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The Use of Factors to Discover Potential Systems or Linearizations

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Abstract. Factors of a given system of PDEs are solutions of an adjoint system of PDEs related to the system's Fréchet derivative. In this paper, we introduce the notion of potential conservation laws, arising from specific types of factors, which lead to useful potential systems. Point symmetries of a potential system could yield nonlocal symmetries of the given system and its linearization by a noninvertible mapping.

We also introduce the notion of linearizing factors to determine necessary conditions for the existence of a linearization of a given system of PDEs.

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1. Introduction

Consider a system of N partial differential equations (PDEs) $R\{u\}$ given by

$$G^\sigma\left(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}\right) = 0, \quad \sigma = 1, 2, \ldots, N, \tag{1.1}$$

with independent variables $x = (x_1, x_2, \ldots, x_n)$ and dependent variables $u = (u^1, u^2, \ldots, u^m)$; $\underset{j}{u}$ denotes the set of coordinates corresponding to all jth-order partial derivatives of u with respect to x (a coordinate in $\underset{j}{u}$ is denoted by

$$u^\gamma_{i_1 i_2 \ldots i_j} \equiv \frac{\partial^j u^\gamma}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_j}}$$

with $\gamma = 1, 2, \ldots, m$; $i_j = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k$).

DEFINITION 1.1. *A symmetry* of a given system of PDEs $R\{u\}$ is a transformation mapping any solution of $R\{u\}$ into another solution.

This definition is strictly topological and, in particular, coordinate-free. Consequently, $R\{u\}$ should admit a wide range of continuous symmetries: its family of solutions is expected to be invariant under a wide range of continuous deformations. For the rest of this paper we consider continuous symmetries characterized by infinitesimal generators whose forms allow such symmetries to be discovered and utilized algorithmically.

DEFINITION 1.2. A *(Lie) point symmetry* admitted by $R\{u\}$ is characterized by an infinitesimal generator of the form

$$\mathbf{X} = \sum_{\mu=1}^{m} \eta^{\mu}\left(x, u, \underset{1}{u}\right) \frac{\partial}{\partial u^{\mu}} \tag{1.2}$$

with η linear in the coordinates of $\underset{1}{u}$:

$$\eta^{\mu} = \alpha^{\mu}(x, u) - \sum_{i=1}^{n} \xi_i(x, u) u_i^{\mu}. \tag{1.3}$$

An infinitesimal generator (1.2) corresponds to a one-parameter Lie group of point transformations

$$\begin{aligned} x_i^* &= x_i + \varepsilon \xi_i(x, u) + \mathrm{O}(\varepsilon^2), \quad i = 1, 2, \ldots, n, \\ u^{\mu^*} &= u^{\mu} + \varepsilon \alpha^{\mu}(x, u) + \mathrm{O}(\varepsilon^2), \quad \mu = 1, 2, \ldots, m. \end{aligned} \tag{1.4}$$

Under the action of (1.2), a solution $u = \theta(x)$ of $R\{u\}$ is mapped into the one-parameter family of solutions

$$u = \Phi(x; \varepsilon) = \mathrm{e}^{\varepsilon \mathrm{U}} u|_{u=\theta(x)}, \tag{1.5}$$

where U is the prolongation operator given by

$$\mathrm{U} = \mathbf{X} + (D_i \eta^{\mu}) \frac{\partial}{\partial u_i^{\mu}} + \cdots + (D_{i_1} D_{i_2} \cdots D_{i_j} \eta^{\mu}) \frac{\partial}{\partial u^{\mu}_{i_1 i_2 \cdots i_j}} + \cdots$$

in terms of total differential operators

$$D_i = \frac{\partial}{\partial x_i} + u_i^{\gamma} \frac{\partial}{\partial u^{\gamma}} + \cdots + u^{\gamma}_{i i_1 i_2 \cdots i_l} \frac{\partial}{\partial u^{\gamma}_{i_1 i_2 \cdots i_l}} + \cdots, \quad i = 1, 2, \ldots, n.$$

(Summation over a repeated index is assumed throughout this paper.)

Lie [1–7] gave an algorithm to find the infinitesimal generators of a given system $R\{u\}$: If $u^* = u + \varepsilon \eta(x, u, \underset{1}{u}) + \mathrm{O}(\varepsilon^2)$, then

$$\begin{aligned} &G^{\sigma}\left(x, u^*, \underset{1}{u^*}, \underset{2}{u^*}, \ldots, \underset{k}{u^*}\right) \\ &\quad = G^{\sigma}\left(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}\right) + \varepsilon \sum_{\rho=1}^{m} \mathcal{L}_{\rho}^{\sigma}[u] \eta^{\rho} + \mathrm{O}(\varepsilon^2), \end{aligned}$$

where $\mathcal{L}[u]$ is the Fréchet derivative of $R\{u\}$. One can prove the following theorem:

THEOREM 1.3. **X** *is admitted by* $R\{u\}$, *including its differential consequences, if and only if the equations*

$$\sum_{\rho=1}^{m} \mathcal{L}_\rho^\sigma[u]\eta^\rho = 0, \quad \sigma = 1, 2, \ldots, N, \tag{1.6}$$

are satisfied for any solution $u = \theta(x)$ *of* $R\{u\}$.

The determining Equations (1.6) form an overdetermined linear system of PDEs with $n + m$ unknowns $\alpha^1, \alpha^2, \ldots, \alpha^m$; $\xi_1, \xi_2, \ldots, \xi_n$. There exist various symbolic manipulation programs [8–14] which perform one or more of the following functions automatically and/or interactively: set up determining equations, find the dimension (if finite) of their solution space, and solve them explicitly.

For a given system $R\{u\}$, point symmetries can yield various applications including the discovery of new solutions from known solutions (Equation (1.5)), the construction of specific invariant solutions [1–7], and the generation of conservation laws through Noether's theorem [4–7]. In addition, one can determine algorithmically whether or not $R\{u\}$ can be linearized by an *invertible* point transformation and construct an explicit linearization when one exists [15, 16, 6].

DEFINITION 1.4. A *local symmetry* admitted by $R\{u\}$ is characterized by an infinitesimal generator of the form

$$\mathbf{X} = \sum_{\mu=1}^{m} \eta^\mu\Big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{p}{u}\Big)\frac{\partial}{\partial u^\mu}. \tag{1.7}$$

DEFINITION 1.5. A local symmetry of the form (1.7) is a *contact symmetry* when $m = p = 1$; a *Lie–Bäcklund* (*higher, higher order, generalized*) *symmetry* [5, 6], when it is not a point or contact symmetry.

The algorithm for determining local symmetries of a given $R\{u\}$ involves solving the corresponding determining Equations (1.6).

DEFINITION 1.6. A *nonlocal symmetry* of $R\{u\}$ is a continuous symmetry admitted by $R\{u\}$ which is not characterized by an infinitesimal generator of local type (1.7).

In order to have algorithms to compute or utilize nonlocal symmetries most effectively, one should be able to characterize them in terms of infinitesimal generators of point symmetries in some coordinate frame. There exists an algorithm to find a class of such nonlocal symmetries (*potential symmetries*) provided system $R\{u\}$ contains at least one PDE expressed as a conservation law. This allows one to introduce potential variables v and a related potential system $S\{u, v\}$ [6, 17, 18]. Consequently, one can extend the known applications and calculations of point symmetries to potential symmetries. Moreover, one can discover algorithmically linearizations by *noninvertible* mappings [6, 19].

In this paper, we develop the notions of potential conservation laws which are useful for discovering potential systems, and linearizing factors which identify linearizable systems. We clarify and extend results presented in recent papers [20–22]. A complete potential symmetry analysis is given for the nonlinear diffusion equation.

2. Potential Symmetries

Suppose one PDE of $R\{u\}$, without loss of generality $G^N = 0$, is a conservation law

$$\sum_{i=1}^{n} D_i f^i\Big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k-1}{u}\Big) = 0.$$

Then $R\{u\}$ is the system given by

$$G^\sigma\Big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}\Big) = 0, \quad \sigma = 1, 2, \ldots, N-1, \tag{2.1}$$

$$\sum_{i=1}^{n} D_i f^i\Big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k-1}{u}\Big) = 0. \tag{2.2}$$

If $n = 2$, let $x_1 = x$, $x_2 = t$. Through (2.2), one can introduce auxiliary potential variables v and form potential system $S\{u, v\}$ given by

$$f^1 = \frac{\partial v}{\partial t}, \qquad f^2 = -\frac{\partial v}{\partial x}. \tag{2.3}$$

If $n \geqslant 3$, then through (2.2), one can introduce n auxiliary potential variables $v = (v^1, v^2, \ldots, v^n)$ and up to $\frac{1}{2}n(n-1)(n-2)$ nontrivial constants $\{\alpha_{ijk}\}$ and form a gauge-dependent auxiliary system (potential system) $S\{u, v\}$ of $N+n-1$ PDEs:

$$\begin{aligned} f^i\Big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k-1}{u}\Big) &= \sum_{j,k=1}^{n} \in_{ijk}\alpha_{ijk}\frac{\partial v^j}{\partial x_k}, \quad i = 1, 2, \ldots, n, \\ G^\sigma\Big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{k}{u}\Big) &= 0, \quad \sigma = 1, 2, \ldots, N-1, \end{aligned} \tag{2.4}$$

where $\in_{ijk}$ is the permutation symbol and $\{\alpha_{ijk}\}$ satisfies conditions

$$\alpha_{ijk} = \alpha_{kji}, \tag{2.5}$$

$$\sum_{j,k=1}^{n} |\in_{ijk}\alpha_{ijk}| \neq 0, \quad i = 1, 2, \ldots, n. \tag{2.6}$$

Note that conditions (2.5), (2.6) effectively mean that for particular choices of gauge, one need only have $n-1$ potential variables.

More generally, one can introduce $\frac{1}{2}n(n-1)$ potential variables $v = (\Psi^{12}, \Psi^{13}, \dots, \Psi^{1n}, \Psi^{23}, \dots, \Psi^{2n}, \dots, \Psi^{n-1,n})$, where Ψ^{ij} $(i<j)$ are components of an antisymmetric tensor, such that

$$f^i\Big(x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{k-1}{u}\Big) = \sum_{i<j}(-1)^j \frac{\partial \Psi^{ij}}{\partial x_j} + \sum_{j<i}(-1)^{i-1}\frac{\partial \Psi^{ji}}{\partial x_j}, \quad i,j = 1,2,\dots,n, \tag{2.7}$$

and form a corresponding potential system of $N+n-1$ PDEs with $m+\frac{1}{2}n(n-1)$ dependent variables $u = (u^1, u^2, \dots, u^m)$, Ψ^{ij} $(i<j)$. Since (2.7) is underdetermined, one can impose suitable constraints (a choice of gauge) on the potentials Ψ^{ij} to make system (2.7) a determined system [18].

If $(u(x), v(x))$ solves $S\{u,v\}$, then $u(x)$ solves $R\{u\}$; if $u(x)$ solves $R\{u\}$, then through integrability conditions (2.3), (2.4), or (2.7) there exists some (nonunique) $v(x)$ such that $(u(x), v(x))$ solves $S\{u,v\}$. Since $v(x)$ is not unique, it follows that an invertible point or contact transformation in (x,u)-space could yield a noninvertible nonlocal transformation in (x,u,v)-space and, vice-versa, an invertible point transformation in (x,u,v)-space could yield a noninvertible nonlocal transformation in (x,u)-space. Consequently, the study of potential system $S\{u,v\}$ through qualitative or quantitative methods which are not coordinate-dependent, may yield new results for $R\{u\}$ and vice-versa. In particular, a symmetry of $S\{u,v\}$ $(R\{u\})$ defines a symmetry of $R\{u\}$ $(S\{u,v\})$; a point symmetry of $S\{u,v\}$ $(R\{u\})$ could induce a nonlocal symmetry of $R\{u\}$ $(S\{u,v\})$.

DEFINITION 2.1. A *potential symmetry* of $R\{u\}$, related to potential system $S\{u,v\}$, is a point symmetry of $S\{u,v\}$ which does not project onto a point symmetry of $R\{u\}$.

The proof of the following theorem follows immediately:

THEOREM 2.2. *A potential symmetry of $R\{u\}$ is a nonlocal symmetry of $R\{u\}$. In particular, suppose*

$$\begin{aligned}\mathbf{X}^S &= [\alpha^\mu(x,u,v) - \xi_i^S(x,u,v)u_i^\mu]\frac{\partial}{\partial u^\mu} + \\ &\quad + [\beta^\nu(x,u,v) - \xi_i^S(x,u,v)v_i^\nu]\frac{\partial}{\partial v^\nu}\end{aligned}$$

is a point symmetry of $S\{u,v\}$. Then $\mathbf{X}^S$ induces a potential symmetry of $R\{u\}$ if and only if at least one component of (α, ξ^S) depend essentially on v; otherwise $\mathbf{X}^S$ projects onto a point symmetry of $R\{u\}$, namely

$$\mathbf{X} = [\alpha^\mu(x,u) - \xi_i^S(x,u)u_i^\mu]\frac{\partial}{\partial u^\mu}.$$

Conversely,

$$\mathbf{X}^R = \left[\alpha^\mu(x,u) - \xi_i^R(x,u)u_i^\mu\right]\frac{\partial}{\partial u^\mu}$$

yields a nonlocal symmetry of $S\{u,v\}$ *if and only if*

$$\mathbf{X}^S = \mathbf{X}^R + \zeta^\nu\left(x,u,\underset{1}{u},\underset{1}{v}\right)\frac{\partial}{\partial v^\nu}$$

is not a point symmetry of $S\{u,v\}$ *for any choice of* ζ.

Now let $v^{(1)} = v$; $S^{(1)} = S\{u, v^{(1)}\}$. Suppose a PDE of $S^{(1)}$ is a conservation law. Then one can introduce further potential variables (and gauge constants and/or constraints) $v^{(2)}$ and form potential system $S^{(2)} = S^{(2)}\{u, v^{(1)}, v^{(2)}\}$ of $N + 2(n-1)$ PDEs with $n^{(2)}$ dependent variables, $N + 2(n-1) \leqslant n^{(2)} \leqslant m + n(n-1)$. Point symmetries of $S^{(2)}$ could yield additional nonlocal symmetries of $R\{u\}$. Continuing this process with other conservation laws one could obtain potential variables $v^{(1)}, v^{(2)}, \ldots, v^{(J)}$ and corresponding potential systems $S^{(1)}, S^{(2)}, \ldots, S^{(J)}\{u, v^{(1)}, v^{(2)}, \ldots, v^{(J)}\}$. Potential system $S^{(J)}$ would involve $N + J(n-1)$ PDEs with $n^{(J)}$ dependent variables, $N + J(n-1) \leqslant n^{(J)} \leqslant m + \frac{1}{2}Jn(n-1)$. At any step $J \geqslant 2$, a point symmetry of $S^{(J)}$ could yield a potential symmetry of $S^{(J-1)}$ which is either a point symmetry or nonlocal symmetry of $R\{u\}$. If a potential symmetry of $S^{(J-1)}$ yields a point symmetry of $R\{u\}$, then a 'lost' point symmetry of $R\{u\}$ is 'recovered'. (A point symmetry of $R\{u\}$ is said to be 'lost' in $S^{(K)}$ if it does not induce a point symmetry of $S^{(K)}$.)

3. Conservation Laws, Potential Conservation Laws, and Potential Systems

Up to now, in order to obtain potential systems, we assumed that at least one PDE of a given system is a conservation law. The question of how to construct conservation laws yielding useful potential systems naturally arises. After defining the adjoint of an operator, we state some known theorems concerning the discovery of conservation laws.

DEFINITION 3.1. The *adjoint* of the differential operator $\mathcal{L}[u]$ is the differential operator $\mathcal{L}^*[u]$ which satisfies

$$\int_\Omega V^\sigma \mathcal{L}_\rho^\sigma[u] W^\rho \,\mathrm{d}x = \int_\Omega W^\rho \mathcal{L}_\sigma^{*\rho}[u] V^\sigma \,\mathrm{d}x \tag{3.1}$$

on any domain $\Omega \subset \mathbf{R}^n$, for every pair of k times differentiable functions

$$\begin{aligned} V(x) &= (V^1(x), V^2(x), \ldots, V^N(x)), \\ W(x) &= (W^1(x), W^2(x), \ldots, W^m(x)) \end{aligned}$$

with compact support in Ω.

In particular, $\mathcal{L}^*[u]$ is the adjoint of $\mathcal{L}[u]$ if $V^\sigma \mathcal{L}^\sigma_\rho[u] W^\rho - W^\rho \mathcal{L}^{*\rho}_\sigma[u] V^\sigma$ is a divergence expression.

THEOREM 3.2. *Suppose there exists a set of factors (multipliers, characteristics) $\{\lambda^\sigma(x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{p}{u})\}$ where the components of $u(x)$ are arbitrary p-times differentiable functions such that*

$$\sum_{\sigma=1}^{N} \lambda^\sigma G^\sigma = \sum_{i=1}^{n} D_i f^i \tag{3.2}$$

holds for some $\{f^i(x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{q}{u})\}$. Then

$$\sum_{\sigma=1}^{N} \mathcal{L}^{*\rho}_\sigma[u] \lambda^\sigma = 0, \quad \rho = 1, 2, \dots, m, \tag{3.3}$$

must hold for any solution of $R\{u\}$ and its differential consequences, where $\mathcal{L}^*[u]$ is the adjoint of the Fréchet derivative $\mathcal{L}[u]$ of $R\{u\}$ [5, 23].

THEOREM 3.3. *If the Fréchet derivative of $R\{u\}$ is selfadjoint, i.e. $\mathcal{L}[u] = \mathcal{L}^*[u]$, then system $R\{u\}$ is the set of Euler–Lagrange equations for some variational principle with Lagrangian L* [5, 23].

THEOREM 3.4 (Noether's Theorem [5, 6, 23–25]). *If $L(x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{l}{u})$ is a Lagrangian for $R\{u\}$ then $\{\lambda^\sigma\}$ yields a set of factors for a conservation law of $R\{u\}$ if both*

$$(1) \quad \mathbf{X} = \sum_{\mu=1}^{m} \lambda^\mu \left(x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{p}{u}\right) \frac{\partial}{\partial u^\mu}$$

is a local symmetry of $R\{u\}$;

$$(2) \quad \sum_{\sigma=1}^{m} \mathcal{M}_\sigma[u] \lambda^\sigma = \sum_{i=1}^{n} D_i A^i$$

for some $\{A^i(x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{q}{u})\}$, where $\mathcal{M}[u]$ is the Fréchet derivative of Lagrangian L.

If (1) and (2) hold then the resulting conservation law is $\sum_{i=1}^n D_i[W^i[u, \lambda] - A^i] = 0$, where

$$\begin{aligned} W^i[u, \lambda] &= \lambda^\gamma \left[\frac{\partial L}{\partial u^\gamma_i} + \cdots + (-1)^{l-1} D_{i_1} \cdots D_{i_{l-1}} \frac{\partial L}{\partial u_{i i_1 \cdots i_{l-1}}} \right] + \\ &\quad + (D_{i_1} \lambda^\gamma) \left[\frac{\partial L}{\partial u^\gamma_{i_1 i}} + \cdots + (-1)^{l-2} D_{i_2} \cdots D_{i_{l-1}} \frac{\partial L}{\partial u^\gamma_{i_1 i i_2 \cdots i_{l-1}}} \right] + \\ &\quad + \cdots + (D_{i_1} \cdots D_{i_{l-1}} \lambda^\gamma) \frac{\partial L}{\partial u^\gamma_{i_1 \cdots i_{l-1} i}}. \end{aligned}$$

Note that if $\mathcal{L}[u]$ is not selfadjoint, then only Theorem 3.2 holds. Unlike Theorem 3.4, it yields no explicit formula for a conservation law of $R\{u\}$.

In principle any conservation law of $R\{u\}$ leads to an associated potential system: Suppose a set of factors $\{\lambda^\sigma(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{p}{u})\}$ exists with $\lambda^M \neq 0$ so that (3.2) holds.

Consider the new system $\widehat{R}_M\{u\}$ given by

$$G^\sigma = 0, \quad \sigma = 1, 2, \ldots, M-1, M+1, \ldots, N,$$
$$\sum_{i=1}^{n} D_i f^i = 0. \tag{3.4}$$

It follows that each solution of $R\{u\}$ is a solution of $\widehat{R}_M\{u\}$. On the other hand, each solution of $\widehat{R}_M\{u\}$ is a solution of $R\{u\}$ or factor system $\widetilde{R}_M\{u\}$ given by

$$G^\sigma = 0, \quad \sigma = 1, 2, \ldots, M-1, M+1, \ldots, N,$$
$$\lambda^M\left(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{p}{u}\right) = 0. \tag{3.5}$$

Suppose there are solutions of $\widetilde{R}_M\{u\}$ which are not solutions $R\{u\}$. (This can only happen when $\lambda^M = 0$ has a solution $u(x)$.) Since the solution set of $\widehat{R}_M\{u\}$ is the union of the solution sets of $\widetilde{R}_M\{u\}$ and $R\{u\}$, one would expect $\widehat{R}_M\{u\}$ to lose symmetries of $R\{u\}$. This leads to the consideration of only certain types of factors in order to discover useful potential systems:

DEFINITION 3.5. A *potential factor* is a factor which does not vanish for any $u(x)$, i.e.

$$\lambda^M\left(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{p}{u}\right) = 0 \tag{3.6}$$

has no solutions $u(x)$.

DEFINITION 3.6. A *potential conservation law* of $R\{u\}$ is a conservation law of $R\{u\}$ arising from a set of factors with at least one potential factor.

DEFINITION 3.7. Let $\widehat{R}_M\{u\}$ be a system (3.4) associated with a potential conservation law of $R\{u\}$ with potential factor λ^M. A corresponding potential system (see (2.3), (2.4), or (2.7)) $\widehat{S}_M\{u, v\}$ is a *useful potential system.*

If a potential system arising from $R\{u\}$ is not a useful potential system, then one would expect it to yield no potential symmetries of $R\{u\}$. For example, let $R\{u\}$ be the linear wave equation

$$u_{xx} - x^{-4} u_{tt} = 0. \tag{3.7}$$

The factor $\lambda = 2u_t$ yields

$$2u_t(u_{xx} - x^{-4}u_{tt}) = D_x(2u_x u_t) - D_t((u_x)^2 + x^{-4}(u_t)^2)$$

and, hence, one obtains conservation law and system $\widehat{R}^1\{u\}$ given by

$$D_x(2u_x u_t) - D_t((u_x)^2 + x^{-4}(u_t)^2) = 0. \tag{3.8}$$

Correspondingly, one has the factor system $\widetilde{R}^1\{u\}$ given by

$$u_t = 0 \tag{3.9}$$

and potential system $\widehat{S}^1\{u, v\}$ given by

$$v_t = 2u_x u_t, \qquad v_x = x^{-4}(u_t)^2 + (u_x)^2. \tag{3.10}$$

On the other hand, factor $\lambda = 1$ yields conservation law

$$D_x(u_x) - D_t(x^{-4}u_t) = 0 \tag{3.11}$$

with corresponding factor system $\widetilde{R}^2\{u\}$ given by the equation

$$1 = 0 \tag{3.12}$$

and potential system $\widehat{S}^2\{u, v\}$ given by

$$v_t = u_x, \qquad v_x = x^{-4}u_t. \tag{3.13}$$

Conservation law (3.8) is not a potential conservation law; in particular $u_t = 0$ has solutions (thus $\lambda = u_t$ is not a potential factor) and almost all solutions of $u_t = 0$ do not solve Equation (3.7). On the other hand, conservation law (3.11) is a potential conservation law ($\widetilde{R}^2\{u\} \equiv R\{u\}$; there are no solutions $u(x)$ of Equation (3.12)).

It is interesting to compare the point symmetries of $R\{u\}$, $\widehat{S}^1\{u, v\}$, and $\widehat{S}^2\{u, v\}$ given by (3.7), (3.10), and (3.13), respectively:

(1) $R\{u\}$ admits an infinite-parameter group which leads to its mapping to the wave equation $u_{xt} = 0$ [17, 6].

(2) $\widehat{S}^1\{u, v\}$ admits

$$\mathbf{X}_1^{\widehat{S}^1} = u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v},$$

$$\mathbf{X}_2^{\widehat{S}^1} = [u + 2(tu_t - xu_x)]\frac{\partial}{\partial u} + 2(tv_t - xv_x)\frac{\partial}{\partial v},$$

$$\mathbf{X}_3^{\widehat{S}^1} = (xu - x^2u_x)\frac{\partial}{\partial u} + (u^2 - x^2v_x)\frac{\partial}{\partial v},$$

$$\mathbf{X}_4^{\widehat{S}^1} = u\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial v},$$

$$\mathbf{X}_5^{\widehat{S}^1} = x\frac{\partial}{\partial u} + 2u\frac{\partial}{\partial v},$$

$$\mathbf{X}_6^{\widehat{S}^1} = \frac{\partial}{\partial u},$$

$$\mathbf{X}_7^{\widehat{S}^1} = \frac{\partial}{\partial v}. \tag{3.14}$$

($\mathbf{X}_4^{\widehat{S}^1}, \mathbf{X}_5^{\widehat{S}^1}, \mathbf{X}_6^{\widehat{S}^1}$, and $\mathbf{X}_7^{\widehat{S}^1}$ are trivial since independent variables are invariant.)

Note that each infinitesimal generator of $\widehat{S}^1\{u,v\}$ is admitted by $\widetilde{R}\{u\}$ ($u_t = 0$) as well as $R\{u\}$. From the form of (3.14) we see that $\widehat{S}^1\{u,v\}$ yields no potential symmetries of $R\{u\}$, as to be expected.

(3) $\widehat{S}^2\{u,v\}$ admits a four-parameter group [17, 6] given by infinitesimal generators

$$\begin{aligned}
\mathbf{X}_1^{\widehat{S}^2} &= u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v}, \qquad \mathbf{X}_2^{\widehat{S}^2} = (tu_t - xu_x)\frac{\partial}{\partial u} + (tv_t - xv_x - 2v)\frac{\partial}{\partial v},\\
\mathbf{X}_3^{\widehat{S}^2} &= [3tu - xv + (t^2 + x^{-2})u_t - 2xtu_x]\frac{\partial}{\partial u} + \\
&\quad + [(t^2 + x^{-2})v_t - 2xtv_x - (tv + x^{-1}u)]\frac{\partial}{\partial v},\\
\mathbf{X}_4^{\widehat{S}^2} &= u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}.
\end{aligned}$$

Infinitesimal generator $\mathbf{X}_3^{\widehat{S}^2}$ yields a potential symmetry of $R\{u\}$.

Factor $\lambda = 2u_t$ yields a conservation law and corresponding potential system $\widehat{S}\{u,v\}$ where $R\{u\}$ is the wave equation

$$u_{xx} - \frac{1}{c^2(x)}u_{tt} = 0.$$

For any wave speed $c(x)$ one can show that this potential system has no potential symmetries [26] and that the point symmetries of $\widehat{S}\{u,v\}$ project onto point symmetries of both $R\{u\}$ and $u_t = 0$. This is to be expected, since $\lambda = 2u_t$ is not a potential factor of $R\{u\}$.

For physical equations one can have factors depending on u yielding potential conservation laws. For example, consider the equation of one-dimensional planar gas dynamics $R\{u\}$ given by ($u = (v, p, \rho)$):

$$G^1 = \rho_t + v\rho_x + \rho v_x = 0, \tag{3.15a}$$

$$G^2 = \rho(v_t + vv_x) + p_x = 0, \tag{3.15b}$$

$$G^3 = \rho(p_t + vp_x) + B(p, \rho)v_x = 0, \tag{3.15c}$$

where $B(p, \rho)$ satisfies some constitutive relation. Let $\lambda = (\lambda^1, \lambda^2, \lambda^3) = (v, 1, 0)$. Then the resulting conservation law is a potential conservation law since $\lambda^2 = 1$ is a potential factor. It yields $\widehat{R}_2\{u\}$ given by the system of PDEs (3.15a, c) and

$$vG^1 + G^2 = D_t(\rho v) + D_x(p + \rho v^2) = 0. \tag{3.16}$$

4. Complete Potential Symmetry Analysis of the Nonlinear Diffusion Equation

As a prototypical example we consider the nonlinear diffusion equation $R\{u\}$ given by

$$u_t = \left(K(u)u_x\right)_x. \tag{4.1}$$

4.1. Potential Systems of $R\{u\}$

First we determine all potential systems of (4.1) arising from potential conservation laws for any diffusivity $K(u)$. The Fréchet derivative of (4.1) is given by

$$\begin{aligned}\mathcal{L}[u] &= K(u)D_x^2 + 2K'(u)u_x D_x - D_t + \left[K''(u)u_x^2 + K'(u)u_{xx}\right] \\ &= D_x^2 \cdot K(u) - D_t.\end{aligned} \tag{4.2}$$

The adjoint of (4.2) is $\mathcal{L}^*[u] = K(u)D_x^2 + D_t \neq \mathcal{L}[u]$. From Theorem 3.2, if a factor $\lambda(x, t, u, u_x, u_t)$ yields a conservation law of (4.1), then

$$\mathcal{L}^*[u]\lambda = K(u)D_x^2\lambda + D_t\lambda = 0 \tag{4.3}$$

must hold for any solution of (4.1), including its differential consequences. One can show that for *any* $K(u)$ the only solutions of (4.3) are $\lambda = c_1 x + c_2$ for arbitrary constants c_1, c_2.

Hence there are at most two potential systems arising from potential factors $\lambda = 1$, $\lambda = x$; factor $\lambda = 1$ obviously yields a potential conservation law from the form of (4.1); factor $\lambda = x$ yields the potential conservation law $x[u_t - (K(u)u_x)_x] = D_t[xu] - D_x[x(L(u))_x - L(u)] = 0$, where $K(u) = L'(u)$. Consequently, we obtain useful potential systems

$$S^1\{u, v\}: \begin{cases} v_x = u, \\ v_t = \left(L(u)\right)_x, \end{cases} \tag{4.4}$$

and

$$S^2\{u, V\}: \begin{cases} V_x = xu, \\ V_t = x\left(L(u)\right)_x - L(u). \end{cases} \tag{4.5}$$

4.1.1. *Potential Systems of* $S^1\{u,v\}$

The Fréchet derivative of $S^1\{u,v\}$ is the operator

$$\mathcal{L}[u,v] = \begin{bmatrix} 1 & -D_x \\ K'(u)u_x + K(u)D_x & -D_t \end{bmatrix}$$

with adjoint given by

$$\mathcal{L}^*[u,v] = \begin{bmatrix} 1 & -K(u)D_x \\ D_x & D_t \end{bmatrix}.$$

Then two cases arise out when solving

$$\mathcal{L}^*[u,v] \begin{bmatrix} \lambda^1(x,t,u,v) \\ \lambda^2(x,t,u,v) \end{bmatrix} = 0 \tag{4.6}$$

where $(u(x,t),\ v(x,t))$ is any solution of $S^1\{u,v\}$:

(1) $K(u)$ *arbitrary*: Here one can show that the only solution of (4.6) is $(\lambda^1,\lambda^2) = (0,1)$, leading to potential system

$$T^1\{u,v,w\}\colon \quad \begin{cases} v_x = u, \\ w_x = v, \\ w_t = L(u). \end{cases} \tag{4.7}$$

The Fréchet derivative of (4.7) is

$$\mathcal{L}[u,v,w] = \begin{bmatrix} 1 & -D_x & 0 \\ 0 & 1 & -D_x \\ K(u) & 0 & -D_t \end{bmatrix}$$

with adjoint

$$\mathcal{L}^*[u,v,w] = \begin{bmatrix} 1 & 0 & K(u) \\ D_x & 1 & 0 \\ 0 & D_x & D_t \end{bmatrix}.$$

Then one can show that the system

$$\mathcal{L}^*[u,v,w] \begin{bmatrix} \lambda^1(x,t,u,v,w) \\ \lambda^2(x,t,u,v,w) \\ \lambda^2(x,t,u,v,w) \end{bmatrix} = 0, \tag{4.8}$$

where $(u(x,t), v(x,t), w(x,t))$ is any solution of (4.7), only has the trivial solution $(\lambda^1,\lambda^2,\lambda^3) = (0,0,0)$.

(2) $K(u) = u^{-2}$: Here $(\lambda^1, \lambda^2) = (u^{-1}F^1(v,t), F^2(v,t))$, where $(F^1(v,t), F^2(v,t))$ are arbitrary functions satisfying the linear system

$$F^1 = \frac{\partial F^2}{\partial v}, \qquad \frac{\partial F^1}{\partial v} = -\frac{\partial F^2}{\partial t}. \tag{4.9}$$

In Section 5, we show how these factors indicate the linearization of the system

$$v_x = u, \qquad v_t = u^{-2}u_x. \tag{4.10}$$

4.1.2. *Potential Systems of* $S^2\{u, V\}$

The Fréchet derivative of $S^2\{u, V\}$ is the operator

$$\mathcal{L}[u, V] = \begin{bmatrix} x & -D_x \\ x(K'(u)u_x + K(u)D_x) - K(u) & -D_t \end{bmatrix}$$

with adjoint given by

$$\mathcal{L}^*[u, V] = \begin{bmatrix} x & -K(u)(2 + xD_x) \\ D_x & D_t \end{bmatrix}.$$

Then two cases arise when seeking factors $(\lambda^1(x,t,u,V), \lambda^2(x,t,u,V))$ satisfying the corresponding adjoint Equations (3.3):

(1) $K(u)$ *arbitrary*: Here the only solution of (3.3) is $(\lambda^1, \lambda^2) = (0, x^{-2})$. These factors yield the potential conservation law ($x^{-2} \neq 0$ if $x \in \mathbf{R}$)

$$x^{-2}[V_t - x(L(u))_x + L(u)] = D_t[x^{-2}V] - D_x[x^{-1}L(u)] = 0,$$

leading to potential system

$$T^2\{u, V, W\}\colon \begin{cases} W_x = x^{-2}V, \\ W_t = x^{-1}L(u), \\ V_x = xu. \end{cases}$$

It is unnecessary to seek factors for $T^2\{u, V, W\}$, since one can show that $T^1\{u, v, w\}$ and $T^2\{u, V, W\}$ are equivalent through the mapping

$$v = x^{-1}V + W, \qquad w = xW. \tag{4.11}$$

However, $S^1\{u, v\}$ and $S^2\{u, V\}$ are not invertibly equivalent since, as will be seen in Section 4.2, for any $K(u)$ these systems admit point symmetry Lie algebras of different dimension.

(2) $K(u) = u^{-2}$: Here

$$(\lambda^1, \lambda^2) = \left(\frac{1}{xu}, \frac{V}{x^2}\right)$$

is the only other solution of Equations (3.3). Obviously λ^1 is a potential factor. These factors yield the potential conservation law

$$\frac{1}{xu}(V_x - xu) + \frac{V}{x^2}(V_t - u^{-1} - xu^{-2}u_x) = D_x\left[\frac{V}{xu} - x\right] + D_t\left[\frac{V^2}{2x^2}\right] = 0,$$

which, in turn, yields systems

$$\widehat{S}_1^2\colon\ \{u, V\}\colon\quad \begin{cases} V_t = u^{-1} + xu^{-2}u_x, \\ D_x\left(\frac{V}{xu} - x\right) + D_t\left(\frac{V^2}{2x^2}\right) = 0, \end{cases} \tag{4.12}$$

$$\widehat{S}_2^2\colon\ \{u, V\}\colon\quad \begin{cases} V_x = xu, \\ D_x\left(\frac{V}{xu} - x\right) + D_t\left(\frac{V^2}{2x^2}\right) = 0. \end{cases} \tag{4.13}$$

Since λ^2 is not a potential factor, only $\widehat{S}_1^2$ leads to a useful potential system, given by

$$\overline{T}_1^2\{u, V, \mathcal{W}\}\colon\quad \begin{cases} V_t = u^{-1} + xu^{-2}u_x, \\ \mathcal{W}_t = 2\left(x - \frac{V}{xu}\right), \\ \mathcal{W}_x = \frac{V^2}{x^2}. \end{cases} \tag{4.14}$$

The Fréchet derivative of $\overline{T}_1^2\{u, V, \mathcal{W}\}$ is given by

$$\mathcal{L}[u, V, \mathcal{W}] = \begin{bmatrix} xu^{-2}D_x - (u^{-2} + 2xu^{-3}u_x) & -D_t & 0 \\ \frac{2}{xu^2}V & -2(xu)^{-1} & -D_t \\ 0 & 2x^{-2}V & -D_x \end{bmatrix}$$

with adjoint

$$\mathcal{L}^*[u, V, \mathcal{W}] = \begin{bmatrix} -u^{-2}(2 + xD_x) & \frac{2}{xu^2}V & 0 \\ D_t & -2(xu)^{-1} & 2x^{-2}V \\ 0 & D_t & D_x \end{bmatrix}.$$

It turns out that the only solution of

$$\mathcal{L}^*[u, V, \mathcal{W}] \begin{bmatrix} \lambda^1(x, t, u, V, \mathcal{W}) \\ \lambda^2(x, t, u, V, \mathcal{W}) \\ \lambda^3(x, t, u, V, \mathcal{W}) \end{bmatrix} = 0,$$

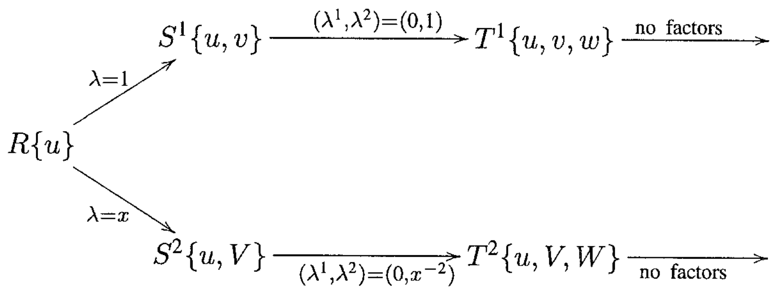
when $(u(x,t), V(x,t), \mathcal{W}(x,t))$ solves (4.14), is $(\lambda^1, \lambda^2, \lambda^3) = (x^{-2}, 0, 0)$. The resulting potential conservation law yields system

$$\overline{U}_1^2\{u, V, \mathcal{W}, \mathcal{Z}\}\colon\quad \begin{cases} \mathcal{W}_x = x^{-2}V^2, \\ \mathcal{W}_t = 2\left(x - \frac{V}{xu}\right), \\ \mathcal{Z}_x = x^{-2}V, \\ \mathcal{Z}_t = -\frac{1}{xu}. \end{cases}$$

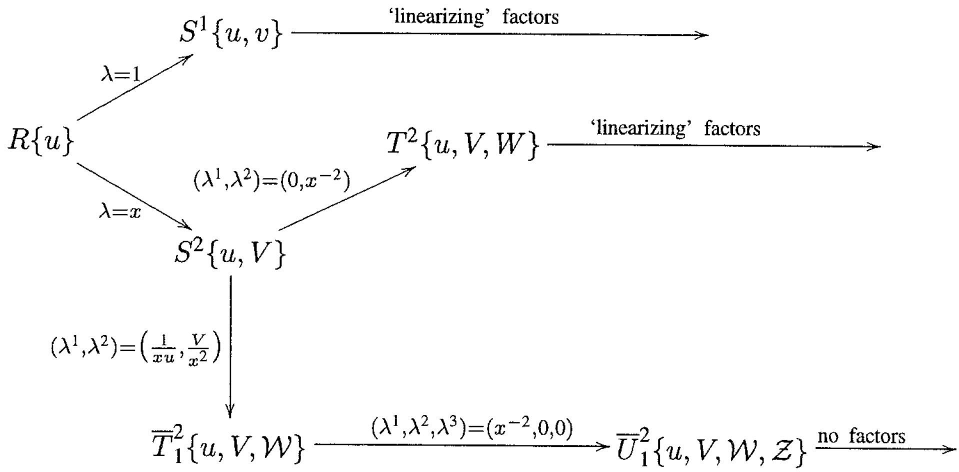
One can show that this potential system admits only trivial factors of the form $\lambda^\sigma(x,t,u,V,\mathcal{W},\mathcal{Z})$.

The factors and potential systems arising for the nonlinear diffusion equation $R\{u\}$ are summarized by the following diagrams:

Case I: $K(u)$ arbitrary



Case II: $K(u)=u^{-2}$



4.2. Symmetry Classification of $R\{u\}$

The infinitesimal generators of symmetries of $R\{u\}$ arising from point symmetries of systems $R\{u\}$, $S^1\{u,v\}$, $T^1\{u,v,w\}$, $S^2\{u,V\}$, $\overline{T}_1^2\{u,V,\mathcal{W}\}$, and $\overline{U}_1^2\{u,V,\mathcal{W},\mathcal{Z}\}$ depend on the form of diffusivity $K(u)\neq$ const, modulo scaling and translations in u.

4.2.1. *Point Symmetries of $R\{u\}$*

(1) *$K(u)$ arbitrary:*

$$\mathbf{X}_1^R = u_x\frac{\partial}{\partial u},\qquad \mathbf{X}_2^R = u_t\frac{\partial}{\partial u},\qquad \mathbf{X}_3^R = (xu_x+2tu_t)\frac{\partial}{\partial u}.$$

(2) $K(u) = u^{\lambda}$:

$$\mathbf{X}_1^R,\ \mathbf{X}_2^R,\ \mathbf{X}_3^R,\ \mathbf{X}_4^R = (2u - \lambda x u_x)\frac{\partial}{\partial u}.$$

(3) $K(u) = u^{-4/3}$:

$$\mathbf{X}_1^R, \ldots, \mathbf{X}_4^R,\ \mathbf{X}_5^R = (3xu + x^2 u_x)\frac{\partial}{\partial u}.$$

4.2.2. *Point Symmetries of* $S^1\{u, v\}$

(1) $K(u)$ *arbitrary*:

$$\mathbf{X}_1^{S^1} = \mathbf{X}_1^R + v_x\frac{\partial}{\partial v}, \qquad \mathbf{X}_2^{S^1} = \mathbf{X}_2^R + v_t\frac{\partial}{\partial v},$$

$$\mathbf{X}_3^{S^1} = \mathbf{X}_3^R + (xv_x + 2tv_t - v)\frac{\partial}{\partial v}, \qquad \mathbf{X}_4^{S^1} = \frac{\partial}{\partial v}.$$

(2) $K(u) = u^{\lambda}$:

$$\mathbf{X}_1^{S^1}, \ldots, \mathbf{X}_4^{S^1},\ \mathbf{X}_5^{S^1} = \mathbf{X}_4^R + ((2+\lambda)v - \lambda x v_x)\frac{\partial}{\partial v}.$$

(3) $K(u) = \frac{1}{1+u^2}\mathrm{e}^{a \arctan u}$, $a = \text{const}$:

$$\mathbf{X}_1^{S^1}, \ldots, \mathbf{X}_4^{S^1},\ \mathbf{X}_5^{S^1} = [(u^2+1) + vu_x + atu_t]\frac{\partial}{\partial u} + [x + vv_x + atv_t]\frac{\partial}{\partial v}.$$

(4) $K(u) = u^{-2}$:

$$\mathbf{X}_2^{S^1}, \ldots, \mathbf{X}_5^{S^1},\ \mathbf{X}_\infty^{S^1} = [F^1(v,t)u_x + u^2F^2(v,t)]\frac{\partial}{\partial u} + F^1(v,t)v_x\frac{\partial}{\partial v},$$

where $(F^1(v,t),\ F^2(v,t))$ is an arbitrary solution of the linear system

$$\frac{\partial F^1}{\partial t} = \frac{\partial F^2}{\partial v}, \qquad \frac{\partial F^1}{\partial v} = F^2. \tag{4.15}$$

4.2.3. *Point Symmetries of* $T^1\{u, v, w\}$

(1) $K(u)$ *arbitrary*:

$$\mathbf{X}_1^{T^1} = \mathbf{X}_1^{S^1} + w_x\frac{\partial}{\partial w}, \qquad \mathbf{X}_2^{T^1} = \mathbf{X}_2^{S^1} + w_t\frac{\partial}{\partial w},$$

$$\mathbf{X}_3^{T^1} = \mathbf{X}_3^{S^1} + (xw_x + 2tw_t - 2w)\frac{\partial}{\partial w}, \qquad \mathbf{X}_4^{T^1} = \mathbf{X}_4^{S^1} + x\frac{\partial}{\partial w},$$

$$\mathbf{X}_5^{T^1} = \frac{\partial}{\partial w}.$$

(2) $K(u) = u^\lambda$:

$$\mathbf{X}_1^{T^1}, \dots, \mathbf{X}_5^{T^1},\ \mathbf{X}_6^{T^1} = \mathbf{X}_5^{S^1} + [2(1+\lambda)w - \lambda x w_x]\frac{\partial}{\partial w}.$$

(3) $K(u) = \frac{1}{1+u^2}\,\mathrm{e}^{a \arctan u}$, $a = \text{const}$:

$$\mathbf{X}_1^{T^1}, \dots, \mathbf{X}_5^{T^1},\ \mathbf{X}_6^{T^1} = \mathbf{X}_5^{S^1} + [\tfrac{1}{2}(x^2 - v^2) + vw_x + atw_t]\frac{\partial}{\partial w}.$$

(4) $K(u) = u^{-2}$:

$$\mathbf{X}_2^{T^1}, \dots, \mathbf{X}_6^{T^1},\ \mathbf{X}_\infty^{T^1} = \mathbf{X}_\infty^{S^1} + [F^3(v,t) - vF^1(v,t) + F^1(v,t)w_x]\frac{\partial}{\partial w},$$

where $(F^1(v,t), F^2(v,t), F^3(v,t))$ is an arbitrary solution of the linear system

$$\frac{\partial F^3}{\partial v} = F^1, \qquad \frac{\partial F^3}{\partial t} = F^2, \qquad \frac{\partial F^1}{\partial v} = F^2. \tag{4.16}$$

(5) $K(u) = u^{-4/3}$:

$$\mathbf{X}_1^{T^1}, \dots, \mathbf{X}_6^{T^1},\ \mathbf{X}_7^{T^1} = \mathbf{X}_5^{R} + (xv - w + x^2 v_x)\frac{\partial}{\partial v} + (x^2 w_x - xw)\frac{\partial}{\partial w},$$

(6) $K(u) = u^{-2/3}$:

$$\mathbf{X}_1^{T^1}, \dots, \mathbf{X}_6^{T^1},\ \mathbf{X}_7^{T^1} = (3uv + wu_x)\frac{\partial}{\partial u} + (v^2 + wv_x)\frac{\partial}{\partial v} + ww_x\frac{\partial}{\partial w}.$$

4.2.4. *Point Symmetries of* $S^2\{u, V\}$

(1) *$K(u)$ arbitrary*:

$$\mathbf{X}_1^{S^2} = \mathbf{X}_2^{R} + V_t\frac{\partial}{\partial V}, \qquad \mathbf{X}_2^{S^2} = \mathbf{X}_3^{R} + (xV_x + 2tV_t - 2V)\frac{\partial}{\partial V},$$

$$\mathbf{X}_3^{S^2} = \frac{\partial}{\partial V}.$$

(2) $K(u) = u^\lambda$:

$$\mathbf{X}_1^{S^2},\ \mathbf{X}_2^{S^2},\ \mathbf{X}_3^{S^2},\ \mathbf{X}_4^{S^2} = \mathbf{X}_4^{R} + (2(\lambda+1)V - \lambda x V_x)\frac{\partial}{\partial V}.$$

(3) $K(u) = u^{-4/3}$:

$$\mathbf{X}_1^{S^2}, \ldots, \mathbf{X}_4^{S^2}, \ \mathbf{X}_5^{S^2} = \mathbf{X}_5^R + x^2 V_x \frac{\partial}{\partial V}.$$

4.2.5. *Point Symmetries of* $\overline{T}_1^2\{u, V, \mathcal{W}\}$

Here $K(u) = u^{-2}$ with admitted infinitesimal generators

$$\mathbf{X}_1^{\overline{T}_1^2} = \mathbf{X}_1^{S^2} + \mathcal{W}_t \frac{\partial}{\partial \mathcal{W}}, \qquad \mathbf{X}_2^{\overline{T}_1^2} = \mathbf{X}_2^{S^2} + (x\mathcal{W}_x + 2t\mathcal{W}_t - 3\mathcal{W}) \frac{\partial}{\partial \mathcal{W}},$$

$$\mathbf{X}_3^{\overline{T}_1^2} = \mathbf{X}_4^{S^2} + 2(x\mathcal{W}_x - \mathcal{W}) \frac{\partial}{\partial \mathcal{W}}, \qquad \mathbf{X}_4^{\overline{T}_1^2} = \frac{\partial}{\partial \mathcal{W}}.$$

4.2.6. *Point Symmetries of* $\overline{U}_1^2\{u, V, \mathcal{W}, \mathcal{Z}\}$

Here $K(u) = u^{-2}$ with admitted infinitesimal generators

$$\mathbf{X}_1^{\overline{U}_1^2} = \mathbf{X}_1^{\overline{T}_1^2} + \mathcal{Z}_t \frac{\partial}{\partial \mathcal{Z}}, \qquad \mathbf{X}_2^{\overline{U}_1^2} = \mathbf{X}_2^{\overline{T}_1^2} + (x\mathcal{Z}_x + 2t\mathcal{Z}_t - \mathcal{Z}) \frac{\partial}{\partial \mathcal{Z}},$$

$$\mathbf{X}_3^{\overline{U}_1^2} = \mathbf{X}_3^{\overline{T}_1^2} + 2x\mathcal{Z}_x \frac{\partial}{\partial \mathcal{Z}}, \qquad \mathbf{X}_4^{\overline{U}_1^2} = \mathbf{X}_4^{\overline{T}_1^2}, \qquad \mathbf{X}_5^{\overline{U}_1^2} = \frac{\partial}{\partial \mathcal{Z}}.$$

We now analyze the above symmetries in view of the material presented in Sections 2 and 3:

When $K(u) = u^{-4/3}$, the point symmetry $\mathbf{X}_5^R$ is 'lost' in $S^1\{u, v\}$, since it induces no point symmetry of $S^1\{u, v\}$. In particular, $\mathbf{X}_5^R$ induces a nonlocal symmetry of $S^1\{u, v\}$ which is represented by the infinitesimal generator

$$\mathbf{X}_5^{S^1} = \mathbf{X}_5^R + (x^2 v_x + xv - D_x^{-1} v) \frac{\partial}{\partial v}.$$

On the other hand, $S^1\{u, v\}$ yields potential symmetries of $R\{u\}$ given by $\mathbf{X}_5^{S^1}$ when $K(u) = \frac{1}{1+u^2} e^{a \arctan u}$, and by $\mathbf{X}_\infty^{S^1}$ when $K(u) = u^{-2}$. The latter symmetry leads directly to the linearization of $R\{u\}$ by a noninvertible mapping [6, 15, 16].

For any $K(u)$, $T^1\{u, v, w\}$ 'covers' $R\{u\}$ and $S^1\{u, v\}$, since the point symmetries of $T^1\{u, v, w\}$ project onto all point symmetries of both $R\{u\}$ and $S^1\{u, v\}$. In particular, the point symmetry $\mathbf{X}_5^R$, 'lost' in $S^1\{u, v\}$, is 'recovered' as a point symmetry of $T^1\{u, v, w\}$. Moreover, if $K(u) = u^{-2/3}$, the point symmetry $\mathbf{X}_7^{T^1}$ yields a potential symmetry of $S^1\{u, v\}$ and a (new) nonlocal symmetry of $R\{u\}$.

For any $K(u)$, the point symmetry $\mathbf{X}_1^R$ is 'lost' in $S^2\{u, V\}$, since it induces no point symmetry of $S^2\{u, V\}$. One can show that $\mathbf{X}_1^R$ induces a nonlocal symmetry of $S^2\{u, V\}$ which is represented by the infinitesimal generator

$$\mathbf{X}^{S^2} = \mathbf{X}_1^R + (xu - D_x^{-1}u)\frac{\partial}{\partial V}.$$

All other point symmetries of $R\{u\}$ induce point symmetries of $S^2\{u, V\}$ and, in turn, $S^2\{u, V\}$ yields no potential symmetries of $R\{u\}$.

Since $T^2\{u, V, W\}$ is equivalent to $T^1\{u, v, w\}$, through mapping (4.11) it follows that each point symmetry of $T^1\{u, v, w\}$ accordingly maps into a point symmetry of $T^2\{u, V, W\}$. In particular, it is interesting to note that the point symmetry $\mathbf{X}_1^R$ 'lost' in $S^2\{u, V\}$ is 'recovered' as the point symmetry

$$\mathbf{X}^{T^2} = \mathbf{X}_1^R + (V_x - x^{-1}V - W)\frac{\partial}{\partial V} + (W_x + x^{-1}W)\frac{\partial}{\partial W}$$

of $T^2\{u, V, W\}$.

Finally, the potential systems $\overline{T}_1^2\{u, V, \mathcal{W}\}$, $\overline{U}_1^2\{u, V, \mathcal{W}, \mathcal{Z}\}$, which only arise for $K(u) = u^{-2}$, are disappointing since they do not 'recover' $\mathbf{X}_1^R$ as a point symmetry, their point symmetries yield no nonlocal symmetries of R{u}, and, unlike $T^2\{u, V, W\}$, do not lead directly to the linearization of $R\{u\}$.

5. Linearizing Factors

Suppose $R\{u\}$ is a linear system of PDEs given by

$$G^\sigma = \sum_{\rho=1}^{m} L_\rho^\sigma[x]u^\rho = 0, \quad \sigma = 1, 2, \ldots, N. \tag{5.1}$$

Let $L^*[x]$ be the adjoint of linear operator $L[x]$. From the definition of the adjoint, the proof of the following theorem is obvious:

THEOREM 5.1. *A set of factors* $\lambda(x) = (\lambda^1(x), \lambda^2(x), \ldots, \lambda^N(x))$ *yields a conservation law for* (5.1) *if and only if*

$$\sum_{\sigma=1}^{N} L_\sigma^{*\rho}[x]\lambda^\sigma(x) = 0, \quad \rho = 1, 2, \ldots, m.$$

Theorem 5.1 combined with the observation that conservation laws are invariant under contact transformations [27] leads one to consider u-dependent factors which yield conservation laws. In particular, if $R\{x, u\}$ is linearizable by an invertible contact transformation it is necessary that it admit u-dependent factors of the form

$$\lambda^\sigma\Big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{p}{u}\Big) = \mathbf{A}_\rho^\sigma\Big(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{p}{u}\Big)F^\rho(X),$$

where

$$\mathbf{A}_\rho^\sigma\left(x,u,\underset{1}{u},\underset{2}{u},\ldots,\underset{p}{u}\right),\quad \sigma,\rho=1,2,\ldots,N,$$

are specific functions of the components of $(x,u,\underset{1}{u},\underset{2}{u},\ldots,\underset{p}{u})$ and $F(X) = (F^1(X),F^2(X),\ldots,F^N(X))$ are arbitrary functions satisfying a linear system

$$\sum_{\sigma=1}^{N} L_\sigma^{*\rho}[X]F^\sigma=0,\quad \rho=1,2,\ldots,m; \tag{5.2}$$

$X=(X_1(x,u),X_2(x,u),\ldots,X_n(x,u))$ yields independent variables for a resulting linear system (X can depend on components of $\underset{1}{u}$ in the scalar case) given by

$$\sum_{\rho=1}^{m} L_\rho^\sigma[X]U^\rho=0,\quad \sigma=1,2,\ldots,N,$$

with dependent variables $U=(U^1,U^2,\ldots,U^m)$; $L[X]$ is the adjoint of linear operator $L^*[x]$. This yields necessary conditions for linearizing $R\{u\}$ [22] and leads to the following definition:

DEFINITION 5.2. Factors $\lambda^\sigma=\lambda^\sigma(x,u,\underset{1}{u},\underset{2}{u},\ldots,\underset{p}{u})$, $\sigma=1,2,\ldots,N$, are *linearizing factors* for $R\{u\}$ provided corresponding adjoint Equations (3.3) can be expressed in the form (5.2).

If a given system $R\{u\}$ admits linearizing factors, then it is unnecessary to determine a corresponding conservation law since such a conservation law does not help in finding an explicit linearization of $R\{u\}$. In particular one must still apply specific symmetry algorithms [6, 15, 16] to construct linearizations when they exist. We now consider four examples:

5.1. Nonlinear diffusion equation

The nonlinear diffusion system (4.4), for $K(u)=u^{-2}$, admits linearizing factors with arbitrary functions satisfying (4.9). The application of linearization algorithms [6, 15, 16] yields the mapping of (4.4) to the heat equation, which is the adjoint equation of (4.9).

5.2. Burgers' equation

Burgers' equation $u_{xx}-uu_x-u_t=0$, written in conservation form ($\lambda=2$) $D_x(2u_x-u^2)-D_t(2u)=0$, yields potential system $S\{u,v\}$ given by

$$v_x=2u,\qquad v_t=2u_x-u^2. \tag{5.3}$$

One can show that (5.3) admits linearizing factors

$$(\lambda^1, \lambda^2) = \mathrm{e}^{-v/4}\Big(\tfrac{1}{2}\, uF^1(x,t) + F^2(x,t),\ F^1(x,t)\Big),$$

where $(F^1(x,t),\ F^2(x,t))$ is an arbitrary solution of the linear system

$$\frac{\partial F^1}{\partial x} = F^2, \qquad \frac{\partial F^1}{\partial t} = -\frac{\partial F^2}{\partial x}. \tag{5.4}$$

Again the application of linearization algorithms [6, 15, 16] yields the mapping of (5.3) to the heat equation, which is the adjoint equation of (5.4).

5.3. Nonlinear telegraph equation

The nonlinear telegraph system

$$v_t = u_x, \qquad v_x = u^{-2}u_t + 1 - u^{-1} \tag{5.5}$$

admits linearizing factors $(\lambda^1, \lambda^2) = (F^1(X,T),\ u^{-1}F^2(X,T))$, where $(X, V) = (x - v, t - \log u)$, and $(F^1(X,T),\ F^2(X,T))$ is an arbitrary solution of the linear system

$$\frac{\partial F^1}{\partial X} + \frac{\partial F^2}{\partial T} - F^2 = 0, \qquad \frac{\partial F^1}{\partial T} + \frac{\partial F^2}{\partial X} = 0. \tag{5.6}$$

The application of linearization algorithms [6, 15, 16] yields the mapping of (5.5) to a linear system, which is the adjoint of (5.6).

5.4. Nonlinear diffusion equation revisited

For arbitrary $K(u)$, the nonlinear diffusion system (4.7) admits no factors of the form $\lambda^\sigma(x,t,u,v,w)$, $\sigma = 1,2,3$. However, when $K(u) = u^{-2}$, it does admit linearizing factors

$$(\lambda^1, \lambda^2, \lambda^3) = (u^{-1}F^1(v,t), u^{-2}u_xF^1(v,t) - F^2(v,t), uF^3(v,t)),$$

where $(F^1(v,t), F^2(v,t), F^3(v,t))$ is an arbitrary solution of the linear system

$$\frac{\partial F^1}{\partial v} = F^2, \qquad \frac{\partial F^1}{\partial t} = -\frac{\partial F^2}{\partial v}, \qquad F^3 = -F^1. \tag{5.7}$$

Again, application of linearization algorithm [6, 15, 16] leads to the mapping of the nonlinear diffusion system (4.7), when $K(u) = u^{-2}$, to the adjoint of system (5.7), namely the linear heat equation system.

6. Remarks

Other approaches to obtain nonlocal symmetries by 'covering systems' [28–30] or to obtain linearizations [31] appear to be restricted to PDEs with two independent variables. The approaches presented in this paper to obtain useful potential systems or linearizing factors clearly extend to systems of PDEs with three or more independent variables. A specific example of such a potential system yielding potential symmetries will be presented in a future paper.

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