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Wärme- und Stoffübertragung

An analysis on forced convection in a channel filled with a Brinkman-Darcy porous medium: Exact and approximate solutions

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Abstract. An analysis was made to investigate non-Darcian fully developed flow and heat transfer in a porous channel bounded by two parallel walls subjected to uniform heat flux. The Brinkman-extended Darcy model was employed to study the effect of the boundary viscous frictional drag on hydrodynamic and heat transfer characteristics. An exact expression has been derived for the Nusselt number under the uniform wall heat flux condition. Approximate results were also obtained by exploiting a momentum integral relation and an auxiliary relation implicit in the Brinkman-extended Darcy model. Excellent agreement was confirmed between the approximate and exact solutions even in details of velocity and temperature profiles.

Analyse von Zwangskonvektion in einem mit porösem Brinkman-Darcy-Medium gefüllten Kanal

Zusammenfassung. Diese Studie untersucht voll entwickelte „Non-Darcy"-Strömung und Wärmetransport in einem porösen Kanal, der durch zwei parallele Wände gebildet und einem einheitlichen Wärmestrom ausgesetzt wird. Das erweiterte „Brinkman-Darcy-Modell" wurde angewandt, um den Effekt des viskosen Reibungswiderstands der Wände auf die hydrodynamischen Charakteristiken und die Wärmeübertragung zu untersuchen. Ein exakter Ausdruck wurde für die Nusselt-Zahl unter einheitlichem Wandwärmestrom abgeleitet. Annähernde Ergebnisse wurden durch die Auswertung einer Impuls-Integral-Beziehung und einer im erweiterten „Brinkman-Darcy-Modell" implizierten Hilfsfunktion erhalten. Ausgezeichnete Übereinstimmung zwischen den genäherten und den exakten Lösungen konnten sogar in den Geschwindigkeits- und Temperaturprofilen festgestellt werden.

Nomenclature

C_f	drag coefficient
f	velocity profile function
h	half channel height
K	permeability
Nu	Nusselt number
p	pressure
Re	Reynolds number
q_w	wall heat flux
T	temperature
u	flow velocity
x	coordinate in the flow direction
y	coordinate normal to the channel wall
α	porous media shape parameter
ε	porosity
η	dimensionless coordinate, y/h
θ	temperature profile function
μ	fluid viscosity
ζ	velocity profile shape factor
ϱ	fluid density

Subscripts

B	bulk mean
c	channel center-line
w	wall

1 Introduction

In view of possible applications in geophysical and energy related engineering problems, fluid flow and heat transfer within porous media has recently attracted considerable attention [1]. Most analytical studies were carried out using the well-known "Darcy flow model", for the model leads to considerable simplification of mathematical treatments. The Darcy flow model, however, does not account for the no-slip condition which must obviously be satisfied at all impermeable solid boundaries. Naturally, such a boundary viscous effect becomes more significant for materials with higher porosities such as fibrous media and foam materials.

Brinkman [2] extended the Darcy model by adding the viscous shear stress term to the Darcy term. This Brinkman-extended Darcy model was successfully employed by Chan et al. [3], Tong and Subramanian [4], Lauriat and Prasad [5], Sen [6] and Vasseur and Robillard [7] for the problems associated with free convection within fluid saturated porous media. Forced convection in Brinkman-Darcy porous media, so far, has received little attention, in spite of its importance in possible applications in heat transfer enhancement [8]. Kaviany [9] attacked what appears to be one of the most fundamental and important forced convection problems, namely, the forced convection through a porous channel bounded by isothermal parallel plates. He, however, did not treat the case in which the walls are subjected to constant heat flux.

In the first half of this paper, we shall extend Kaviany's analysis to the case of constant heat flux, and derive a closedform analytical expression for the Nusselt number. Then, in the remaining part of this study, we shall treat fully developed channel flow by following the approximate solution procedure, previously proposed for the Darcian fluid free convection [10] and examine the approximate results against the exact solution.

2 Governing equations and boundary conditions

Figure 1 shows a two-dimensional porous channel and its coordinates (x, y). We shall assume, the fluid and solid matrix in the channel are in thermal equilibrium, and treat them as a continuum. The governing equations for the fully developed flow, namely, the Brinkman-extended Darcy model and energy equation, are given by

$$\frac{\mu}{\varepsilon}\frac{d^2u}{dy^2}-\frac{\mu}{K}u=\frac{dP}{dx}, \tag{1}$$

$$\varrho\, Cp\, u\frac{\partial T}{\partial x}=k\frac{\partial^2 T}{\partial y^2}. \tag{2}$$

Since the velocity and temperature fields are symmetric about the channel center line, only the upper half of the channel will be taken into consideration. The appropriate boundary conditions for Eqs. (1) and (2) are:

$$y=0:\quad \frac{du}{dy}=0,\quad \frac{\partial T}{\partial y}=0 \tag{3a, b}$$

$$y=h:\quad u=0,\quad \frac{\partial T}{\partial y}=\frac{q_w}{k} \tag{3c, d}$$

where u is the Darcian (apparent) velocity in the x-direction, while T is the local temperature. Both upper and lower impermeable walls are subjected to constant heat flux q_w. ε and K are the porosity and permeability of the porous medium, respectively; ϱ, μ and Cp are the density, viscosity, and heat capacity of the fluid, respectively; p, the local pressure; k, the equivalent thermal conductivity of the fluid-saturated porous medium.

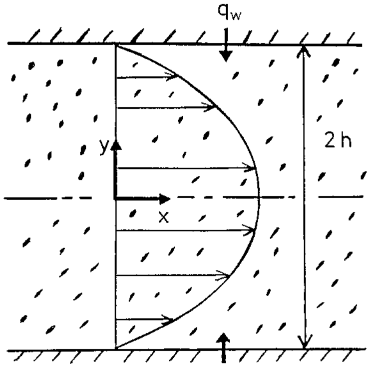

Fig. 1. A channel and coordinate system

The first left hand-side term of Eq. (1) (Brinkman term) denotes the viscous effect (which is significant near the wall) while the second term (Darcy term) expresses the effect of the bulk friction due to the presence of solid matrix.

3 Exact solution

Kaviany [9] integrated the momentum Eq. (1) using the boundary conditions given by Eqs. (3a) and (3c), and obtained the following exact expression for the axial velocity u normalized by its bulk mean u_B:

$$\frac{u}{u_B}=\frac{\alpha(\cosh\alpha-\cosh\alpha\,\eta)}{\alpha\cosh\alpha-\sinh\alpha}, \tag{4}$$

where

$$\eta=y/h \tag{5a}$$

and

$$\alpha=h/(K/\varepsilon)^{1/2} \tag{5b}$$

is the dimensionless parameter defined as the ratio of the channel half height h to the boundary layer length scale $(K/\varepsilon)^{1/2}$. Moreover, the total drag coefficient C_f, as the sum of contributions from the boundary viscous drag (Brinkman term) C_{fB} and the bulk frictional drag (Darcy term) C_{fD}, can be given by

$$C_f\equiv 2h\left(-\frac{dp}{dx}\right)/\varrho(u_B/\varepsilon)^2=C_{fB}+C_{fD}=\frac{8}{Re}\frac{\alpha^3\cosh\alpha}{\alpha\cosh\alpha-\sinh\alpha} \tag{6}$$

where

$$C_{fB}\equiv 2\frac{\mu}{\varepsilon}\frac{du}{dy}\bigg|_{y=0}/\varrho(u_B/\varepsilon)^2=\frac{8}{Re}\frac{\alpha^2\sinh\alpha}{\alpha\cosh\alpha-\sinh\alpha} \tag{7a}$$

and

$$C_{fD}\equiv 2\frac{\mu}{K}u_B h/\varrho(u_B/\varepsilon)^2=\frac{8\alpha^2}{Re}. \tag{7b}$$

Let us treat the energy Eq. (2), and derive an analytical expression. Upon integrating Eq. (2) once over the range, $0\leqq y\leqq h$, we have

$$\varrho\, Cp\, u_B\, h\frac{dT_B}{dx}=q_w \tag{8}$$

where T_B is the bulk mean temperature. The foregoing equation may be used to eliminate $\partial T/\partial x$ $(=dT_B/dx)$ from Eq. (2):

$$\frac{u}{u_B}=\frac{d^2\theta}{d\eta^2} \tag{9}$$

where

$$\theta=\frac{k(T-T_w)}{q_w h}$$

is the dimensionless temperature referenced to the local wall temperature T_w. The boundary conditions given by Eqs. (3 b) and (3 d) can be replaced in terms of the dimensionless variables as

$$\eta=0: \quad \frac{d\theta}{d\eta}=0 \tag{11 a}$$

$$\eta=1: \quad \theta=0. \tag{11 b}$$

Substituting the Kaviany's solution (Eq. (4)) into (9), we may integrate Eq. (9) with aid of (11 a) and (11 b), to obtain the dimensionless temperature distribution as follows:

$$\theta=\frac{1}{\alpha(\alpha\cosh\alpha-\sinh\alpha)} \times\left[\frac{\alpha^2}{2}(\eta^2-1)\cosh\alpha-\cosh(\alpha\eta)+\cosh\alpha\right]. \tag{12}$$

After some integration, we finally obtain the following analytical expression for the Nusselt number of our primary concern:

$$Nu=\frac{4hq_w}{k(T_B-T_w)}=4/\int_0^1\theta(u/u_B)\,d\eta =\frac{48\alpha(\alpha\cosh\alpha-\sinh\alpha)^2}{2\alpha(\alpha^2-6)\cosh 2\alpha+15\sinh 2\alpha+2\alpha(\alpha^2-9)}. \tag{13}$$

4 Approximate solution

Cheng [11] applied the classical integral method to the problem of the Darcian fluid free convection over flat surfaces. The authors subsequently improved their approximate method by introducing an auxiliary relation implicit in the energy equation in a differential form [10], and then extended the method to analyze various Darcian flow cases such as free convection over non-isothermal curved surfaces with and without thermal stratification [10, 12] and combined free and forced convection over non-isothermal curved surfaces [13].

In what follows, we shall propose an approximate solution procedure similar to the previous one successfully employed for the Darcian fluid flows, to analyze forced convection of the Brinkman-Darcy fluid within a porous channel.

The Brinkman-extended Darcy law (Eq. (1)) can be rewritten in a dimensionless form as

$$\frac{d^2f}{d\eta^2}-\alpha^2 f=\frac{\varepsilon h^2}{\mu u_c}\frac{dp}{dx} \tag{14}$$

where

$$f=u/u_c \tag{15}$$

and u_c is the velocity along the duct center line at $y=0$.

Let us integrate the foregoing equation over the upper half of the channel utilizing Eq. (3 a):

$$\left.\frac{df}{d\eta}\right|_{\eta=1}-\alpha^2\int_0^1 f\,d\eta=\frac{\varepsilon h^2}{\mu u_c}\frac{dp}{dx}. \tag{16}$$

Furthermore, we shall introduce the following auxiliary relation by writing Eq. (14) at the wall.

$$\left.\frac{d^2f}{d\eta^2}\right|_{\eta=1}=\frac{\varepsilon h^2}{\mu u_c}\frac{dp}{dx} \tag{17}$$

where the boundary condition given by Eq. (3 c) has been implemented. Upon combining Eqs. (16) and (17), we have

$$\left.\frac{df}{d\eta}\right|_{\eta=1}-\alpha^2\int_0^1 f\,d\eta=\left.\frac{d^2f}{d\eta^2}\right|_{\eta=1}. \tag{18}$$

The foregoing integral momentum equation may be used to estimate the boundary and bulk frictional drags. For that purpose, we shall introduce a one parameter family of velocity profiles as follows:

$$f=1-\eta^\zeta. \tag{19}$$

When $\zeta=2$, Eq. (19) reduces to the parabolic function for the Poiseuille flow. Since we expect ζ to be greater than 2, Eq. (19) automatically satisfies the boundary conditions given by Eqs. (3 a) and (3 c). Substituting Eq. (19) into (18), and carrying out differentiation and integration, we obtain a quadratic equation for the shape factor ζ, which may easily be solved for ζ as

$$\zeta=\frac{1}{2}(1+\sqrt{9+4\alpha^2}) \tag{20}$$

Upon noting

$$u_c/u_B=1/\int_0^1 f\,d\eta=(1+\zeta)/\zeta=\frac{3+\sqrt{9+4\alpha^2}}{1+\sqrt{9+4\alpha^2}} \tag{21}$$

the drag coefficients C_f, C_{fB} and C_{fD} can be evaluated from

$$C_f\equiv C_{fB}+C_{fD}=8(\zeta^2-1)/Re=4(3+2\alpha^2+\sqrt{9+4\alpha^2})/Re \tag{22}$$

where

$$C_{fB}=8(\zeta+1)/Re=4(3+\sqrt{9+4\alpha^2})/Re \tag{23 a}$$

and

$$C_{fD}=8\alpha^2/Re. \tag{23 b}$$

Furthermore, substitution of Eqs. (19) and (21) into the energy Eq. (9) gives

$$\frac{1+\zeta}{\zeta}(1-\eta^\zeta)=\frac{d^2\theta}{d\eta^2}. \tag{24}$$

The foregoing equation can be integrated twice using the boundary conditions, namely, Eqs. (11 a) and (11 b), which yields

$$\theta=\frac{1}{\zeta(\zeta+2)}(1-\eta^{\zeta+2})-\frac{1+\zeta}{2\zeta}(1-\eta^2). \tag{25}$$

Finally, we obtain an approximate formula for the Nusselt number integrating the product $f\theta$ over the channel, namely

$$Nu=4\Big/\left(\frac{1+\zeta}{\zeta}\right)\int_0^1 f\theta\,d\eta=\frac{12(\zeta+3)(2\zeta+3)}{2\zeta^2+13\zeta+17}. \tag{26}$$

5 Results and discussion

Figure 2 shows the effect of the parameter α on the center line velocity level. As α increases u_c/u_B asymptotically decreases to unity, indicating the establishment of a slug flow. The approximate solution (shown by a solid line) appears to be in good agreement with the exact solution (shown by a dashed line). Details of velocity profiles are shown in Fig. 3 for $\alpha=1$, 5 and 15. The velocity profiles generated by Eqs. (19) and (20), in fact, closely follow the profiles given by the exact solution, Eq. (4). The friction groupings, $C_f Re$, $C_{fB} Re$ and $C_{fD} Re$ are plotted in Fig. 4 over the range $0.1 \leqq \alpha \leqq 10^3$. For $\alpha<1$, the bulk frictional drag (Darcy's pressure drop) is almost negligible, and the flow behaves just like the Poiseuille flow. For $\alpha>10$, on the other hand, the bulk frictional drag predominates over the boundary viscous drag. As a result, the curve of the total drag ($C_f Re$) overlaps onto the line of the bulk frictional drag ($C_{fD} Re$) for large α. The difference between the approximate and exact solution is hardly discernible in the figure. In Fig. 5, the dimensionless temperature profiles given by Eqs. (25) and (20) are compared against those of the exact expression, Eq. (12), derived in this study. It is seen that the temperature field becomes more uniform as the velocity profile becomes flatter for large α. Again, the approximate solution appears to be in good accord with the exact solution. Finally, the Nusselt numbers for the case of constant heat flux are plotted in Fig. 6 following the resulting exact and approximate formulas, namely, Eqs. (13) and (26). (The results obtained by Kaviany [9] for the case of constant wall temperature are also presented for reference.) Reasonable agreement between the approximate and exact formulas can be confirmed from the figure. It can be shown that the two formulas possess the same asymptotic values, namely, $Nu=140/17$ for the Poiseuille flow (at $\alpha=0$) and $Nu=12$ for a Darcy flow (at $\alpha\to\infty$).

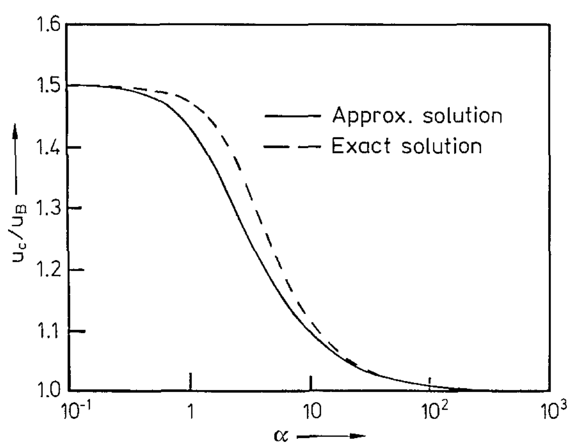

Fig. 2. Center-line velocity

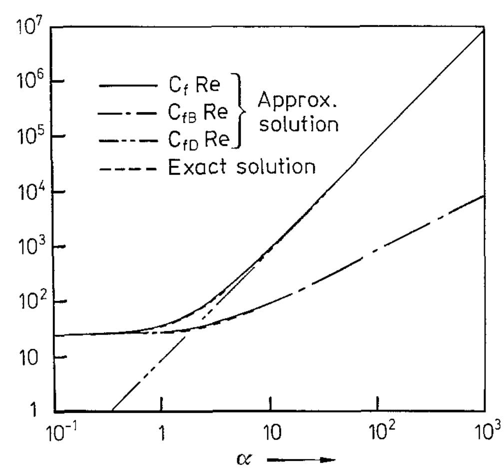

Fig. 4. Drag coefficients

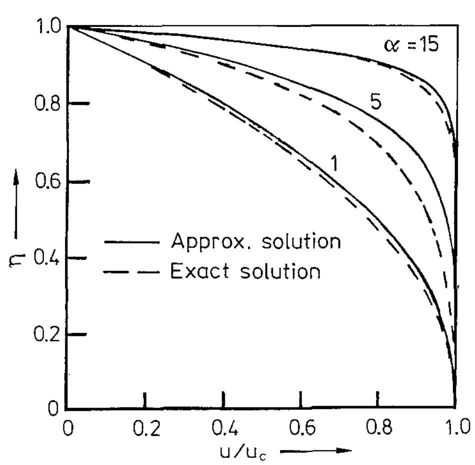

Fig. 3. Velocity profiles

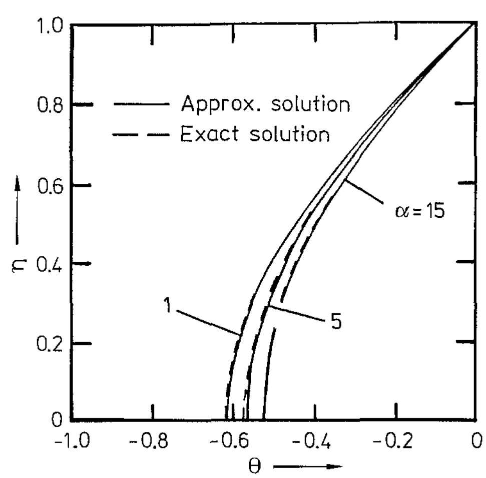

Fig. 5. Temperature profiles

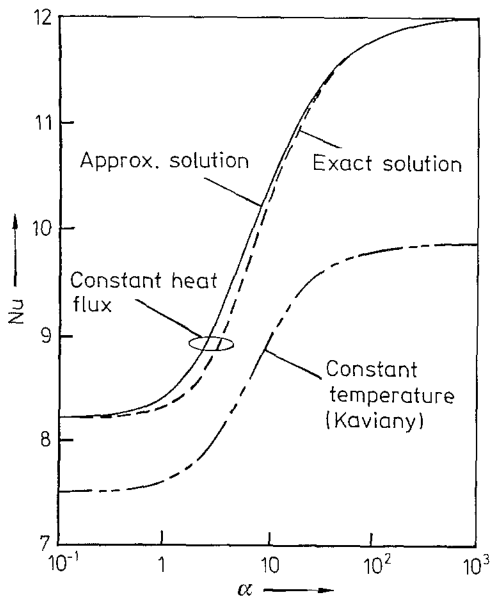

Fig. 6. Nusselt number

6 Concluding remarks

An analysis was made for forced convection in a two-dimensional porous channel subjected to uniform heat flux. The Brinkman-extended Darcy model was used to account for the boundary viscous effect on the hydrodynamic and heat transfer aspects. An approximate solution method which bases on the momentum integral relation was found quite accurate for evaluating the frictional drags and Nusselt numbers. Even the details of velocity and temperature profiles generated by the approximate formula agree quite well with those based on the exact solution.

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