11], their Boussinesq counterpart will probably involve the trigonal analog of triangular numbers, cf. [12], and again signify vanishing properties of Jacobian theta functions.

Research partly supported by NSF Grant DMS-8802712.

References

1. Segal, G. and Wilson, G.: *Publ. Math. Inst. Hautes Études Sci.* **61** (1985), 5.
2. Treibich, A. and Verdier, J.-L.: Variétés de Krichever des solitons elliptiques de KP, *Proc. Indo-French Conference on Geometry*, Hindustan Book Agency, Delhi, 1993.
3. Treibich, A.: *Duke Math. J.* **59** (1989), 611.
4. Treibich, A. and Verdier, J.-L.: Solitons elliptiques, The Grothendieck Festschrift III, Birkhäuser, Boston, 1990, 437.
5. Verdier, J.-L.: *Algebraic Analysis* **2** (1988), 901.
6. Previato, E. and Verdier, J.-L.: Boussinesq elliptic solitons: the cyclic case, *Proc. Indo-French Conference on Geometry*, Hindustan Book Agency, Delhi, 1993, 173.
7. Previato, E.: The Calogero–Moser–Krichever system and elliptic Boussinesq solutions, *Proc. CRM Workshop*, Les publications CRM, Monréal, 1990, 57.
8. Deligne, P.: *Equations différentielles à points singuliers réguliers*, Lecture Notes in Math. 163, Springer, Berlin, Heidelberg, 1970.
9. Novikov, S. P.: *Soviet Math. Dokl.* **25** (1982), 535.
10. Krichever, I. M.: *Functional Anal. Appl.* **14** (1980), 282.
11. Treibich, A. and Verdier, J.-L.: Revêtements tangentiels et sommes de 4 nombres triangulaires, *C.R. Acad. Sci. Paris, Série I* **311** (1990).
12. Adler, M. and van Moerbeke, P.: The trigonal case outside the big cell, private communication.