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A Survey of Truncation Error Analysis for Padé and Continued Fraction Approximants

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Abstract. To compute the value of a function $f(z)$ in the complex domain by means of a converging sequence of rational approximants $\{f_n(z)\}$ of a continued fraction and/or Padé table, it is essential to have sharp estimates of the truncation error $|f(z) - f_n(z)|$. This paper is an expository survey of constructive methods for obtaining such truncation error bounds. For most cases dealt with, $\{f_n(z)\}$ is the sequence of approximants of a continued fractoin, and each $f_n(z)$ is a (1-point or 2-point) Padé approximant. To provide a common framework that applies to rational approximant $f_n(z)$ that may or may not be successive approximants of a continued fraction, we introduce linear fractional approximant sequences (LFASs). Truncation error bounds are included for a large number of classes of LFASs, most of which contain representations of important functions and constants used in mathematics, statistics, engineering and the physical sciences. An extensive bibliography is given at the end of the paper.

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1. Introduction

Many important functions $f(z)$ of mathematical physics, chemistry, engineering, and statistics are represented by convergent sequences $\{f_n(z)\}$ of rational functions that are entries of a (1-point or multipoint) Padé table for $f(z)$. In most cases of practical interest $\{f_n(z)\}$ is the sequence of approximants of a continued fraction (see, e.g., [1], [37], [45] and references contained therein). One reason for the importance of Padé tables and related continued fractions is that sequences of their approximants may converge in larger regions of the complex plane **C** than the power series expansion, which may not converge at all. Also the algorithmic character of continued fractions and Padé approximants provides efficient methods for the computation of special functions.

To compute the value $f(z) = \lim_{n\to\infty} f_n(z)$ at a point $z \in \mathbf{C}$ (using $\{f_n(z)\}$), it is essential to have realistic upper bounds for the truncation error $|f(z) - f_n(z)|$ that results from replacing the true value $f(z)$ by an approximant $f_n(z)$. Following the

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advent of high-speed computers, an extensive literature on truncation error analysis for Padé and continued fraction approximants has developed. This paper is a survey of constructive methods for obtaining such truncation error bounds. References to a large number of the original research publications on this subject are contained in the bibliography.

Three types of truncation error estimates are considered. *A posteriori bounds* for the truncation error $|f(z)-f_n(z)|$ are determined after calculating the approximants $f_0(z), f_1(z), \dots, f_n(z)$ and related expressions. *A priori bounds* are expressed in terms of z and parameters defining $f_n(z)$; they can be used to appraise the truncation error at the start of the computations. A third type of error estimation describes asymptotically the speed of convergence of $\{f_n(z)\}$. This paper contains examples of all three types of error bounds. However, emphasis is given to *a posteriori* bounds, since they generally give the sharpest error estimates.

In some cases dealt with in this paper the approximant sequence $\{f_n(z)\}$ is not the sequence of approximants of a continued fraction (cf., sections 3.2.3 and 3.2.4). In order to treat all of the approximant sequences $\{f_n(z)\}$ with a uniform framework, we introduce *linear fractional approximant sequences* (LFASs). An LFAS F is an ordered pair

$$F = \langle\langle\{\langle a_j, b_j, c_j, d_j\rangle\}, \{w_j\}\rangle, \{f_n\}\rangle, \tag{1.1a}$$

where the *elements* a_j, b_j, c_j, d_j and converging factors w_j are complex numbers (possibly functions of a complex variable z) satisfying

$$a_j d_j - b_j c_j \neq 0, \quad j = 0, 1, 2, \dots. \tag{1.1b}$$

The *nth approximant* $f_n = v_n(F)$ of F is given by

$$f_n := v_n(F) := T_n(F, w_n), \quad n = 0, 1, 2, 3, \dots, \tag{1.1c}$$

where $\{T_n(F, w)\}$ and the *generating sequence* $\{t_n^F(w)\}$ are defined by

$$t_j^F(w) := \frac{a_j + c_j w}{b_j + d_j w}, \quad j = 0, 1, 2, \dots, \tag{1.1d}$$

and

$$\begin{aligned} T_0(F, w) &:= t_0^F(w), \\ T_n(F, w) &:= T_{n-1}(F, t_n^F(w)), \quad n = 1, 2, 3, \dots. \end{aligned} \tag{1.1e}$$

An LFAS F is said *to converge to a value* $v(F) \in \widehat{\mathbf{C}} = \mathbf{C} \cup [\infty]$, if its sequence of approximants $\{v_n(F)\}$ converges to $v(F)$; i.e., $v(F) = \lim_{n\to\infty} v_n(F)$. To indicate explicitly the association with F of its elements and converging factors we write $a_j(F), b_j(F), c_j(F), d_j(F)$ and $w_j(F)$. For convenience we sometimes use the abbreviated notation

$$\Gamma_j(F) := \langle a_j(F), b_j(F), c_j(F), d_j(F)\rangle, \quad j = 0, 1, 2, \dots. \tag{1.2}$$

The *LFAS algorithm* $\mathcal{A}(F)$ is the mapping of the ordered pair $\langle\{\Gamma_j(F)\},\{w_j(F)\}\rangle$ to $\{f_n\}=\{v_n(F)\}$. If F depends on a complex variable z, we may write $F(z)$, $v(F(z))$ and $v_n(F(z))$.

To obtain upper bounds for the truncation error $|v(F)-v_n(F)|$, it is useful to work with special families $\mathcal{F}$ of LFASs that contain F. For that purpose we consider *sequences of element regions* $\Omega=\{\Omega_j\}$ and *converging factors* $W=\{w_j\}$ satisfying

$$\phi\neq\Omega_j\in\mathbf{C}^4,\quad j=0,1,2,\ldots, \tag{1.3a}$$

where

$$\Gamma_j=\langle a_j,b_j,c_j,d_j\rangle\in\Omega_j\quad\text{implies } a_jd_j-b_jc_j\neq 0, \tag{1.3b}$$

and

$$w_j\in\mathbf{C},\quad j=0,1,2,\ldots. \tag{1.3c}$$

For each such pair of sequences (Ω,W), we define the family of LFASs

$$\mathcal{F}=\mathcal{F}(\Omega,W):=[\text{LFASs } F:\ \Gamma_j(F)\in\Omega_j\quad\text{and}\quad w_j(F)=w_j,\ j\geq 0]. \tag{1.4}$$

For brevity we write $\mathcal{F}$ instead of $\mathcal{F}(\Omega,W)$ if the dependence of $\mathcal{F}$ on (Ω,W) is clearly understood. We also use subfamilies of $\mathcal{F}$ defined, for each $F\in\mathcal{F}$ and $n=0,1,2,3,\ldots$, by

$$\mathcal{F}_n(F):=[G\in\mathcal{F}:\ \Gamma_j(G)=\Gamma_j(F),\quad j=0,1,2,\ldots,n]. \tag{1.5}$$

If $G\in\mathcal{F}_n(F)$, we say that G is *n-equivalent to F*, and we call $\mathcal{F}_n(F)$ the *n-th equivalence class of F in $\mathcal{F}$*. For each $f\in\mathcal{F}$ and integer $n\geq 0$, we define the *n-th limit region $L_n(F,\mathcal{F})$ for $\mathcal{F}_n(F)$* by

$$L_n(F,\mathcal{F}):=c(\ell_n(F,\mathcal{F}))\quad(c(S)\text{ denotes closure of } S), \tag{1.6a}$$

where

$$\ell_n(F,\mathcal{F}):=[\lambda\in\mathbf{C}:\ \lambda=\lim_{j\to\infty}v_{m_j}(G)\ \text{for } G\in\mathcal{F}_n(F)\text{ and subsequence }\{m_j\}]. \tag{1.6b}$$

If F converges to a finite value $v(F)$, then $L_n(F,\mathcal{F})$ is not empty, since $v(F)\in\ell_n(F,\mathcal{F})\subseteq L_n(F,\mathcal{F})$. The concept of limit region was first used in the context of truncation error analysis by L. Lorentzen, M. Overholt, W. J. Thron and H. Waadeland (see, e.g., [48]). Our definition (1.6) differs from their's in that we allow $\ell_n(F,\mathcal{F})$ to contain finite limits of subsequences $\{f_{m_j}(G)\}$ with $G\in\mathcal{F}_n(F)$ even if $\{v_m(G)\}$ diverges. For a given family $\mathcal{F} = \mathcal{F}(\Omega, W)$ and for a finitely convergent $F \in \mathcal{F}$, we define the *best bound* $\beta_n(F, \mathcal{F})$ *of the truncation error* $|v(F) - v_n(F)|$ *for* $v_n(F)$ *with respect to* $\mathcal{F}$ by

$$\beta_n(F,\mathcal{F}) := \sup[|\lambda - v_n(F)| : \lambda \in L_n(F,\mathcal{F})]. \tag{1.7}$$

Clearly $|v(F) - v_n(F)| \le \beta_n(F,\mathcal{F})$, since $v(F) \in L_n(F,\mathcal{F})$. The term "best" for $\beta_n(F,\mathcal{F})$ is based on the fact that the values $\lambda \in \ell_n(F,\mathcal{F})$ are all possible candidates for $v(F)$, if we assume that our knowledge about F is limited to the following: (a) $F \in \mathcal{F}$, (b) F is finitely convergent, and (c) the only known elements of F are $\Gamma_j(F)$, $j = 0,1,2,\dots,n$. One can readily see that a given LFAS F can belong to many families $\mathcal{F}^{(\alpha)}$, $\alpha \in A$. If F is finitely convergent and $F \in \mathcal{F}^{(1)} \subseteq \mathcal{F}^{(2)}$, then

$$\beta_n(F,\mathcal{F}^{(1)}) \le \beta_n(F,\mathcal{F}^{(2)}). \tag{1.8}$$

Thus an LFAS F may have different "best" error bounds $\beta_n(F,\mathcal{F}^{(\alpha)})$ corresponding to different families $\mathcal{F}^{(\alpha)}$. It is therefore advantageous to use the smallest family $\mathcal{F}$ that is feasible.

For a given LFAS F, the linear fractional transformations $T_n(F,w)$ defined by (1.1) can be expressed in the form

$$T_n(F,w) = \frac{A_n + wC_n}{B_n + wD_n}, \quad n = 0,1,2,\dots, \tag{1.9}$$

where the $A_n = A_n(F)$, $B_n = B_n(F)$, $C_n = C_n(F)$ and $D_n = D_n(F)$ are defined by the *difference equations*

$$\begin{aligned}
&\text{a)}\quad A_0 := a_0, \quad b_0 := b_0, \quad C_0 := c_0, \quad D_0 := d_0\\
&\text{b)}\quad A_n := a_nC_{n-1} + b_nA_{n-1}, \quad C_n := c_nC_{n-1} + d_nA_{n-1},\\
&\qquad\qquad n = 1,2,3,\dots,\\
&\text{c)}\quad B_n := a_nD_{n-1} + b_nB_{n-1}, \quad D_n := c_nD_{n-1} + d_nB_{n-1},\\
&\qquad\qquad n = 1,2,3,\dots.
\end{aligned} \tag{1.10}$$

They satisfy the *determinant formulas*

$$A_nD_n - B_nC_n = (-1)^n \prod_{j=0}^{n} (a_jd_j - b_jc_j) \ne 0, \quad n = 0,1,2,\dots \tag{1.11}$$

(see, e.g., [37], Section 2.2).

An LFAS F in (1.1) reduces to a *continued fraction* (CF)

$$F = a_0 + \mathop{\mathrm{K}}_{j=1}^{\infty} \left(\frac{a_j}{b_j}\right) = a_0 + \frac{a_1}{b_1} \genfrac{}{}{0pt}{}{}{+} \frac{a_2}{b_2} \genfrac{}{}{0pt}{}{}{+} \frac{a_3}{b_3} \genfrac{}{}{0pt}{}{}{+} \cdots, \tag{1.12a}$$

if the elements a_j, b_j, c_j, d_j in (1.12) and converging factors w_j satisfy

$$a_0 \in \mathbf{C}, \quad b_0 = 1, \quad c_0 = 1, \quad d_0 = 0, \tag{1.12b}$$

$$a_j \neq 0, \quad b_j \in \mathbf{C}, \quad c_j = 0, \quad d_j = 1, \quad j = 1, 2, 3, \ldots, \tag{1.12c}$$

and

$$w_j = 0, \quad j = 0, 1, 2, \ldots. \tag{1.12d}$$

The *nth approximant of a CF* (1.12) is then

$$v_n(F) := T_n(F, 0) =: a_0 + \frac{a_1}{b_1} \, \frac{a_2}{+ \, b_2} + \cdots + \frac{a_n}{b_n}. \tag{1.13}$$

An LFAS F in (1.1) reduces to a *modified continued fraction* (MCF)

$$F = a_0 + \mathop{\mathrm{K}}_{j=1}^{\infty} (a_j, b_j; w_j) \tag{1.14}$$

if the elements satisfy (1.12 b,c). The *n-th approximant of a MCF* (1.14) is given by

$$v_n(F) := T_n(F, w_n) =: a_0 + \frac{a_1}{b_1} \, \frac{a_2}{+ \, b_2} + \cdots + \frac{a_{n-1}}{b_{n-1}} \, \frac{a_n}{+ \, b_n + w_n}. \tag{1.15}$$

For CFs (1.12) and MCFs (1.14), the difference equations (1.10) reduce to

$$\begin{aligned} A_{-1} &:= 1, \quad A_0 := a_0, \quad B_{-1} := 0, \quad B_0 := 1, \\ A_n &= b_n A_{n-1} + a_n A_{n-2}, \quad n = 1, 2, 3, \ldots, \\ B_n &= b_n B_{n-1} + a_n B_{n-2}, \quad n = 1, 2, 3, \ldots. \end{aligned} \tag{1.16}$$

Here $C_n = A_{n-1}$ and $D_n = B_{n-1}$, $n \geq 1$. Throughout this paper, when referring to CFs and MCFs, we make use of the familiar notation in (1.12) and (1.14), respectively.

Special classes of LFASs that are dealt with here (Section 3), which are neither CFs nor MCFs, are those associated with *normalized Carathéodory functions* (*$\mathcal{C}$-functions*)

$$\mathcal{C} := [f : f \quad \text{is analytic and } \operatorname{Re} f(z) > 0 \text{ for } |z| < 1, f(0) > 0] \tag{1.17}$$

and *normalized Schur functions* (*$\mathcal{S}$-functions*)

$$\mathcal{S} := [f : f \text{ is analytic and } |f(z)| < 1 \text{ for } |z| < 1, -1 < f(0) < 1]. \tag{1.18}$$

Associated with $\mathcal{C}$-functions are the LFASs $F = C[\{\delta_j\}z]$ with generating sequences $\{t_j^F(w)\}$ of the form

$$t_0^F(w) := \delta_0 \frac{1 - w}{1 + w}, \quad t_j^F(w); = z \frac{\bar{\delta}_j + w}{1 + \delta_j w}, \quad j = 1, 2, 3, \ldots \tag{1.19a}$$

where

$$\delta_0 > 0 \quad \text{and} \quad \delta_j \in \mathbf{C}, \quad 0 \leq |\delta_j| < 1, \quad j = 1, 2, 3, \ldots, \tag{1.19b}$$

and with converging factors

$$w_j = 0, \quad j = 0, 1, 2, \ldots. \tag{1.19c}$$

Associated with $\mathcal{S}$-functions are LFASs $F = S[\{\gamma_j\}, z]$ with generating sequences of the form

$$\begin{aligned} t_j^F(w) := \frac{\gamma_j + zw}{1 + \bar{\gamma}_j zw}, \quad \gamma_0 \in \mathbf{R}, \quad |\gamma_0| < 1, \\ \gamma_j \in \mathbf{C}, \quad |\gamma_j| < 1, \quad j = 1, 2, 3, \ldots, \end{aligned} \tag{1.20a}$$

and converging factors

$$w_j = 0, \quad j = 0, 1, 2, \ldots. \tag{1.20b}$$

Sequences of value regions $V = \{V_n\}$ corresponding to sequences of element regions $\Omega = \{\Omega_j\}$ and converging factors $W = \{w_j\}$ are discussed in Section 2. Methods for obtaining truncation error bounds based on sequences of value regions $\{V_n\}$ are developed (Theorems 2.5 and 2.6). For many special families $\mathcal{F}(\Omega, W)$ of LFASs, we are able to determine best truncation error bounds $\beta_n(F, \mathcal{F})$ by using best sequences of value regions. Applications of the method are described in Sections 3 and 4. In Section 3 the method is applied to the following 7 special families of LFASs:

$$\begin{aligned} \mathcal{F}^{W(\rho)} := [K(a_j/1) : 0 \neq |a_j| \leq \rho(1-\rho), \quad a_j \in \mathbf{C} \quad \text{for } j \geq 1], \\ 0 < \rho \leq \frac{1}{2}, \quad \text{(Worpitzky)} \end{aligned} \tag{1.21}$$

$$\begin{aligned} \mathcal{F}^{SP(\rho)} := [K(1/b_j) : |b_j| \geq \rho + 1/\rho, \quad b_j \in \mathbf{C} \quad \text{for } j \geq 1], \\ 0 < \rho \leq 1, \quad \text{(Śleszyński–Pringsheim)} \end{aligned} \tag{1.22}$$

$$\begin{aligned} \mathcal{F}^{St(z)} := [K(a_j z/1) : a_j > 0 \quad \text{for } j \geq 1, \\ 0 \neq z \in \mathbf{C}, \quad |\arg z| < \pi], \quad \text{(Stieltjes)} \end{aligned} \tag{1.23}$$

$$\begin{aligned} \mathcal{F}^{T(z)} := [K(F_j z/(1 + G_j z)) : F_j, G_j > 0, \quad 0 \neq z \in \mathbf{C}, \\ |\arg z| < \pi], \quad \text{(Thron)} \end{aligned} \tag{1.24}$$

$$\begin{aligned} \mathcal{F}^{J(z)} := [K(-\alpha_j^2/(\beta_j + z)) : -\alpha_1^2 = 1, \quad \beta_1 \in \mathbf{R}; \\ 0 \neq \alpha_j \in \mathbf{R}, \quad \beta_j \in \mathbf{R} \quad \text{for } j \geq 2; \\ \operatorname{Im} z \neq 0], \quad \text{(real J-fractions)} \end{aligned} \tag{1.25}$$

$$\mathcal{F}^{PPC(z)} := [C[\{\delta_j\}, z] : \{t_j^F(w)\} \text{ and } w_j \text{ in (1.19)}, \quad |z| < 1], \quad \text{(Carathéodory)} \tag{1.26}$$

$$\mathcal{F}^{Sh(z)} := [S[\{\gamma_j\}, z] : \{t_j^F(w)\} \text{ and } w_j \text{ in (1.20)}, \quad |z| < 1], \quad \text{(Schur)} \tag{1.27}$$

In Section 4 the value region method is applied to the following 4 special families of LFASs that are limit k-periodic CFs or MCFs:

$$K(a_j, 1, x_1), \quad a_j \to a \in \mathbf{C} - (-\infty, -1/4], \quad \text{as } j \to \infty, \tag{1.28a}$$

$$K(a_j/1), \quad a_j \to 0, \quad \text{as } j \to \infty, \tag{1.28b}$$

$$K(a_j, 1, w_j), \quad a_j \to \infty, \quad \text{as } j \to \infty, \tag{1.29}$$

$$K(1/b_j), \quad b_j \to \infty, \quad \text{as } j \to \infty, \tag{1.30}$$

$$K(1, b_j, w_j), \quad b_{4i+j} \to \beta_j, \quad j = 1, 2, 3, 4, \quad \text{as } i \to \infty. \tag{1.31}$$

Section 5 deals with asymptotically best truncation error bounds for limit periodic LFASs (including limit periodic MCFs). Due to constraints of space and time we have had to omit some topics and results on the subject of this paper. Among the omissions is a formal discussion about simple sequences developed in [35]. Some examples of simple sequences are given in (3.18), (3.35) and Section 3.3. We have also omitted applications of truncation error bounds to particular special functions and results from computational experiments. Examples of such applications and experiments can be found in many of the papers given in the references. Before continuing with Section 2 we summarize some additional definitions and notation that are subsequently used.

For $m = 0, 1, 2, \ldots$, the *m-th tail of an LFAS F* (see (1.1)) is the LFAS, denoted by $F^{(m)}$, with elements $a_j^{(m)}, b_j^{(m)}, c_j^{(m)}, d_j^{(m)}$ and converging factors $w_j^{(m)}$ defined by

$$\Gamma_j^{(m)} := \langle a_j^{(m)}, b_j^{(m)}, c_j^{(m)}, d_j^{(m)} \rangle := \langle a_{m+j}, b_{m+j}, c_{m+j}, d_{m+j} \rangle \tag{1.32a}$$

and

$$w_j^{(m)} ;= w_{m+j}. \tag{1.32b}$$

We note that $F^{(0)} = F$,

$$t_j^{F^{(m)}}(w) = t_{m+j}^F(w), \quad m = 0, 1, 2, \ldots \quad \text{and} \quad j = 0, 1, 2, \ldots, \tag{1.33a}$$

and

$$T_1(F^{(m)}, w) = t_{m+1}^F(w),$$
$$T_n(F^{(m)}, w) = T_{n-1}(F^{(m)}, t_{m+n}^F(w)), \quad m = 0, 1, 2, \ldots, \tag{1.33b}$$

It follows that, for $m = 0, 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$,

$$T_n(F^{(m)}, w) = t^F_{m+1} \circ t^F_{m+2} \circ \cdots \circ t^F_{m+n}(w), \tag{1.34}$$

$$T_{m+n}(F, w) = T_m(F, T_n(F^{(m)}, w), \tag{1.35}$$

$$v_n(F^{(m)}) := T_n(F^{(m)}, w_n^{(m)}) = T_n(F^{(m)}, w_{m+n}), \tag{1.36}$$

$$\begin{aligned} v_{m+n}(F) &:= T_{m+n}(F, w_{m+n}) \\ &= T_m(F, T_n(F^{(m)}, w_{m+n}) = T_m(F, v_n(F^{(m)}), \end{aligned} \tag{1.37}$$

and

$$v(F) := \lim_{n\to\infty} v_n(F) = T_m(F, v(F^{(m)})), \quad m = 0, 1, 2, \ldots. \tag{1.38}$$

A sequence $\{\tau_n\}$, where $\tau_n \in \widehat{\mathbf{C}}$, is called a *tail sequence of an LFAS F* if, for some $\tau \in \widehat{\mathbf{C}}$,

$$\tau_m = T_m^{-1}(F, \tau), \quad m = 0, 1, 2, \ldots. \tag{1.39}$$

An example of a tail sequence of a LFAS F is given by

$$\tau_m := v(F^{(m)}), \quad m = 0, 1, 2, \ldots, \tag{1.40}$$

provided, of course, that the tails $F^{(m)}$ are convergent. The sequence $\{v(F^{(m)}\}$ is called the *right* (i.e., correct) *tail sequence of* F since the τ in (1.39) is given by $\tau = v(F)$ (see (1.38)). Another important tail sequence, called the *critical tail sequence*, is defined by

$$\tau_m := -h_m(F) := T_m^{-1}(F, \infty), \quad m = 0, 1, 2, \ldots. \tag{1.41}$$

It follows from (1.9) and (1.41) that

$$h_m(F) = B_m(F)/D_m(F), \quad m = 0, 1, 2, \ldots. \tag{1.42}$$

2. Truncation Error Bounds from Value Regions

Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ denote a given non-empty family (1.4) of LFASs. Let $\{U_n(\mathcal{F})\}_{n=-1}^{\infty}$ be defined by

$$\begin{aligned} U_n(\mathcal{F}) := [&t^F_{n+1} \circ t^F_{n+2} \circ \cdots \circ t^F_{n+m}(w_{n+m}) : \\ &F \in \mathcal{F} \quad \text{and} \quad m = 1, 2, 3, \ldots]. \end{aligned} \tag{2.1}$$

We begin with the following

THEOREM 2.1. *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a given non-empty family* (1.4) *of LFASs. Then*

$$[t_n^F(w_n) : F \in \mathcal{F}] \subseteq U_{n-1}(\mathcal{F}), \quad n = 0, 1, 2, 3, \ldots, \tag{2.2a}$$

and

$$[t_n^F(U_n(\mathcal{F})) : F \in \mathcal{F}] \subseteq U_{n-1}(\mathcal{F}), \quad n = 0, 1, 2, 3, \ldots. \tag{2.2b}$$

Proof. Condition (2.2a) is an immediate consequence of (2.1). To prove (2.2b) let $F \in \mathcal{F}, n \in [0, 1, 2, \ldots]$ and $u \in U_n(\mathcal{F})$ be given. Then by (2.1) there exists a $G \in \mathcal{F}_n(F)$ and an integer $m \geq 1$ such that

$$u = t_{n+1}^G \circ t_{n+2}^G \circ \cdots \circ t_{n+m}^G(w_{n+m}). \tag{2.3}$$

It follows from this, (2.1) and $t_n^F(w) = t_n^G(w)$ that

$$t_n^F(u) = t_n^G \circ t_{n+1}^G \circ \cdots \circ t_{n+m}^G(w_{n+m}) \in U_{n-1}(\mathcal{F}). \tag{2.4}$$

Q.E.D.

A sequence $\{V_n\}_{n=-1}^{\infty}$ of non-empty subsets of $\hat{\mathbf{C}}$ is called *a sequence of value regions with respect to* $\mathcal{F} = \mathcal{F}(\Omega, W)$ if the following conditions are satisfied:

$$[t_n^F(w_n) : F \in \mathcal{F}] \subseteq V_{n-1}, \quad n = 0, 1, 2, 3, \ldots, \tag{2.5a}$$

$$[t_n^F(V_n) : F \in \mathcal{F}] \subseteq V_{n-1}, \quad n = 0, 1, 2, 3, \ldots. \tag{2.5b}$$

The family of all sequences of value regions $\{V_n\}$ with respect to $\mathcal{F}$ is denoted by $\mathcal{V}(\mathcal{F})$. It is clear that

$$\{U_n(\mathcal{F})\}_{n=-1}^{\infty} \in \mathcal{V}(\mathcal{F}). \tag{2.6}$$

From our next result (Theorem 2.2) we see that $\{U_n(\mathcal{F})\}$ is the "smallest" sequence in $\mathcal{V}(\mathcal{F})$. We therefore call $\{U_n(F)\}$ the *best sequence of value regions with respect to* $\mathcal{F}$.

THEOREM 2.2. *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a non-empty family of LFASs. If $\{V_n\} \in \mathcal{V}(\mathcal{F})$, then*

$$U_n(\mathcal{F}) \subseteq V_n, \quad n = -1, 0, 1, 2, \ldots. \tag{2.7}$$

Proof. Let $\{V_j\} \in \mathcal{V}(\mathcal{F})$ and $n \in [-1, 0, 1, 2, \ldots]$ and $u \in U_n(\mathcal{F})$ be given. Then there exists a $G \in \mathcal{F}$ and an integer $m \geq 1$ such that u can be expressed by (2.3). If $m = 1$, then $u = t_{n+1}^G(w_{n+1}) \in V_n$ by (2.5a). If $m = 2$, then $u = t_{n+1}^G \circ t_{n+2}^G(w_{n+2}) \in t_{n+1}^G(V_{n+1})$ by (2.5a) and hence $u \in t_{n+1}^G(V_{n+1}) \subseteq V_n$, by (2.5b). Continuing in this manner one can show (by induction) that all expressions of the form (2.3) are in V_n. This proves (2.7). Q.E.D.

Some elementary but useful properties of value regions are summarized in our next result (Theorem 2.3). A proof is an immediate consequence of the above definitions and hence is omitted.

THEOREM 2.3. *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a non-empty family of LFASs. Then*

(A) *If $\{V_n\}_{n=-1}^{\infty}$ is a family of non-empty subsets of $\widehat{\mathbf{C}}$ such that*

$$w_n \in V_n, \quad n = 0, 1, 2, \ldots, \tag{2.8}$$

and (2.2b) holds, then $\{V_n\} \in \mathcal{V}(\mathcal{F})$.

(B) *If $\{V_n\} \in \mathcal{V}(\mathcal{F})$, then $\{c(V_n)\} \in \mathcal{V}(\mathcal{F})$, where $c(V_n)$ denotes the closure of V_n.*

(C) *If $\{V_n^{(\alpha)}\} \in \mathcal{V}(\mathcal{F})$ for all α in an index set A, then*

$$\left\{\bigcap_{\alpha \in A} V_n^{(\alpha)}\right\} \in \mathcal{V}(\mathcal{F}). \tag{2.9}$$

An approach for obtaining truncation error bounds by use of value regions is based on the following:

THEOREM 2.4. *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a given non-empty family of LFASs. Let $F \in \mathcal{F}$ converge to a finite value $v(F) = \lim v_n(F)$. Let $\{V_n\} \in \mathcal{V}(\mathcal{F})$ and let k be a non-negative integer such that*

$$w_n \in c(V_n), \quad n = k, k+1, k+2, \ldots. \tag{2.10}$$

Then

$$|v(F) - v_n(F)| \leq \operatorname{diam} T_n(F, c(V_n)), \quad n = k, k+1, k+2, \ldots. \tag{2.11}$$

REMARKS (Remarks to Theorem 2.4). Determination of truncation error bounds by use of Theorem 2.4 involves the following steps: (a) First we obtain a sequence $\{V_n\} \in \mathcal{V}(\mathcal{F})$ such that (2.10) holds for some $k \geq 0$. (b) Next we find a description of the set $T_n(F, c(V_n))$ such that its diameter ($\operatorname{diam} T_n(F, c(V_n))$) can be computed. Many examples that illustrate these steps are described in Sections 3 and 4.

Proof of Theorem 2.4. By Theorem 2.3(B), $\{c(V_n)\} \in \mathcal{V}(\mathcal{F})$. Thus an application of (2.5b) yields

$$\begin{aligned} &T_n(F, c(V_n)) = T_{n-1}(F, t_n^F(c(V_n))) \subseteq T_{n-1}(F, c(V_{n-1})), \\ &\quad n = 1, 2, 3, \ldots. \end{aligned} \tag{2.12}$$

Hence $\{T_n(F, c(V_n))\}$ is a nested sequence of non-empty closed subsets of $\widehat{\mathbf{C}}$. From this, (1.1c) and (2.10) we obtain, for all $n \geq k$ and $m \geq 0$,

$$\begin{aligned} v_{n+m}(F) &:= T_{n+m}(F, w_{n+m}) \\ &\in T_{n+m}(F, c(V_{n+m})) \subseteq T_n(F, c(V_n)), \end{aligned} \tag{2.13}$$

and hence

$$|v_{n+m}(F) - v_n(F)| \leq \operatorname{diam} T_n(F, c(V_n)), \quad m = 0, 1, 2, \ldots. \tag{2.14}$$

Assertion (2.11) follows from (2.14). Q.E.D.

We note in passing that many important convergence theorems for LFASs have been proved by first establishing (2.14) and then showing that $\lim_{n\to\infty} \operatorname{diam} T_n(F, c(V_n)) = 0$ (see, e.g., [37] and [45]). Every closed set that contains the set

$$[v_{n+m}(G) : G \in \mathcal{F}_n(F), \quad m \geq 0] \tag{2.15}$$

is called an *nth inclusion region for F with respect to $\mathcal{F}$*. We denote the family of all such regions by $I_n(F, \mathcal{F})$. Clearly $T_n(F, c(V_n)) \in I_n(F, \mathcal{F})$ for all $\{V_n\} \in \mathcal{V}(\mathcal{F})$ and $F \in \mathcal{F}$. Since

$$T_n(F, c(U_n(\mathcal{F}))) = c[v_{n+m}(G) : G \in \mathcal{F}_n(F), \quad m \geq 0], \tag{2.16}$$

$T_n(F, c(U_n(\mathcal{F})))$ is called the *best nth inclusion region for F with respect to $\mathcal{F}$*. Henrici and Pfluger [21] were the first to use inclusion regions in their development of truncation error bounds for S-fractions (see (1.23) and Section 3). In our next result (Theorem 2.5) we show that, subject to stated sufficient conditions, the best truncation error bound $\beta_n(F, \mathcal{F})$ (see (1.7)) can be expressed in terms of $T_n(F, c(U_n(\mathcal{F})))$.

THEOREM 2.5. *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a non-empty family of LFASs. Let $F \in \mathcal{F}$ be convergent to a finite value $v(F)$ and let n be a non-negative integer such that $T_n(F, c(U_n(\mathcal{F})))$ is bounded. Then*

$$\beta_n(F, \mathcal{F}) = \sup[|\lambda - v_n(F)| : \lambda \in T_n(F, c(U_n(\mathcal{F})))], \tag{2.17}$$

provided that at least one of the following conditions holds:

$$\textit{For } m \geq n+1, \quad w_m \in U_m(\mathcal{F}). \tag{2.18}$$

$$\textit{For } m \geq n+1, \quad w_m \in c(U_m(\mathcal{F})) \tag{2.19a}$$

and for every $k \geq 1$, there exists a sequence $\{G_j\}$ of finitely converging LFASs such that

$$G_j \in \mathcal{F}_n(F) \quad \textit{for } j \geq 1 \quad \textit{and} \quad w_{n+k} = \lim_{j\to\infty} v(G_j^{(n+k)}). \tag{2.19b}$$

Proof. In view of the definition of $\beta_n(F, \mathcal{F})$ in (1.7) and of (2.17) it suffices to show that, if the conditions of Theorem 2.5 hold, then

$$L_n(F, \mathcal{F}) = T_n(F, c(U_n(\mathcal{F}))), \tag{2.20}$$

where the nth limit region $L_n(F,\mathcal{F})$ for $\mathcal{F}_n(F)$ is defined by (1.6).

First we suppose that condition (a) *holds.* Let $\lambda \in T_n(F, U_n(\mathcal{F}))$ be given. Then by the definition of $U_n(\mathcal{F})$ in (2.1), there exists a $G_1 \in \mathcal{F}_n(F)$ and an integer $m_1 \geq n+1$ such that

$$\lambda = T_n(F, t_{n+1}^{G_1} \circ t_{n+2}^{G_1} \circ \cdots \circ t_{m_1}^{G_1}(w_{m_1})) = v_{m_1}(G_1).$$

Since by (2.18) $w_{m_1} \in U_{m_1}(\mathcal{F})$, there exists a $G_2 \in \mathcal{F}_{m_1}(G_1)$ and an $m_2 \geq m_1+1$ such that

$$w_{m_1} = t_{m_1+1}^{G_2} \circ t_{m_1+2}^{G_2} \circ \cdots \circ t_{m_2}^{G_2}(w_{m_2})$$

and hence $\lambda = v_{m_2}(G_2)$ and $w_{m_2} \in U_{m_2}(\mathcal{F})$. Continuing in this manner, we obtain a sequence $\{G_j\}$ of LFASs and a sequence of integers $\{m_j\}$ such that, for each $j \geq 1$,

$$\begin{aligned} &G_{j+1} \in \mathcal{F}_{m_j}(G_j), \quad m_{j+1} \geq m_j + 1, \\ &w_{m_j} = t_{m_j+1}^{G_{j+1}} \circ t_{m_j+2}^{G_{j+1}} \circ \cdots \circ t_{m_{j+1}}^{G_{j+1}}(w_{m_{j+1}}), \end{aligned} \tag{2.21a}$$

and hence

$$\lambda = v_{m_j}(G_j) \quad \text{for } j = 1,2,3,\ldots. \tag{2.21b}$$

From the definition of $\mathcal{F} = \mathcal{F}(\Omega, W)$ in (1.4) and from (2.21a) it follows that there exists a $G \in \mathcal{F}$ such that

$$G \in \mathcal{F}_{m_j}(G_j) \quad \text{for } j = 1,2,3,\ldots, \quad \text{and} \quad G \in \mathcal{F}_n(F). \tag{2.22}$$

Therefore by (2.21) and (2.22), $\lambda = v_{m_j}(G)$ for $j \geq 1$ so that

$$\lambda = \lim_{j\to\infty} v_{m_j}(G) \in L_n(F,\mathcal{F}).$$

We have shown that $T_n(F, U_n(\mathcal{F})) \subseteq L_n(F,\mathcal{F})$ and since $L_n(F,\mathcal{F})$ is a closed set we have

$$T_n(F, c(U_n(\mathcal{F}))) \subseteq L_n(F,\mathcal{F}). \tag{2.23}$$

To prove that the inclusion in (2.23) holds in the opposite direction, we let $\lambda \in \ell_n(F,\mathcal{F})$ be given. Then by the definition of $\ell_n(F,\mathcal{F})$ in (1.6a), there exists a $G \in \mathcal{F}_n(F)$ and a subsequence of natural numbers $\{m_j\}$ (with $m_1 \geq n+1$) such that

$$\lambda = \lim_{j\to\infty} v_{m_j}(G). \tag{2.24}$$

Therefore, for all $j \geq 1$,

$$v_{m_j}(G) := T_{m_j}(G, w_{m_j}) = T_n(F, t_{n+1}^G \circ \cdots \circ t_{m_j}^G(w_{m_j})) \in T_n(F, U_n(\mathcal{F})).$$

Hence

$$\lambda = \lim_{j \to \infty} v_{m_j}(G) \in T_n(F, c(U_n(\mathcal{F}))),$$

which shows that $\ell_n(F, \mathcal{F}) \subseteq T_n(F, c(U_n(\mathcal{F})))$ and so

$$L_n(F, \mathcal{F}) \subseteq T_n(F, c(U_n(\mathcal{F}))), \tag{2.25}$$

since the right side of (2.25) is a closed set. The equation (2.20) follows from (2.23) and (2.25).

Next we suppose that condition (b) *holds.* Let $\lambda \in T_n(F, U_n(\mathcal{F}))$ be given. Then by the definition of $U_n(\mathcal{F})$ in (2.1) and of $\mathcal{F}_n(F)$ in (1.5), there exists a $G \in \mathcal{F}_n(F)$ and a positive integer k such that

$$\lambda = T_n(F, t_{n+1}^G \circ t_{n+2}^G \circ \cdots \circ t_{n+k}^G(w_{n+k})). \tag{2.26}$$

By condition (b) there exists a sequence $\{G_j\}$ (depending upon n and k) such that for all $j = 1, 2, 3, \ldots$,

$$G_j \in \mathcal{F}_{n+k}(G) \quad \text{and} \quad w_{n+k} = \lim_{j \to \infty} v(G_j^{(n+k)}). \tag{2.27}$$

To verify the first relation in (2.27) we note that condition (b) of the hypothesis places no restrictions on the elements $\mathcal{C}_m(G_j)$ for $n+1 \leq m \leq n+k$. Hence we can set $\mathcal{C}_m(G_j) = \mathcal{C}_m(G)$ for $n+1 \leq m \leq n+k$ and $j \geq 1$. Therefore by (2.26) and (2.27)

$$\begin{aligned} \lambda &= T_{n+k}(G, w_{n+k}) = \lim_{j \to \infty} T_{n+k}(G, v(G_j^{(n+k)})) \\ &= \lim_{j \to \infty} T_{n+k}(G_j, v(G_j^{(n+k)})), \quad (\text{since } G_j \in \mathcal{F}_{n+k}(G) \subseteq \mathcal{F}_n(F)) \\ &= \lim_{j \to \infty} v(G_j), \quad \text{by (1.38).} \end{aligned}$$

It follows that

$$\lambda = \lim_{j \to \infty} v(G_j) \in L_n(F, \mathcal{F})$$

and hence

$$T_n(F, U_n(\mathcal{F})) \subseteq L_n(F, \mathcal{F}).$$

Since the right side of the last inclusion is a closed set, we obtain

$$T_n(F, c(U_n(\mathcal{F}))) \subseteq L_n(F, \mathcal{F}). \tag{2.28}$$

Finally, we note that $v(F) \in \ell_n(F, \mathcal{F})$ so that $\ell_n(F, \mathcal{F})$ is not empty. Let $\lambda \in \ell_n(F, \mathcal{F})$ be given. Then by definition of $\ell_n(F, \mathcal{F})$ in (1.6a), there exists a $G \in \mathcal{F}_n(F)$ and a subsequence $\{m_j\}$ of the natural numbers such that

$$\lambda = \lim_{j \to \infty} v_{m_j}(G). \tag{2.29}$$

Without loss of generality we may assume that $m_1 > n$ and let $k_j := m_j - n$, $j \geq 1$. From this and (2.29) it follows that

$$\lambda = \lim_{j \to \infty} T_n(F, t_{n+1}^G \circ t_{n+2}^G \circ \cdots \circ t_{n_{k_j}}^G (w_{n_{k_j}})) \in T_n(F, c(U_n(\mathcal{F}))),$$

since $t_{n+1}^G \circ \cdots \circ t_{n+k_j}^G (w_{n+k_j}) \in U_n(\mathcal{F})$ for all $j \geq 1$. Therefore $\ell_n(F, \mathcal{F}) \subseteq T_n(F, c(U_n(\mathcal{F})))$ and, since the right side of this inclusion is a closed set, we obtain

$$L_n(F, \mathcal{F}) \subseteq T_n(F, c(U_n(\mathcal{F}))). \tag{2.30}$$

The relations (2.28) and (2.30) imply (2.20) and this completes our proof. Q.E.D.

Our next result (Theorem 2.6) provides explicit and easily computable bounds for the truncation error $|v(F) - v_n(F)|$ when one has value regions V_n that are closed circular disks centered at the corresponding converging factors w_n. If in addition the hypotheses of Theorem 2.5 hold, then $V_n = c(U_n(\mathcal{F}))$ and hence the explicit error bound is the best bound $\beta_n(F, \mathcal{F})$.

THEOREM 2.6. *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a non-empty family of LFASs. Let $\{V_m\}$ be a sequence of value regions corresponding to $\mathcal{F}$ such that for some integer $k \geq 0$ and some sequence of positive numbers $\{\rho_m\}_{m=k}^{\infty}$,*

$$V_m := [u \in \mathbf{C} : |u - w_m| \leq \rho_m], \quad m = k, k+1, k+2, \ldots. \tag{2.31}$$

Let $F \in \mathcal{F}$ have a finite value $v(F)$ and let n be an integer such that $n \geq k$ and the nth inclusion region $T_n(F, V_n)$ is a closed circular disk (and hence bounded). Let $D_n(F)$ and $h_n(F)$ be defined as in (1.10) *and* (1.41), *respectively. Then:*

(A)

$$\begin{aligned} |v(F) - v_n(F)| &\leq \sup[|\lambda - v_n(V)| : \lambda \in T_n(F, V_n)] \\ &= \frac{\rho_n \prod_{j=0}^{n} |a_j(F) d_j(F) - b_j(F) c_j(F)|}{|D_n(F)|^2 \cdot |w_n + h_n(F)|(|w_n + h_n(F)| - \rho_n)}. \end{aligned} \tag{2.32}$$

(B) *If, in addition*

$$V_n = c(U_n(\mathcal{F})) \tag{2.33}$$

and the hypotheses of Theorem 2.5 hold, then the expression on the right side of (2.32) *is the best truncation error bound $\beta_n(F, \mathcal{F})$ for F with respect to $\mathcal{F}$.*

Proof. (A): By (1.41) $T_n(F, -h_n(F)) = \infty$. Therefore, since $T_n(F, V_n)$ is a bounded closed circular disk, we obtain

$$-h_n(F) \notin V_n.$$

Let $u_n \in \mathbf{C}$ denote the point of intersection of the circular boundary $\partial T_n(F, V_n)$ and the line segment $[w_n, -h_n(F)]$. From the defining relations for value regions (2.5) (see also the proof of Theorem 2.4) we see that $\{T_m(F, V_m)\}_{m=n}^{\infty}$ is a nested sequence of closed circular disks and, for $m = 0, 1, 2, \ldots$,

$$v_{n+m}(F) := T_{n+m}(F, w_{n+m}) \in T_{n+m}(F, V_{n+m}) \subseteq T_n(F, V_n).$$

Hence

$$v(F) = \lim_{m\to\infty} v_{n+m}(F) \in T_n(F, V_n).$$

Let $\lambda \in T_n(F, V_n)$ be given and let $u \in V_n$ be chosen so that $\lambda = T_n(F, u)$. Then by (1.9) we obtain

$$\begin{aligned} |\lambda - v_n(F)| &= |T_n(F, u) - T_n(F, w_n)| \\ &= \left| \frac{A_n(F) + C_n(F)u}{B_n(F) + D_n(F)u} - \frac{A_n(F) + C_n(F)w_n}{B_n(F) + D_n(F)w_n} \right| \\ &= \left| \frac{(w_n - u)(A_n(F)D_n(F) - B_n(F)C_n(F))}{(B_n(F) + D_n(F)u)(B_n(F) + D_n(F)w_n)} \right|. \end{aligned}$$

Using this with the determinant formulas (1.11) and $B_n(F) = h_n(F)D_n(F)$ from (1.42) yields

$$|\lambda - v_n(F)| = \frac{|w_n - u| \prod_{j=0}^{n} |a_j(F)d_j(F) - b_j(F)c_j(F)|}{|D_n(F)|^2 \cdot |u + h_n(F)| \cdot |w_n + h_n(F)|}. \tag{2.34}$$

It is readily seen that

$$\max_{u\in V_n} |w_n - u| = \rho_n \quad \text{and} \quad \min_{u\in V_n} |u + h_n(F)| = |w_n + h_n(F)| - \rho_n > 0, \tag{2.35}$$

where the extremum in both cases is attained with $u = u_n$. An application of (2.35) to (2.34) gives (2.32).

(B) follows immediately from part (A) proved above and Theorem 2.5. Q.E.D.

One can use Theorems 2.5 and 2.6 to obtain best truncation error bounds $\beta_n(F, \mathcal{F})$ by determining a simple explicit (geometrical or analytical) description of $c(U_n(\mathcal{F}))$ and of its image $T_n(F, c(U_n(\mathcal{F})))$. Applications of that kind are given in Sections 3 and 4 for a number of important special families $\mathcal{F}$ of LFASs (see (1.21) to (1.31)). For some of these applications, $c(U_n(\mathcal{F}))$ can be determined by direct elementary methods. For other families we have made use of the following:

THEOREM 2.7. *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a given non-empty family of LFASs, such that every $F \in \mathcal{F}$ and its tails $F^{(m)}$ converge to finite values $v(F^{(m)})$, $m \geq 0$. Let $\{V_n\}$ be a sequence of value regions with respect to $\mathcal{F}$ such that, for some integer $k \geq 0$,*

$$[t_n^F(V_n) : F \in \mathcal{F}] = V_{n-1}, \quad n = k+1, k+2, \ldots, \tag{2.36}$$

and such that

$$\lim_{n\to\infty} \left\{ \sup_{F\in\mathcal{F}} [\operatorname{diam} T_n(F^{(m)}, V_{n+m})] \right\} = 0, \quad m = k+1, k+2, \ldots. \tag{2.37}$$

Then

$$c(V_n) = c(U_n(\mathcal{F})), \quad n = k, k+1, k+2, \ldots. \tag{2.38}$$

Proof. By Theorem 2.2, $c(U_n(\mathcal{F})) \subseteq c(V_n)$ for $n \geq 0$. Thus it suffices to show that

$$c(V_n) \subseteq c(U_n(\mathcal{F})), \quad n = k, k+1, k+2, \ldots. \tag{2.39}$$

Let $n \geq k$ and $u_n \in V_n$ be given. We show that $u_n \in c(U_n(\mathcal{F}))$. From (2.36) there exists an $F_1 \in \mathcal{F}$ and a $u_{n+1} \in V_{n+1}$ such that $u_n = t_{n+1}^{F_1}(u_{n+1})$. Again by (2.36) there exists an $F_2 \in \mathcal{F}$ and a $u_{n+2} \in V_{n+2}$ such that $u_{n+1} = t_{n+2}^{F_2}(u_{n+2})$. Continuing in this manner we see that there exist sequences $\{u_j\}$ and $\{F_j\}$ such that, for $j = 1, 2, 3, \ldots$,

$$F_j \in \mathcal{F}, \quad u_{n+j-1} \in V_{n+j-1}, \quad \text{and} \quad u_{n+j-1} = t_{n+j}^{F_j}(u_{n+j}). \tag{2.40}$$

Let $F \in \mathcal{F}$ be defined by $\Gamma_{n+j}(F) := \Gamma_{n+j}(F_j)$ for $j \geq 1$. From this and (2.40) we have

$$u_{n+j-1} = t_{n+j}^F(u_{n+j}), \quad j = 1, 2, 3, \ldots,$$

and hence by (1.34), for all $m = 1, 2, 3, \ldots$,

$$u_n = t_{n+1}^F \circ t_{n+2}^F \circ \cdots \circ t_{n+m}^F(u_{n+m}) = T_m(F^{(n)}, u_{n+m}). \tag{2.41}$$

By (2.5) and (1.33)

$$\begin{aligned} T_m(F^{(n)}, V_{n+m}) &= T_{m-1}(F^{(n)}, t_{m+n}^F(V_{n+m})) \\ &\subseteq T_{m-1}(F^{(n)}, V_{n+m-1}), \end{aligned} \tag{2.42}$$

and hence $\{T_m(F^{(n)}, V_{m+n})\}_{m=1}^{\infty}$ is a nested sequence of non-empty subsets of $\widehat{\mathbb{C}}$. For $m \geq 1$ and $j \geq 1$, we obtain by (1.33) and (2.5)

$$\begin{aligned} v_{m+j}(F^{(n)}) &:= T_{m+j}(F^{(n)}, w_{n+m+j}) \\ &= T_{m+j-1}(F^{(n)}, t_{n+m+j}^F(w_{n+m+j})) \\ &\in T_{m+j-1}(F^{(n)}, V_{n+m+j-1}) \subseteq \cdots \subseteq T_m(F^{(n)}, V_{n+m}). \end{aligned} \tag{2.43}$$

Therefor

$$v(F^{(n)}) := \lim_{j\to\infty} v_{m+j}(F^{(n)}) \in T_n(F^{(n)}, c(V_{n+m})),\\ m = 1, 2, 3, \ldots. \tag{2.44a}$$

By (2.41) and the fact that $u_{n+m} \in V_{n+m}$ for $m \geq 0$, we have

$$u_n = T_n(F^{(n)}, u_{n+m}) \in T_m(F^{(n)}, V_{n+m}), \quad m = 1, 2, 3, \ldots. \tag{2.44b}$$

Thus we conclude from (2.43), (2.44) and the hypothesis (2.37) that

$$u_n = v(F^{(n)}). \tag{2.45}$$

We also have from the definition of $U_n(\mathcal{F})$ in (2.1) that

$$v_m(F^{(n)}) := T_m(F^{(n)}, w_{n+m}) \in U_n(\mathcal{F}) \tag{2.46}$$

so that

$$v(F^{(n)}) = \lim_{m\to\infty} v_m(F^{(n)}) \in c(U_n(\mathcal{F})). \tag{2.47}$$

Combining (2.45) with (2.47) yields

$$u_n \in c(U_n(\mathcal{F})).$$

We have shown that $V_n \subseteq c(U_n(\mathcal{F}))$, from which (2.38) follows. Q.E.D.

3. Special Families of LFASs with Simple Value Regions

In Sections 3 and 4 we apply Theorem 2.5 to obtain best truncation error bounds $\beta_n(F, \mathcal{F})$ for a number of important special families $\mathcal{F}$ of LFASs. Other truncation error bounds are included which, though not best, are sharp enough to be useful and are easy to compute. Sequences of value regions $V = \{V_j\}$ with respect to families $\mathcal{F}(\Omega, W)$ play an essential role in these two sections. The procedure used to determine families $\mathcal{F}(\Omega, W)$ and associated value regions $\{V_j\}$ is a generalization of an approach developed for continued fractions. It rests on the observation first made in [44] that, starting in a "natural" way with element regions $\{\Omega_j\}$ and converging factors $\{w_j\}$, it may be very difficult to find corresponding value regions $\{V_j\}$ (or $\{U_j(\mathcal{F})\}$ in (2.1)). A simpler approach is to start with sequences $\{V_j\}$ and $\{w_j\}$ and determine a corresponding sequence $\{\Omega_j\}$, which may lead to null sets $\Omega_j = \emptyset$, $j \geq 1$. One way of doing this for continued fractions (with $w_j = 0$, $j \geq 0$) was used in [49] and [52] for special cases, and then was formalized by Lane [42] for circular disks V_n and arbitrary continued fractions $K(a_n/b_n)$. We refer to the generalization of this procedure, described in Section 3.1, as the $VW\Omega$-method.

3.1. THE $VW\Omega$-METHOD

Starting with a sequence $V = \{V_j\}$ of non-empty subsets of $\widehat{\mathbf{C}}$ and a sequence of complex numbers $W = \{w_j\}$ satisfying

$$w_j \in V_j, \quad j = 0, 1, 2, \ldots, \tag{3.1}$$

we determine a sequence $\Omega = \{\Omega_j\}$ of subsets of $\mathbf{C}^4$ by

$$\Omega_j := [\Gamma_j := \langle a_j, b_j, c_j, d_j \rangle \in \mathbf{C}^4 : \frac{a_j + c_j V_j}{b_j + d_j V_j} \subseteq V_{j-1}], \quad j = 0, 1, 2, \ldots, \tag{3.2a}$$

with the restriction that

$$a_j d_j - b_j c_j \neq 0 \quad \text{for all } \Gamma_j = \langle a_j, b_j, c_j, d_j \rangle \in \Omega_j \tag{3.2b}$$

We call this procedure the *$VW\Omega$-method.* It follows from (1.1) and (2.5) that $\{V_j\}$ is a sequence of value regions with respect to the family $\mathcal{F}(\Omega, W)$ of LFASs (1.4) provided

$$\Omega_j \neq \emptyset, \quad j = 0, 1, 2, \ldots. \tag{3.3}$$

In practice, conditions (3.2b) are ensured by imposing special conditions for the generating sequence

$$t_j^F(w) := \frac{a_j + c_j w}{b_j + d_j w}, \quad j = 0, 1, 2, \ldots. \tag{3.4}$$

As an illustration of the above we start with

$$V_j := V_0 := [u \in \mathbf{C} : 0 \leq |u| \leq 1/2], \quad w_j := 0, \quad j = -1, 0, 1, \ldots, \tag{3.5a}$$

and generating functions of the form

$$t_0^F(w) := w, \quad t_j^F(w) := \frac{a_j}{1 + w}, \quad a_j \neq 0, \quad j = 0, 1, 2, \ldots. \tag{3.5b}$$

Then (3.1) and (3.2b) are satisfied and (3.2a) reduces to

$$\Omega_0 := [\Gamma_0 = \langle 0, 1, 1, 0 \rangle \in \mathbf{C}^4] \tag{3.6a}$$

and

$$\Omega_j := \left[\Gamma_j = \langle a_j, 1, 0, 1 \rangle \in \mathbf{C}^4 : \frac{a_j}{1 + V_0} \subseteq V_0\right], \quad j = 1, 2, 3, \ldots. \tag{3.6b}$$

It is readily shown from (3.5a) and (3.6b) that

$$\Omega_j = \Omega_1 = [\Gamma_j = \langle a_j, 1, 0, 1\rangle \in \mathbf{C}^4 : \ 0 < |a_j| \le 1/4], \quad j = 1, 2, 3, \ldots. \tag{3.7}$$

In the example described above $\mathcal{F} = \mathcal{F}(\Omega, W)$ is the family of all continued fractions (CFs)

$$F = \mathop{\mathrm{K}}_{j=1}^{\infty}\left(\frac{a_j}{1}\right) \quad \text{such that } a_j \in E = [a \in \mathbf{C} : \ 0 < |a_j| \le 1/4]. \tag{3.8}$$

In 1865 Julius Worpitzky [66] proved that all CFs (3.8) converge to finite values. The set E is therefore called a *simple convergence region* for CFs of the form $K(a_j/1)$ and this set E is the first known example of a convergence region for CFs (see [28] for a discussion of Worpitzky's contributions to CF theory and his times). Best truncation error bounds $\beta_n(F, \mathcal{F})$ for the family of CFs (3.8) are given in Section 3.2.

In most (but not all) of the special families $\mathcal{F}(\Omega, W)$ of LFASs that have been studied extensively, the determination of $\{\Omega_j\}$ defined by (3.2) is simplified (as in the preceding example) by holding constant all but one of the components in $\Gamma_j = \langle a_j, b_j, c_j, d_j\rangle \in \Omega_j$. The determination of $\{\Omega_j\}$ in (3.2) can be (and usually is) simplified further by choosing regions V_j whose boundaries are circles or lines in $\mathbf{C}$, or else intersections of such regions. An additional simplification is attained when $\{V_j\}$, $\{w_j\}$ and $\{\Omega_j\}$ are all constant sequences; that is,

$$V_j = V_0, \quad w_j = w_1, \quad \Omega_j = \Omega_1 \quad \text{for all } j \ge 1. \tag{3.9}$$

When this occurs we use the terms *simple value region* V_0 and *simple element region* Ω_1. In the following Section 3.2 we consider families $\mathcal{F}$ with simple value regions V_0 such that $c(V_0)$ is a closed circular disk and the corresponding Ω_1 is a simple element region.

3.2. FAMILIES OF LFASS WITH SIMPLE CIRCULAR DISK VALUE REGIONS

We begin this section with a result that is an immediate consequence of Theorem 2.6.

THEOREM 3.1. *Let* $\mathcal{F} = \mathcal{F}(\Omega, W)$ *be a family of LFAS continued fractions* (*CFs*)

$$\mathop{\mathrm{K}}_{j=1}^{\infty}\left(\frac{a_j}{b_j}\right) = \frac{a_1}{b_1} \+ \frac{a_2}{b_2} \+ \frac{a_3}{b_3} \+ \cdots, \quad (c_j = 0, d_j = 1, w_j = 0, \tag{3.10}$$

see (1.12). *Let* $\{V_j\}$ *be a sequence of value regions with respect to* $\mathcal{F}$ *such that for some integer* $k \ge 0$ *and sequence of positive numbers* $\{\rho_j\}_{j=k}^{\infty}$,

$$V_j = [u \in \mathbf{C} : \ |u| \le \rho_j], \quad j = k, k+1, k+2, \ldots.$$

Let $F \in \mathcal{F}$ have a finite value $v(F)$ and let n be a given positive integer with $n > k$. Let $v_n(F)$ and $h_n(F)$ be defined by (1.15) *and $h_n(F) := B_n(F)/B_{n-1}(F)$. If the nth inclusion region $T_n(F, V_n)$ is a bounded circular disk, then:*

(A)

$$|v(F) - v_n(F)| \le \sup_\lambda[|\lambda - v_n(F)| : \lambda \in T_n(F, V_n)]$$

$$= \frac{\rho_n \prod_{j=1}^{n} |a_j(F)|}{(|h_n(F)| - \rho_n) \cdot |B_n(F) B_{n-1}(F)|} \tag{3.11}$$

$$= \frac{\rho_n}{(|h_n(F)| - \rho_n)} |v_n(F) - v_{n-1}(F)|.$$

(B) *If, in addition,*

$$V_n = c(U_n(\mathcal{F})) \tag{3.12}$$

and the hypotheses of Theorem 2.5 hold, then the expressions on the right side of (3.11) *give the best truncation error bound $\beta_n(F, \mathcal{F})$ for F with respect to $\mathcal{F}$.*

A number of special families $\mathcal{F} = \mathcal{F}(\Omega, W)$ have best value regions $\{U_m(\mathcal{F})\}$ and converging factors $\{w_m\}$ satisfying

$$U_m(\mathcal{F}) = U_0(\mathcal{F}), \quad m = 0, 1, 2, \ldots, \tag{3.13a}$$

and

$$w_m = 0 \in c(U_0(\mathcal{F})) := [u \in \mathbf{C} : |u| \le \rho], \quad \rho > 0, \quad m \ge 0. \tag{3.13b}$$

We give results for four such families in this section.

3.2.1. Worpitzky Family $\mathcal{F}^{W(\rho)}$

For $0 < \rho \le \frac{1}{2}$, we call

$$\mathcal{F}^{W(\rho)} := \left[\operatorname*{K}_{j=1}^{\infty} (a_j/1) : a_j \in \mathbf{C}, \quad 0 < |a_j| \le \rho(1 - \rho), \quad j \ge 1 \right] \tag{3.14}$$

the *ρ-Worpitzky family of* LFASs. Since $0 < \rho(1 - \rho) \le 1/4$, it follows from Worpitzky's convergence region result (see (3.8)) that every $F \in \mathcal{F}^{W(\rho)}$ has a finite value $v(F)$. The family $\mathcal{F}^{W(\rho)}(\Omega, W)$ has element regions

$$\Omega_0 := \langle 0, 1, 1, 0 \rangle \quad \text{so that} \quad t_0^F(w) \equiv w \quad \text{for all } F \in \mathcal{F}^{W(\rho)}, \tag{3.15a}$$

and

$$\Omega_j = \Omega_1 = \langle E_a(\rho), 1, 0, 1 \rangle, \quad \text{where } E_a(\rho) := [u \in \mathbf{C} : 0 < |u| \le \rho(1 - \rho)], \tag{3.15b}$$

and converging factors $w_j = 0$, $j \ge 0$. Our results for $\mathcal{F}^{W(\rho)}$ are summarized by the following:

THEOREM 3.2 ($\mathcal{F}^{W(\rho)}$). *Let ρ satisfying $0 < \rho \leq 1/2$ be given. Then:*
(A) *$\mathcal{F}^{W(\rho)}$ has a simple best value region*

$$U_m(\mathcal{F}^{W(\rho)}) = U_0(\mathcal{F}^{W(\rho)}) = [u \in \mathbf{C} : 0 < |u| < \rho], \quad m = -1, 0, 1, 2, \ldots. \tag{3.16}$$

(B) *For each $F \in \mathcal{F}^{W(\rho)}$ and each positive integer n, the best truncation error bound $\beta_n(F, \mathcal{F}^{W(\rho)})$ for $v_n((F)$ with respect to $\mathcal{F}^{W(\rho)}$ is given by*

$$\begin{aligned}\beta_n(F, \mathcal{F}^{W(\rho)}) &= \frac{\rho \prod_{j=1}^{n} |a_j(F)|}{(|h_n(F)| - \rho) \cdot |B_n(F) B_{n-1}(F)|} \\ &= \frac{\rho}{(|h_n(F)| - \rho)} |v_n(F) - v_{n-1}(F)|.\end{aligned} \tag{3.17}$$

(C) *If, in addition, $0 < \rho < 1/2$, then*

$$|v(F) - v_n(F)| \leq \left(\frac{2\rho}{1 - 2\rho}\right) |v_n(F) - v_{n-1}(F)|, \quad n = 2, 3, 4, \ldots. \tag{3.18}$$

REMARKS . It follows from (3.16) and the definition of $U_m(\mathcal{F}^{W(\rho)})$ in (2.1) that, if $0 < \rho \leq 1/2$, then

$$|v(F)| \leq \rho \quad \text{for all } F \in \mathcal{F}^{W(\rho)}.$$

Proof of Theorem 3.2. (A): Let $\{V_j(\rho)\}$ be defined by

$$V_j(\rho) := V_0(\rho) := [u \in \mathbf{C} : 0 < |u| < \rho], \quad j = -1, 0, 1, 2, \ldots. \tag{3.19}$$

To prove (A) it suffices to show that

$$\{V_j(\rho)\} \in \mathcal{V}(\mathcal{F}^{W(\rho)}) \quad \text{and} \quad V_0(\rho) \subseteq U_0(\mathcal{F}^{W(\rho)}). \tag{3.20}$$

We prove $\{V_j(\rho)\} \in \mathcal{V}(\mathcal{F}^{W(\rho)})$ by verifying that conditions (2.5) hold. Condition (2.5a) is an immediate consequence of (3.14), (3.19) and $\rho(1-\rho) < \rho$. Condition (2.5b) (with $n \geq 1$) is equivalent to

$$\frac{1 + V_0(\rho)}{a} \subseteq \frac{1}{V_0(\rho)} \quad \text{for all } a \in E_a(\rho) := [z \in E : 0 < |z| \leq \rho(1-\rho)],$$

which can be readily proven. To show that $V_0(\rho) \subseteq U_0(\mathcal{F}^{W(\rho)})$ we let u denote an arbitrary point in V_0. For each $n \geq 0$, let g_n denote the nth approximant of the CF

$$1 + K\left(\frac{-\rho(1-\rho)}{1}\right) = 1 - \frac{\rho(1-\rho)}{1} + \frac{-\rho(1-\rho)}{1} + \frac{-\rho(1-\rho)}{1} + \cdots, \tag{3.21}$$

so that

$$g_0 := 1 \quad \text{and} \quad g_n := 1 - \frac{\rho(1-\rho)}{g_{n-1}}, \quad \text{for } n = 1, 2, 3, \ldots. \tag{3.22}$$

We now prove (by induction) that

$$\frac{1}{2} \leq 1 - \rho < g_n < g_{n-1} \leq 1, \quad n = 1, 2, 3, \ldots. \tag{3.23}$$

Since $g_0 := 1$ and $g_1 = 1 - \rho(1-\rho)$, one can see that (3.23) holds for $n = 1$. As our induction hypothesis we assume that

$$1 - \rho < g_k < g_{k-1} < 1, \quad k = 2, 3, \ldots, n-1, \tag{3.24}$$

for some positive integer n. Then $1 - \rho < g_{n-1}$ implies

$$g_n = 1 - \frac{\rho(1-\rho)}{g_{n-1}} > 1 - \frac{\rho(1-\rho)}{1-\rho} = 1 - \rho. \tag{3.25}$$

Furthermore,

$$\begin{aligned} g_{n-1} - g_n &= g_{n-1} - \left(1 - \frac{\rho(1-\rho)}{g_{n-1}}\right) \\ &= \frac{\rho(1-\rho) - g_{n-1}(1-g_{n-1})}{g_{n-1}} > 0 \end{aligned} \tag{3.26}$$

iff

$$g_{n-1}(1 - g_{n-1}) < \rho(1-\rho). \tag{3.27}$$

This inequality holds since $\frac{1}{2} \leq (1-\rho) < g_{n-1} < 1$ and $f(x) := x(1-x)$ is decreasing on the interval $\frac{1}{2} \leq x \leq 1$. We have established (3.23). Worpitzky's theorem ensures that the CF (3.21) converges to a finite value $g = \lim_{n\to\infty} g_n$. Therefore from the recurrence relations (3.22) we see that g satisfies the quadratic equation

$$g = 1 - \frac{\rho(1-\rho)}{g},$$

whose roots are ρ and $(1-\rho)$. From this and (3.23) we conclude that $\{g_n\}_{n=0}^{\infty}$ decreases monotonically to the limit g, with

$$\frac{1}{2} \leq g = 1 - \rho < 1. \tag{3.28}$$

Let $\varepsilon := \rho - |u|$, $\varepsilon_n := g_n - (1-\rho)$, $n \geq 0$, and let $n_0 \geq 1$ be chosen so that

$$\varepsilon_{n_0} \leq \frac{\varepsilon(1-\rho)}{\rho - \varepsilon}. \tag{3.29}$$

We then define a by

$$|a| := (\rho - \varepsilon)[(1-\rho) + \varepsilon_{n_0}] \quad \text{and} \quad \arg a := \arg u. \tag{3.30}$$

It follows from (3.30) that

$$\frac{|a|}{g_{n_0}} = |u| \quad \text{and} \quad |a| = \rho(1-\rho) - [\varepsilon(1-\rho) - \varepsilon_n(\rho - \varepsilon)] \le \rho(1-\rho),$$

and hence by (2.1)

$$u = \frac{a}{g_{n_0}} \in U_0(\mathcal{F}^{W(\rho)}).$$

This completes the proof of (A).

(B): It follows from conditions (2.2), that $\{T_n(F, U_0(\mathcal{F}^{W(\rho)}))\}$ is a nested sequence of subsets of $\mathbf{C}$ if $F \in \mathcal{F}^{W(\rho)}$. Therefore since for $n \ge 1$,

$$T_n(F, c(U_0(\mathcal{F}^{W(\rho)}))) \subseteq \cdots \subseteq T_1(F, c(U_0(\mathcal{F}^{W(\rho)}))) \subseteq c(U_o(\mathcal{F}^{W(\rho)})),$$

we see that $T_n(F, c(U_0(\mathcal{F}^{W(\rho)})))$ is a bounded, closed circular disk. We wish to apply Theorem 3.1(B). For that purpose it suffices to verify that condition (b) of Theorem 2.5 holds. Let $n \ge 1$ and $k \ge 1$ be given. Then for each $j \ge 1$, we define a CF $G_j \in \mathcal{F}^{W(\rho)}$ as follows:

$$a_m(G_j) := a_m(F) \quad \text{for } m = 1, 2, \ldots, n+k,$$

$$a_{n+k+1}(G_j) := \frac{1}{j} \quad \text{and} \quad a_m(G_j) = -\rho(1-\rho) \quad \text{for } m \ge n+k+2.$$

Condition (b) of Theorem 2.5 follows from the fact that the CF (3.21) has value $1-\rho$, and

$$\lim_{j\to\infty} v(G_j^{(n+k)}) = \lim_{j\to\infty} \left(\frac{1/j}{1-\rho}\right) = 0 =: w_{n+k}.$$

Assertion (B) follows then from Theorem 3.1(B).

(C) follows immediately from (B) and the fact that $T_n(F, -h_n(F)) = \infty$, so that $-h_n(F) \notin c(V_0(1/4)) = c(U_0(\mathcal{F}^{W(1/4)}))$ and hence $|h_n(F)| > 1/2$. Q.E.D.

3.2.2. Pringsheim–Śleszyński Family $\mathcal{F}^{PS(\rho)}$

For $0 < \rho \le 1$, we call

$$\mathcal{F}^{PS(\rho)} := \left[\operatorname*{K}_{j=1}^{\infty} \left(\frac{1}{b_j}\right) : b_j \in \mathbf{C}, \quad \rho + \frac{1}{\rho} \le |b_j| < \infty \right] \tag{3.31}$$

the ρ-Pringsheim–Śleszyński family of LFASs. Since $\rho + (1/\rho) \geq 2$, it follows from the Pringsheim–Śleszyński criterion (see, e.g., [37], Theorem 4.35 and [59]) that every $F \in \mathcal{F}^{PS(\rho)}$ has a finite value $v(F)$. The family $\mathcal{F}^{PS(\rho)}$ has element regions

$$\Omega_0 := \langle 0, 1, 1, 0 \rangle \quad \text{so that} \quad t_0^F(w) \equiv w \quad \text{for all } F \in \mathcal{F}^{PS(\rho)} \tag{3.32a}$$

and, for $j \geq 1$,

$$\begin{aligned} &\Omega_j := \Omega_1 := \langle 1, E_b(\rho), 0, 1 \rangle, \\ &\quad \text{where} \quad E_b(\rho) := \left[u \in \mathbf{C} :\ \rho + \frac{1}{\rho} \leq |u| < \infty \right], \end{aligned} \tag{3.32b}$$

and converging factors $w_j = 0$, $j \geq 0$. Our results for $\mathcal{F}^{PS(\rho)}$ are summarized in the following:

THEOREM 3.3 ($\mathcal{F}^{PS(\rho)}$). *Let ρ satisfying $0 < \rho \leq 1$ be given. Then:*

(A) *$\mathcal{F}^{PS(\rho)}$ has a simple best value region $U_0(\mathcal{F}^{PS(\rho)})$ satisfying*

$$c(U_0(\mathcal{F}^{PS(\rho)})) = [u \in \mathbf{C} :\ 0 \leq |u| \leq \rho]. \tag{3.33}$$

(B) *For each $F \in \mathcal{F}^{PS(\rho)}$ and each positive integer n, the best truncation error bound $\beta_n(F, \mathcal{F}^{PS(\rho)})$ for $v_n(F)$ with respect to $\mathcal{F}^{PS(\rho)}$ is given by*

$$\begin{aligned} \beta_n(F, \mathcal{F}^{PS(\rho)}) &= \frac{\rho}{(|h_n(F)| - \rho) \cdot |B_n(F) B_{n-1}(F)|} \\ &= \frac{\rho}{(|h_n(F)| - \rho)} |v_n(F) - v_{n-1}(F)|. \end{aligned} \tag{3.34}$$

(C) *If, in addition, $0 < \rho < 1$, then*

$$|v(F) - v_n(F)| \leq \frac{\rho}{1 - \rho} |v_n(F) - v_{n-1}(F)|, \quad n = 2, 3, 4, \ldots. \tag{3.35}$$

REMARK . It follows from (3.33) and the definition of $U_0(\mathcal{F}^{PS(\rho)})$ in (2.1) that, if $0 < \rho \leq 1$, then

$$|v(F)| \leq \rho \quad \text{for } F \in \mathcal{F}^{PS(\rho)}. \tag{3.36}$$

Proof. (A): Let $\{V_j(\rho)\}$ be defined by

$$V_j(\rho) := V_0(\rho) := [u \in \mathbf{C} :\ 0 \leq |u| \leq \rho], \quad j = -1, 0, 1, 2, \ldots. \tag{3.37}$$

To prove (A) it suffices to show that

$$\{V_j(\rho)\} \in \mathcal{V}(\mathcal{F}^{PS(\rho)}) \quad \text{and} \quad V_0(\rho) = c(U_0(\mathcal{F}^{PS(\rho)})). \tag{3.38}$$

We prove $\{V_j(\rho)\} \in \mathcal{V}(\mathcal{F}^{PS(\rho)})$ by verifying conditions (2.5). Condition (2.5a) follows directly from (3.31), (3.37) and $\rho + (1/\rho) > \rho$. Condition (2.5b) is equivalent to

$$\begin{gathered} b + V_0(\rho) \subseteq \frac{1}{V_0(\rho)} \\ \text{for all } b \in E_b(\rho) := \left[u \in \mathbf{C} : \ \rho + \tfrac{1}{\rho} \leq |u| < \infty\right], \end{gathered} \tag{3.39}$$

which can be readily shown. To prove that

$$V_0(\rho) = c(U_0(\mathcal{F}^{PS(\rho)})) \tag{3.40}$$

we make use of Theorem 2.7. First we show that

$$[t_n^F(V_0(\rho)) : \ F \in \mathcal{F}^{PS(\rho)}] = V_0(\rho). \tag{3.41}$$

In view of (2.5b) it suffices to verify

$$V_0(\rho) \subseteq [t_n^F(V_0(\rho)) : \ F \in \mathcal{F}^{PS(\rho)}] \tag{3.42}$$

or, equivalently, $V_0(\rho) \subseteq 1/(E_b(\rho) + V_0(\rho))$; that is,

$$\frac{1}{V_0(\rho)} \subseteq E_b(\rho) + V_0(\rho). \tag{3.43}$$

Let $v \in 1/V_0(\rho)$ be given and let $\varphi := \arg v$, so that $1/\rho \leq |v| < \infty$ and $0 \leq \varphi < 2\pi$.

Let b and u be defined by

$$|b| := |v| + \rho, \quad \arg b := \varphi, \quad u := -\rho e^{i\varphi}.$$

It follows from this that

$$b + u = (|v| + \rho)e^{i\varphi} - \rho e^{i\varphi} = v, \quad b \in E_b(\rho) \quad \text{and} \quad u \in V_0(\rho).$$

This proves (3.43) and hence also (3.41). Condition (2.37) can be written

$$\lim_{n\to\infty} \left\{ \sup_{F\in\mathcal{F}^{PS(\rho)}} [\operatorname{diam} T_n(F^{(m)}, V_0(\rho))] \right\} = 0. \tag{3.44}$$

This is an immediate consequence of a theorem due to Hillam (see, e.g., [12], Theorem 2.7; [22]). Thus (A) follows from Theorem 2.7.

(B): We apply Theorem 3.1(B). For that purpose we note that $T_n(F, V_0(\rho))$ is a bounded circular disk, since by (2.2) $\{T_n(F, V_0(\rho))\}$ is a nested sequence of closed disks and $T_n(F, V_0(\rho)) \subseteq T_{n-1}(F, V_0(\rho)) \subseteq \cdots \subseteq V_0(\rho)$. It suffices to verify that condition (b) of Theorem 2.5 is satisfied. Let $n \geq 1$ and $k \geq 1$ be given. Then for each $j \geq 1$, we define a CF $G_j \in \mathcal{F}_{n+k}^{PS(\rho)}(F)$ as follows:

$$\begin{aligned} &b_m(G_j) := b_m(F), \quad m = 1, 2, \ldots, n+k, \\ &b_{n+k+1}(G_j) := j, \\ &b_m(G_j) := \rho + 1/\rho, \quad m = n+k+2, \quad n+k+3, \ldots. \end{aligned}$$

Then

$$\lim_{j\to\infty} v(G_j^{(n+k)}) = \lim_{j\to\infty} \frac{1}{j + v(G_j^{(n+k+1)})} = O =: w_{n+k},$$

since $v(G_j^{(n+k+1)})$ is the value of the periodic CF $K\left(1/(\rho + \frac{1}{\rho})\right)$ and hence, by (3.36), $|v(G_j^{(n+k+1)})| \leq \rho$.

Therefore (B) follows from Theorem 3.1(B).

(C) follows from Theorem 3.1(A) and the fact that $T_n(F, -h_n(F)) = \infty$, so that $-h_n(F) \notin V_0\left(\frac{1}{2}\right) = [u \in \mathbf{C} : 0 \leq |u| \leq 1]$; hence $|h_n(F)| > 1$. Q.E.D.

3.2.3. Positive Perron–Carathéodory Family $\mathcal{F}^{PPC(z)}$.

Let

$$z \in D := [u \in \mathbf{C} : 0 \leq |u| < 1] \tag{3.45}$$

be given. We define the family $\mathcal{F}^{PPC(z)}$ of LFASs $F = C[\{\delta_j\}, z]$, called *positive Perron–Carathéodory approximant sequences*, as follows:

$$\mathcal{F}^{PPC(z)} := \left[\text{LFASs } F : t_j^F(w) = \frac{a_j + c_j w}{b_j + d_j w}, \quad j = 0, 1, 2, \ldots\right], \tag{3.46a}$$

where the generating sequences $\{t_j^F(w)\}_{j=0}^{\infty}$ have the form

$$t_0^F(w) := \delta_0 \frac{1-w}{1+w}, \quad t_j^F(w) := z\frac{\bar{\delta}_j + w}{1 + \delta_j w}, \quad j = 1, 2, 3, \ldots, \tag{3.46b}$$

where

$$\delta_0 > 0 \quad \text{and} \quad \delta_j \in D, \quad j = 1, 2, 3, \ldots, \tag{3.46c}$$

and the converging factors $w_j = 0$, for $j \geq 0$. To emphasize the dependence of the δ_j on F we may write $\delta_j(F)$. Each $F \in \mathcal{F}^{PPC(z)}$ is related to the *positive Perron–Carathéodory CF (PPC-fraction)*

$$\delta_0 - \frac{2\delta_0}{1} \; \frac{1}{+\bar{\delta}_1 z +} \; \frac{(1-|\delta_1|^2)z}{\delta_1} \; \frac{1}{+\bar{\delta}_2 z +} \; \frac{(1-|\delta_2|^2)z}{\delta_2} \; + \cdots \tag{3.47}$$

in the following way: We define sequences $\{s_n^F(w)\}$ and $\{S_n(F,w)\}$ by

$$s_0^F(w) := \delta_0 + w, \quad s_{2j}^F(w) := \frac{1}{\bar{\delta}_j z + w}, \quad j = 1,2,3,\ldots \tag{3.48a}$$

$$s_1^F(w) := \frac{-2\delta_0}{1+w}, \quad s_{2j+1}^F(w) := \frac{(1-|\delta_j|^2)z}{\delta_j + w}, \quad j = 1,2,3,\ldots, \tag{3.48b}$$

$$S_0(F,w) := s_0^F(w), \quad S_n(F,w) := S_{n-1}(F, s_n^F(w)), \quad n = 1,2,3,\ldots, \tag{3.48c}$$

and we let $P_n(F,z)$ and $Q_n(F,z)$ denote the nth numerator and denominator, respectively, of the CF (3.47). It follows that

$$S_n(F,w) = \frac{P_n(F,z) + wP_{n-1}(F,z)}{Q_n(F,z) + wQ_{n-1}(F,z)}, \quad n = 1,2,3,\ldots, \tag{3.49}$$

$$t_0^F(w) = s_0^F \circ s_1^F(w^{-1}), \quad t_j^F(w) = [s_{2j}^F \circ s_{2j+1}^F(w^{-1})]^{-1}, \quad j = 1,2,3,\ldots, \tag{3.50a}$$

$$T_n(F,w) = S_{2n+1}(F, w^{-1}) = \frac{P_{2n+1}(F,z)w + P_{2n}(F,z)}{Q_{2n+1}(F,z)w + Q_{2n}(F,z)}, \quad n = 0,1,2,\ldots. \tag{3.50b}$$

Therefore, for $n = 0,1,2,\ldots$,

$$v_n(G) := T_n(F,0) = \frac{P_{2n}(F,z)}{Q_{2n}(F,z)}, \tag{3.51}$$

and

$$A_n(F) = P_{2n}(F,z), \quad B_n(F) = Q_{2n}(F,z), \quad C_n(F) = P_{2n+1}(F,z), \quad D_n(F) = Q_{2n+1}(F,z),$$

where A_n, B_n, C_n, D_n are defined by the difference equations (1.10).

The class $\mathcal{C}$ of *normalized Carathéodory functions* is defined by

$$\mathcal{C} := [f : f \quad \text{is analytic and } \operatorname{Re} f(z) > 0 \text{ for } |z| < 1, \quad f(0) > 0]. \tag{3.52}$$

It can be seen that all functions of the form

$$f(z) = \sum_{j=1}^{n} \lambda_j \frac{e^{i\theta_j} + z}{e^{i\theta_j} - z}, \quad \lambda_j > 0 \quad \text{for } 1 \le j \le n, -\pi < \theta_1 < \theta_2 < \cdots < \theta_n = \pi \tag{3.53}$$

are in $\mathcal{C}$. We consider the subclass $\mathcal{C}_c$ of $\mathcal{C}$ defined by

$$\mathcal{C}_c := [f \in \mathcal{C} : f \quad \text{is not of the form (3.53) and } f \text{ is not constant}]. \tag{3.54}$$

For each $F \in \mathcal{F}^{PPC(z)}$, we let $v(F(z))$ denote the value of F considered as a function of z. In [33], Theorem 10.2, it was shown that,

$$F(z) \in \mathcal{F}^{PPC(z)} \Rightarrow f(z) := v(F(z)) \in \mathcal{C}_c,$$

and, conversely, $f(z) \in \mathcal{C}_c$ implies that there exists a unique $F(z) \in \mathcal{F}^{PPC(z)}$ such that $f(z) = v(F(z))$. The following result (Theorem 3.4) gives best truncation error bounds for $v_n(F(z))$.

THEOREM 3.4 ($\mathcal{F}^{PPC(z)}$). *Let $z \in \mathbf{C}$, satisfying $0 < |z| < 1$ be given. Then:*

(A) *The family of LFASs $\mathcal{F}^{PPC(z)}$ has a sequence of best value regions given by*

$$U_{-1}(\mathcal{F}^{PPC(z)}) = \bigcup_{\delta_0 > 0} \Delta(\delta_0), \tag{3.55a}$$

where

$$\begin{aligned}\Delta(\delta_0) &:= [u \in \mathbf{C} : \ |u - \Gamma(\delta_0)| < R(\delta_0)], \quad \Gamma(\delta_0) := \delta_0 \tfrac{1+|z|^2}{1-|z|^2}, \\ R(\delta_0) &:= \tfrac{2\delta_0 |z|}{1-|z|^2},\end{aligned} \tag{3.55b}$$

and

$$U_m(\mathcal{F}^{PPC(z)}) := [u \in \mathbf{C} : \ 0 \le |u| < |z|], \quad m = 0, 1, 2, \ldots. \tag{3.55c}$$

(B) *For each $F \in \mathcal{F}^{PPC(z)}$ and each integer $n \ge 1$, the best truncation error bound for $v_n(F)$ with respect to $\mathcal{F}^{PPC(z)}$ is given by*

$$\beta_n(F, \mathcal{F}^{PPC(z)}) = \frac{2\delta_0 \prod_{j=1}^{n} (1 - |\delta_j|^2) z^{n+1}}{|Q_{2n}(F, z)|(|Q_{2n}(F, z)| - |zQ_{2n+1}(F, z)|)}, \tag{3.56}$$

where Q_{2n} and Q_{2n+1} are defined by (3.49) *and* (3.52).

(C) *For each $F \in \mathcal{F}^{PPC(z)}$ and integer $n \ge 1$*

$$|v(F) - v_n(F)| \le \frac{4\delta_0 |z|^{n+1}}{1 - |z|^2}. \tag{3.57}$$

REMARKS . (1) We have omitted the point $z = 0$ in Theorem 3.4, since, if $z = 0$, $v_n(F(0)) = T_n(F(0), 0) = \delta_0$ for all $n \ge 0$, and hence $v(F(0)) = \delta_0$. (2) $\delta_0 \in \Delta(\delta_0)$, since $\delta_0 > \delta_0(1 - |z|) = \Gamma(\delta_0) - R(\delta_0) > 0$.

Proof. (A): Let $\{V_j\}$ be defined by

$$V_{-1} := \bigcup_{\delta_0 > 0} \Delta(\delta_0), \quad \Delta(\delta_0) \quad \text{defined by (3.55b)}, \tag{3.58a}$$

$$V_j := [u \in \mathbf{C} : \ 0 \le |u| < |z|], \quad j = 0, 1, 2, \ldots. \tag{3.58b}$$

By (3.55) and (3.58)

$$t_0^F(0) = \delta_0(F) \in \Delta(\delta_0) = t_0^F(V_0) \subseteq V_{-1}, \quad \text{for all } \delta_0 > 0. \tag{3.59a}$$

Therefore (2.5) holds for $n = 0$. For all $n \geq 1$,

$$t_n^F(0) = \bar{\delta}_n(F)z \in V_{n-1} \quad \text{for all } 0 \leq |\delta_n(F)| < 1, \tag{3.60a}$$

and

$$t_n^F(V_n) = [u \in \mathbf{C} : \ |u - \Gamma_n| < R_n] \subseteq V_{n-1} \quad \text{for all } F \in \mathcal{F}^{PPC(z)}, \tag{3.60b}$$

since

$$|\Gamma_n| + R_n < |z| < 1 \tag{3.60c}$$

where

$$\Gamma_n := \frac{z\bar{\delta}_n(1 - |z|^2)}{1 - |z|^2|\delta_n|^2}, \quad R_n := \frac{|z|^2(1 - |\delta_n|^2)}{1 - |z|^2|\delta_n|^2} \tag{3.60d}$$

(see, e.g., [39], Lemma 3.2, for more details on proof of (3.60c)).

It follows from (3.60) that (2.5) holds for all $n \geq 1$. Therefore

$$\{V_j\} \in \mathcal{V}(\mathcal{F}^{PPC(z)}). \tag{3.61}$$

We now show that

$$V_j \subseteq U_j(\mathcal{F}^{PPC(z)}), \quad j = -1, 0, 1, 2, \ldots. \tag{3.62}$$

In fact, for $j \geq 0$,

$$\begin{aligned} V_j = V_0 &:= [u \in \mathbf{C} : \ 0 \leq |u| < |z|] = [\bar{\delta}_1 z : \ 0 \leq |\delta_1| < 1] \\ &= [t_1^F(0) : \ F \in \mathcal{F}^{PPC(z)}] \subseteq U_0(\mathcal{F}^{PPC(z)}). \end{aligned}$$

and, since $t_0^F(V_0) = \Delta(\delta_0(F))$, we have

$$\begin{aligned} V_{-1} &:= [\Delta(\delta_0(F)) : \ F \in \mathcal{F}^{PPC(z)}] = [t_0^F(V_0) : \ F \in \mathcal{F}^{PPC(z)}] \\ &\subseteq U_{-1}(\mathcal{F}^{PPC(z)}). \end{aligned}$$

This proves (3.62) and hence (A).

(B): Since the conditions of Theorem 2.6(B) hold (with condition (a) of Theorem 2.5), we have for $n \geq 1$

$$\begin{aligned} \beta_n(F, \mathcal{F}^{PPC(z)}) &= \frac{|z| \prod_{j=0}^{n} |a_j(F)d_j(F) - b_j(F)c_j(F)|}{|D_n(F)|^2 |h_n(F)|(|h_n(F)| - |z|)} \\ &= \frac{2\delta_0 \prod_{j=1}^{n}(1 - |\delta_j|^2)|z|^{n+1}}{|B_n(F)|(|B_n(F)| - |zD_n(F)|)} \end{aligned}$$

which gives (3.56), using (3.52) and

$$a_0(F) = \delta_0, \quad b_0(F) = 1, \quad c_0(F) = -\delta_0, \quad d_0(F) = 1,$$
$$a_j(F) = \bar{\delta}_j z, \quad b_j(F) = 1, \quad c_j(F) = z, \quad d_j(F) = \delta_j, \quad j = 1, 2, 3, \ldots.$$

This proves (B).

(C): Our proof of (3.57) makes use of Theorem 2.4. In [39], Lemma 3.3, we obtain

$$\operatorname{diam} T_n(F, V_n) = \frac{4\delta_0 \prod_{j=1}^{n}(1 - |\delta_j|^2)|z|^{n+1}}{|Q_{2n}(F, z)|^2 - |zQ_{2n+1}(F, z)|^2}, \quad n = 1, 2, 3, \ldots. \tag{3.63}$$

Using Christoffel–Darboux formulas derived in [40, Section 2], we obtain the inequality

$$|Q_{2n}(F, z)|^2 - |zQ_{2n+1}(F, z)|^2 \geq (1 - |z|^2) \prod_{j=1}^{n}(1 - |\delta_j|^2), \quad n = 1, 2, 3, \ldots. \tag{3.64}$$

Combining (3.63) and (3.64) with Theorem 2.4 yields

$$|v(F) - v_n(F)| \leq \frac{4\delta_0 \prod_{j=1}^{n}(1 - |\delta_j|^2)|z|^{n+1}}{|Q_{2n}(F, z)|^2 - |zQ_{2n+1}(F, z)|^2} \tag{3.65}$$

and hence (3.57). Q.E.D.

REMARK . One can readily show that $\beta_n(F, \mathcal{F}^{PPC(z)})$ is at least as small as the bound given by (3.65). In fact, that statement holds iff

$$|Q_{2n}(F, z)|^2 - |zQ_{2n+1}(F, z)|^2 \leq 2|Q_{2n}(F, z)|(|Q_{2n}(F, z)| - |zQ_{2n+1}(F, z)|). \tag{3.66}$$

Dividing both sides of (3.66) by $|Q_{2n}(F, z)|^2$ and rearranging terms, we obtain the following inequality that is equivalent to (3.66):

$$\left(1 - \left|z \frac{Q_{2n+1}(F, z)}{Q_{2n}(F, z)}\right|\right)^2 \geq 0. \tag{3.67}$$

3.2.4. Positive Schur Family $\mathcal{F}^{Sh(z)}$.

Let

$$z \in D := [u \in \mathbf{C} : 0 \leq |u| < 1] \tag{3.68}$$

be given. We define the family $\mathcal{F}^{Sh(z)}$ of LFASs $F = S[\{\gamma_j\}, z]$, called *positive Schur approximant* sequences, as follows:

$$\mathcal{F}^{Sh(z)} := \left[\text{LFASs } F : \; t_j^F(w) = \frac{\gamma_j + zw}{1 + \bar{\gamma}_j zw}, \quad j = 0, 1, 2, \ldots \right], \tag{3.69a}$$

where

$$\gamma_0 \in \mathbf{R}, \quad |\gamma_0| < 1 \quad \text{and} \quad \gamma_j \in \mathbf{C}, \quad |\gamma_j| < 1, \quad j = 1, 2, 3, \ldots, \tag{3.69b}$$

and converging factors $w_j := 0$, $j = 0, 1, 2, \ldots$. To emphasize dependence of γ_j on F we may write $\gamma_j(F)$. Each $F \in \mathcal{F}^{Sh(z)}$ is related to the *positive Schur CF*

$$\gamma_0 + \frac{(1 - |\gamma_0|^2)z}{\bar{\gamma}_0 z} \, \frac{1}{+ \gamma_1 +} \, \frac{(1 - |\gamma_1|^2)z}{\bar{\gamma}_1 z} \, \frac{1}{+ \gamma_2 + \cdots}, \tag{3.70}$$

in the following way: We define sequences $\{s_n^F(w)\}$ and $\{S_n(F, w)\}$ by

$$s_0^F(w) := \gamma_0 + w, \quad s_{2j}^F(w) := \frac{1}{\gamma_j + w}, \quad j = 1, 2, 3, \ldots, \tag{3.71a}$$

$$s_{2j+1}^F(w) := \frac{(1 - |\gamma_j|^2)z}{\bar{\gamma}_j z + w}, \quad j = 0, 1, 2, \ldots, \tag{3.71b}$$

$$\begin{gathered} S_0(F, w) := s_0^F(w), \quad S_n(F, w) := S_{n-1}(F, s_n^F(w)), \\ n = 1, 2, 3, \ldots, \end{gathered} \tag{3.71c}$$

and let $P_n(F, z)$ and $Q_n(F, z)$ denote the nth numerator and denominator, respectively, of the CF (3.70). It follows that

$$S_n(F, w) = \frac{P_n(F, z) + wP_{n-1}(F, z)}{Q_n(F, z) + wQ_{n-1}(F, z)}, \quad n = 0, 1, 2, \ldots, \tag{3.72}$$

$$\begin{gathered} t_0^F(w) := s_0^F s_1^F(w^{-1}), \\ t_j^F(w) := [s_{2j}^F \circ s_{2j+1}^F(w^{-1})]^{-1}, \quad j = 1, 2, 3, \ldots, \end{gathered} \tag{3.73a}$$

$$\begin{gathered} T_n(F, w) = S_{2n+1}(F, w^{-1}) = \frac{P_{2n+1}(F, z)w + P_{2n}(F, z)}{Q_{2n+1}(F, z)w + Q_{2m}(F, z)}, \\ n = 0, 1, 2, \ldots. \end{gathered} \tag{3.73b}$$

Therefore, for $n = 0, 1, 2, \ldots$,

$$v_n(F) := T_n(F, 0) = S_{2n+1}(F, \infty) = \frac{P_{2n}(F, z)}{Q_{2n}(F, z)}, \tag{3.74}$$

and

$$\begin{aligned} A_n(F) &= P_{2n}(F,z), \quad B_n(F) = Q_{2n}(F,z), \\ C_n(F) &= P_{2n+1}(F,z), \quad D_n(F) = Q_{2n+1}(F,z), \end{aligned} \tag{3.75}$$

where A_n, B_n, C_n, D_n are defined by the difference equations (1.10). The class $\mathcal{S}$ of *normalized Schur functions* is defined by

$$\begin{aligned} \mathcal{S} := [f : \quad & f \text{ is analytic and } |f(z)| \leq 1 \text{ for } |z| < 1, \\ & -1 < f(0) < 1]. \end{aligned} \tag{3.76}$$

It can be seen that all functions of the form

$$\begin{gathered} f(z) = \varepsilon \prod_{j=1}^{n} \frac{z + \omega_j}{1 + \bar{\omega}_j z} \quad |\omega_j| < 1, \\ j = 1, 2, \ldots, n, \quad |\varepsilon| = 1, \quad \varepsilon \prod_{j=1}^{n} \omega_j \in \mathbf{R} \end{gathered} \tag{3.77}$$

are members of $\mathcal{S}$. We consider the subclass $\mathcal{S}_c$ of $\mathcal{S}$ defined by

$$\mathcal{S}_c := [f \in \mathcal{S} : f \quad \text{is not of the form (3.77) and } f \text{ is not constant}]. \tag{3.78}$$

For each $F \in \mathcal{F}^{Sh(z)}$, we let $v(F(z))$ denote the value of F considered as a function of z. In [50] and [32] it is shown that

$$F \in \mathcal{F}^{Sh(z)} \Rightarrow v(F(z)) \in \mathcal{S}_c$$

and, conversely, $f(z) \in \mathcal{S}_c$ implies that there exists a unique $F \in \mathcal{F}^{Sh(z)}$ such that $f(z) = v(F(z))$. The following result (Theorem 3.5) gives best truncation error bounds for $v_n(F(z))$ with respect to $\mathcal{F}^{Sh(z)}$.

THEOREM 3.5 ($\mathcal{F}^{Sh(z)}$). *Let $z \in D := [u \in \mathbf{C} : 0 \leq |u| < 1]$ be given. Then*

(A) *The family of LFASs $\mathcal{F}^{Sh(z)}$ has a sequence of best value regions given by*

$$\begin{aligned} U_{-1}(\mathcal{F}^{Sh(z)}) &= \bigcup_{-1<\gamma_0<1} t_0^F(D) \\ &= \bigcup_{-1<\gamma_0<1} [u \in \mathbf{C} : |u - c(\gamma_0)| < r(\gamma_0)] \\ &\subseteq D \end{aligned} \tag{3.79a}$$

where

$$c(\gamma_0) := \frac{\gamma_0(1 - |z|^2)}{1 - \gamma_0^2|z|^2}, \quad r(\gamma_0) := \frac{(1 - \gamma_0^2)|z|}{1 - \gamma_0^2|z|^2}, \tag{3.79b}$$

and

$$U_m(\mathcal{F}^{Sh(z)}) = D, \quad m = 0, 1, 2, \ldots. \tag{3.79c}$$

(B) *For each* $F \in \mathcal{F}^{Sh(z)}$ *and each integer* $n \geq 0$, *the best truncation error bound for* $v_n(F)$ *with respect to* $\mathcal{F}^{Sh(z)}$ *is given by*

$$\beta_n(F, \mathcal{F}^{Sh(z)}) = \frac{|z|^{n+1} \prod_{j=0}^{n} (1 - |\gamma_j|^2)}{|Q_{2n}(F, z)| \cdot (|Q_{2n}(F, z)| - |Q_{2n+1}(F, z)|)}. \tag{3.80}$$

Proof. (A): A proof of (3.79c) can be found in [31], Lemma 7. The first equality in (3.79a) follows from the definition of U_{-1} in (2.1) and from (3.79c). The second equality in (3.79a) follows from elementary conformal mapping of D by the linear fractional transformation $t_0^F(w)$.

(B) follows immediately from Theorem 2.6(B), since $w_m = 0 \in U_m(\mathcal{F}^{Sh(z)})$, $a_j(F) = \gamma_j$. $b_j(F) = 1$, $c_j(F) = z$, $d_j(F) = \bar{\gamma}_j z$, $h_n(F) = B_n(F)/D_n(F)$, $B_n(F) = Q_{2n}(F, z)$ and $D_n(F) = Q_{2n+1}(F, z)$. Q.E.D.

REMARK . A proof of Theorem 3.5 was given in [31], Theorem 10, using essentially the same methods as employed in Theorem 2.6(B).

3.3. FAMILIES OF LFASS WITH OTHER SIMPLE VALUE REGIONS

In this section we obtain best truncation error bounds for Real J-Fractions, Stieltjes Fractions, Modified Stieltjes Fractions, and Positive T-Fractions. For each of these families of LFASs (CFs), the best value regions are simple and they are half-planes or intersections of half-planes with part or none of the boundaries included. We make use of the following:

THEOREM 3.6. *Let* $\mathcal{F} = \mathcal{F}(\Omega, \{0\})$ *be a family of LFASs of continued fractions (CFs). Let* $\{U_j(\mathcal{F})\}$ *denote the best sequence of value regions corresponding to* $\mathcal{F}$. *Let* $F = \mathrm{K}_{j=1}^{\infty}(a_j/b_j) \in \mathcal{F}$ *be convergent to a finite value* $v(F)$, *let* n *be a positive integer such that* $T_n(F, c(U_n(\mathcal{F})))$ *is bounded and let condition* (a) *or* (b) *of Theorem 2.5 hold. Then*

$$\beta_n(F, \mathcal{F}) = \frac{|v_n(F) - v_{n-1}(F)|}{|h_n(F)| \inf \left[\left| \frac{-1}{h_n(F)} - \frac{1}{u} \right| : u \in c(U_n(\mathcal{F})) \right]}. \tag{3.81}$$

Proof. From Theorem 2.5,

$$\beta_n(F, \mathcal{F}) = \sup[|T_n(F, u) - v_n(F)| : u \in c(U_n(\mathcal{F}))]. \tag{3.82}$$

From (1.1c), (1.9) and (1.11) we have

$$\begin{aligned}|T_n(F,u) - v_n(F)| &= |T_n(F,u) - T_n(F,0)| \\ &= \left|\frac{A_n + uA_{n-1}}{B_n + uB_{n-1}} - \frac{A_n}{B_n}\right| \\ &= \frac{|v_n(F) - v_{n-1}(F)|}{|h_n|\left|\frac{1}{-h_n} - \frac{1}{u}\right|}.\end{aligned} \tag{3.83}$$

From (3.82) and (3.83) we obtain (3.81). Q.E.D.

To apply Theorem 3.6 to a CF $K(a_j/b_j)$, we make use of specific information on the location of $h_n(F)$. From (1.42) and (1.10c), we have

$$h_n(F) = b_n + \frac{a_n}{h_{n-1}(F)} = b_n + \frac{a_n}{b_{n-1}} + \frac{a_{n-1}}{b_{n-2}} + \cdots + \frac{a_2}{b_1}, \quad n = 1, 2, 3, \ldots. \tag{3.84}$$

For each of the following special families of LFASs (CFs), we use information about the value regions to obtain information about the location of $\{h_n(F)\}$, and then we apply geometric arguments to determine

$$\inf\left[\left|\frac{-1}{h_n(F)} - \frac{1}{u}\right| : u \in c(U_n(\mathcal{F}))\right].$$

3.3.1. Real J-Fractions $\mathcal{F}^{J(z)}$.

Let $z \in \mathbf{C} - \mathbf{R}$ be given. The family $\mathcal{F}^{J(z)}$ of *real J-fractions* is defined by

$$\begin{aligned}\mathcal{F}^{J(z)} &:= \mathcal{F}(\Omega, W) \\ &= \left[F : F = \frac{1}{\beta_1 + z} + \frac{-\alpha_1^2}{\beta_2 + z} + \frac{-\alpha_2^2}{\beta_3 + z} + \cdots, \right. \\ &\qquad \left. 0 \neq \alpha_j \in \mathbf{R}, \beta_j \in \mathbf{R}, j \geq 1\right],\end{aligned} \tag{3.85}$$

where $\Omega = \{\Omega_j\}$ and $W = \{w_j\} = \{0\}$,

$$\Omega_1 := \langle 1, [\beta_1 + z : \beta_1 \in \mathbf{R}], 0, 1\rangle \tag{3.86a}$$

$$\Omega_j := \langle [-\alpha^2 : 0 \neq \alpha \in \mathbf{R}], [\beta + z : \beta \in \mathbf{R}], 0, 1\rangle, \quad j = 2, 3, 4, \ldots. \tag{3.86b}$$

The generating sequence $\{t_j^F(w)\}$ for $F \in \mathcal{F}^{J(z)}$ is given by

$$\begin{aligned}&t_0^F(w) := w, \quad t_1^F(w) := \frac{1}{\beta_1 + z + w}, \\ &t_j^F(w) := \frac{-\alpha_{j-1}^2}{\beta_j + z + w}, \quad j = 2, 3, 4, \ldots.\end{aligned} \tag{3.87}$$

THEOREM 3.7. *Let $z \in \mathbf{C} - \mathbf{R}$ be given and let $\mathcal{F}^{J(z)}$ denote the family of real J-fractions* (3.85). *Then:*

(A) *The best sequence of value regions $\{U_n(\mathcal{F}^{J(z)})\}_{n=0}^{\infty}$ with respect to $\mathcal{F}^{J(z)}$ is given by*

$$U_0(\mathcal{F}^{J(z)}) = \left[u \in \mathbf{C} : \left|u + \frac{i}{2\text{Im}\, z}\right| \leq \frac{1}{2|\text{Im}\, z|}, \quad u \neq 0\right], \tag{3.88a}$$

$$U_n(\mathcal{F}^{J(z)}) = V(z) := \begin{cases} H^+ := [u \in \mathbf{C} : \text{Im}\, u > 0], \text{ if } z \in H^+, \\ H^- := [u \in \mathbf{C} : \text{Im}\, u < 0], \text{ if } z \in H^-, \end{cases} \tag{3.88b}$$

$n = 1, 2, 3, \ldots.$

(B) *If $F = F(z) \in \mathcal{F}^{J(z)}$ converges to a finite value $v(F)$ and if n is a positive integer, then the best truncation error bound $\beta_n(F, \mathcal{F}^{J(z)})$ for $v_n(F)$ with respect to $\mathcal{F}^{J(z)}$ is given by*

$$\beta_n(F, \mathcal{F}^{J(z)}) = \frac{|h_n(F(z))|}{|\text{Im}\, h(F(z))|}|v_n(F(z)) - v_{n-1}(F(z))|. \tag{3.89}$$

Proof. (A) follows directly from the definition of $\{U_n(\mathcal{F}^{J(z)})\}$ in (2.1) (see also Theorem 9 in [37]).

(B): We show that condition (b) of Theorem 2.5 holds and then apply Theorem 3.6. From (3.88) it is clear that

$$w_m := 0 \in c(U_m(\mathcal{F}^{J(z)}), \quad \text{for } m = 0, 1, 2, \ldots.$$

Let k be a given positive integer. We then define $\{G_j\}$, for $j \geq 1$, by

$$\begin{aligned} G_j := &\frac{1}{\beta_1(F) + z +} \frac{-\alpha_1^2(F)}{\beta_2(F) + z +} \cdots + \frac{-\alpha_{n+k-1}^2(F)}{\beta_{n+k}(F) + z +} \\ &\frac{-1}{j + z +} \frac{-1}{z +} \frac{-1}{z +} \frac{-1}{z +} \cdots. \end{aligned} \tag{3.90}$$

The periodic CF

$$H := \frac{-1}{z} + \frac{-1}{z} + \frac{-1}{z} + \cdots$$

converges to a finite value $v(H)$ satisfying

$$\left|v(H) - \frac{i}{2\text{Im}\, z}\right| \leq \frac{1}{2|\text{Im}\, z|}.$$

It follows that each G_j converges to a value $v(G_j) \in \mathbf{C}$ and $|z + v(H)| > |\text{Im}\, z| > 0$. Hence

$$|v(G_j^{(n+k)})| = \left|\frac{-1}{j + z + v(H)}\right| \leq \frac{1}{j - |z + v(H)|} \leq \frac{1}{j - |\text{Im}\, z|},$$

for sufficiently large j.

Therefore $\lim_{j\to\infty} v(G_j^{(n+k)}) = 0 = w_m$ for all $m \geq 1$, and so condition (b) of Theorem 2.5 holds. Next we show that $T_n(F, U_n(\mathcal{F}^{J(z)})) = T_n(F, V(z))$ is bounded. By (2.2)

$$T_n(F, V(z)) = T_{n-1}(F, t_n^F(V(z))) \subseteq T_{n-1}(F, V(z)) \subseteq \cdots \\ \subseteq T_1(F, V(z)) \subseteq U_0(F^{J(z)}).$$

From (3.88a), $U_0(\mathcal{F}^{J(z)})$ is a bounded set. By (3.84) and (2.56) we obtain

$$-\frac{1}{h_n(F)} \in \frac{1}{-\beta_n - z - V(z)}$$

and hence

$$-\frac{1}{h_n(F)} \in \left[0 \neq w \in \mathbf{C} : \left|w - \frac{i}{2\mathrm{Im}\, z}\right| \leq \frac{1}{2|\mathrm{Im}\, z|}\right].$$

It follows from this and (3.81) that (3.89) holds. Q.E.D.

3.3.2. Stieltjes Fractions $\mathcal{F}^{St(z)}$.

Let $z \in S_\pi := [u \in \mathbf{C} : 0 \leq |\arg u| < \pi]$ be given. We define the *family $\mathcal{F}^{St(z)}$ of Stieltjes Fractions* by

$$\mathcal{F}^{St(z))} := \mathcal{F}(\Omega, W) = \left[F(z) : F(z) = K\left(\frac{a_j z}{1}\right), \right. \\ \left. a_j > 0, \quad j \geq 1\right], \tag{3.93a}$$

where

$$\Omega = \{\Omega_j\}_{j=0}^{\infty}, \quad W = \{w_j\}_{j=0}^{\infty} = \{0\}, \tag{3.93b}$$

$$\Omega_0 := \langle 0, 1, 1, 0\rangle, \\ \Omega_j := \langle [a_j z : a_j > 0], 1, 0, 1\rangle, \quad j = 1, 2, 3, \ldots. \tag{3.93c}$$

The generating sequence $\{t_j^F(w)\}$ for $F = F(z) = K(a_j z/1) \in \mathcal{F}^{St(z)}$ is given by

$$t_0^F(w) := w, \quad t_j^F(w) := \frac{a_j(F) z}{1 + w}, \quad j = 1, 2, 3, \ldots. \tag{3.94}$$

THEOREM 3.8. *Let $\mathcal{F}^{St(z)}$ be the family of Stieltjes fractions* (3.93) *for a given $z \in S_\pi$. Then:*

(A) *The best sequence of value regions* $\{U_n(\mathcal{F}^{St(z)})\}_{m=0}^{\infty}$ *with respect to* $\mathcal{F}^{St(z)}$ *is given, for* $m \geq 0$, *by*

$$U_m(\mathcal{F}^{St(z)}) = U(z) := \begin{cases} [u \in \mathbf{C} : \ 0 < \arg u \leq \arg z], \\ \quad \text{if } 0 < \arg z < \pi, \\ [u \in \mathbf{C} : \ \arg z \leq \arg u < 0], \\ \quad \text{if } -\pi < \arg z < 0, \\ [u \in \mathbf{C} : \ \arg u = 0], \\ \quad \text{if } \arg z = 0. \end{cases} \tag{3.95}$$

(B) *Let* $F = F(z) := K(a_j z/1) \in \mathcal{F}^{St(z)}$ *converge to a finite value* $v(F)$ *and let* n *be a given positive integer. Then the best truncation error bound* $\beta_n(F, \mathcal{F}^{St(z)})$ *for* $v_n(F)$ *with respect to* $\mathcal{F}^{St(z)}$ *is given by*

$$\beta_n(F, \mathcal{F}^{St(z)}) = K_n(F(z))|v_n(F(z)) - v_{n-1}(F(z))|, \tag{3.96a}$$

where $K_n(F(z))$ *is defined as follows:*

Case (a) $(0 \leq |\arg z| \leq \pi/2)$

$$K_n(F(z)) = 1, \quad \text{if } 0 \leq |\arg z| \leq \tfrac{\pi}{2}. \tag{3.96b}$$

Case (b) *Suppose that* $\pi/2 < |\arg z| < \pi$.

(b_1)

$$K_n(F(z)) = \frac{|h_n(F)|}{|\mathrm{Im}\,(h_n(F))|}, \quad \text{if } 0 < \left|\arg\left(\tfrac{-1}{h_n(F)}\right)\right| \leq \tfrac{\pi}{2}. \tag{3.96c}$$

(b_2)

$$K_n(F(z)) = 1, \quad \text{if } \tfrac{\pi}{2} < \left|\arg\left(-\tfrac{1}{h_n(F)}\right)\right| \leq \tfrac{3\pi}{2} - |\arg z|. \tag{3.96d}$$

(b_3)

$$K_n(F(z)) = |\csc\beta| \quad \text{if } \tfrac{3\pi}{2} - |\arg z| < \left|\arg\left(-\tfrac{1}{h_n(F)}\right)\right| < \pi, \tag{3.96e}$$

where

$$\beta := 2\pi - |\arg z| - \left|\arg\left(-\frac{1}{h_n(F)}\right)\right|. \tag{3.96f}$$

Proof. (A): Suppose $0 < \arg z < \pi$. Let $\{V_j\}$ be defined by (see (3.95))

$$V_j := V := V(z) := [u \in \mathbf{C} : \ 0 < \arg u \leq \arg z], \quad j = 0, 1, 2, \ldots.$$

Let

$$H_1 := [u \in \mathbf{C} : \ \mathrm{Im}\, u > 0], \quad H_2 := [u \in \mathbf{C} : \ \arg z - \pi \leq \arg u \leq \arg z],$$

so that $V = H_1 \cap H_2$. It follows that

$$t_j^G(H_1) \subseteq H_2 \quad \text{and} \quad t_j^G(H_2) \subseteq H_1$$
$$\text{for all } G \in \mathcal{F}^{St(z)}, \quad j = 1,2,3,\ldots,$$

$$t_j^G(V) = t_j^G(H_1) \cap t_j^G(H_2) \subseteq H_1 \cap H_2 = V, \quad j = 0,1,3,\ldots,$$

and

$$t_j^G(0) = a_j(G)z \in V, \quad j = 1,2,3,\ldots.$$

We have shown that $\{V_j\}_{j=0}^{\infty}$ is a sequence of value regions with respect to $\mathcal{F}^{St(z)}$. By an elementary geometrical argument one can show that

$$V = \left[\frac{a_1 z}{1 + a_2 z} : a_1 > 0, \quad a_2 > 0\right] = [t_0^G \circ t_1^G \circ t_2^G(0) : G \in \mathcal{F}^{St(z)}].$$

Therefore $V = V_j = U_j(\mathcal{F}^{St(z)})$, $j = 0,1,2,\ldots$. A similar argument holds for $-\pi < \arg z < 0$ and for $\arg z = 0$. This proves (A).

(B) To apply Theorem 3.6, we verify that condition (b) of Theorem 2.5 holds. Let k be a given positive integer. Let $\{G_j\}_{j=1}^{\infty}$ be defined by

$$G_j := \frac{a_1(F)z}{1} + \frac{a_2(F)z}{1} + \cdots + \frac{a_{n+k}(F)z}{1} + \frac{(1/j)z}{1} + \frac{z}{1} + \frac{z}{1} + \frac{z}{1} + \cdots.$$

It follows that

$$G_j \in \mathcal{F}_{n+k}(F) \quad \text{for } j \geq 1 \quad \text{and} \quad w_{n+k} = 0 = \lim_{j\to\infty} v(G_j^{(n+k)}).$$

Moreover, since $w_m = 0 \in c(U_m(\mathcal{F}^{St(z)}))$ for $m \geq n+1$, condition (b) of Theorem 2.5 holds. We also note that $T_n(F, U(z))$ is a bounded set, since

$$T_n(F, U(z)) \subseteq T_{n-1}(F, U(z)) \subseteq \cdots \subseteq T_1(F, U(z)) = t_1^F(U(z)),$$

and the set $t_1^F(U(z))$ is the intersection of a circular disk and a half-plane, provided $0 < |\arg z| < \pi$.

If $\arg z = 0$, then $z > 0$ and

$$t_1^F(U(z)) = \left[\frac{a_1 z}{1 + u} : 0 < u < \infty\right] = [x \in \mathbf{R}^+ : 0 < x < a_1 z]$$

is bounded. Therefore by Theorem 3.6, $\beta_n(F, \mathcal{F}^{St(z)})$ is given by (3.81). It remains to find estimates for

$$\inf\left[\left(\frac{-1}{h_n(F)} - \frac{1}{u}\right) : u \in U(z)\right].$$

By (3.84)

$$h_n(F) = 1 + \frac{a_n z}{1} + \frac{a_{n-1} z}{1} + \cdots + \frac{a_2 z}{1}.$$

It follows from this and $t_j^G(U(z)) \subseteq U(z)$ for $j \geq 1$ and $G \in \mathcal{F}^{St(z)}$, that

$$h_n(F) \in 1 + U(z) \quad \text{and so} \quad \frac{-1}{h_n(F)} \in \frac{-1}{1+U(z)}.$$

We consider cases for which $0 < \arg z < \pi$. (Similar arguments hold for $-\pi < \arg z < 0$ and $\arg z = 0$ and hence they are omitted.) One can readily show that

$$\frac{1}{U(z)} = [u \in \mathbf{C} : \ \arg z \leq \arg u < 0]$$

and that $-1/(1+U(z))$ is a region in $\mathbf{C}$ bounded by the interval $-1 < u < 0$ and by the circular arc passing through -1 and 0, tangent at $u = -1$ to the line with angle of inclination equal to $\arg z$.

Case (a) If $0 < \arg z \leq \pi/2$, then

$$\inf\left[\left|\left(\frac{-1}{h_n(F)}\right) - \frac{1}{u}\right| : \ u \in U(z)\right] = \frac{1}{|h_n(F)|}$$

and hence (3.96b) follows from (3.81).

Case (b) Suppose that $\pi/2 < \arg z < \pi$.

(b_1) If $0 < |\arg(-1/h_n(F))| \leq \pi/2$, then

$$\inf\left[\left|\left(\frac{-1}{h_n(F)}\right) - \frac{1}{u}\right| : \ u \in U(z)\right] = \left|\operatorname{Im}\left(\frac{-1}{h_n(F)}\right)\right| = \frac{|\operatorname{Im} h_n(F)|}{|h_n(F)|^2}.$$

Hence (3.96c) follows from (3.81).

(b_2) If $\pi/2 < |\arg(-1/h_n(F))| < (3\pi/2) - \arg z$, then

$$\inf\left[\left|\left(\frac{-1}{h_n(F)}\right) - \frac{1}{u}\right| : \ u \in U(z)\right] = \frac{1}{|h_n((F)|},$$

and so (3.96d) follows from (3.81).

(b_3) If $(3\pi/2) - \arg z < \arg(= -1/h_n(F)) < \pi$, then

$$\inf\left[\left|\left(\frac{-1}{h_n(F)}\right) - \frac{1}{u}\right| : \ u \in U(z)\right] = \frac{\left|\sin\left[2\pi - \arg z - \arg\left(\frac{-1}{h_n(F)}\right)\right]\right|}{|h_n(F)|}$$

and hence (3.96e) follows from (3.81). For the computations used to obtain b_2 and b_3, we have used the fact that the ray $\arg u = (3\pi/2) - \arg z$ is perpendicular to the line passing through the ray $\arg u = -\arg z$. Q.E.D.

We state without proof the following useful result originally given by Henrici and Pfluger [21] (see also Theorem 4.4 in [39]).

THEOREM 3.9. ([21]; [39, Theorem 4.4]) *If* $F(z) = K(a_j z/1)$, $a_j > 0$, $j \geq 1$ *is an S-fraction converging to a finite value* $v(F(z))$, *then, for* $n \geq 2$,

$$|v(F(z)) - v_n(F(z))| \leq \begin{cases} |v_n(F(z)) - v_{n-1}(F(z))|, \\ \quad \text{if } |\arg z| \leq \pi/2, \\ \csc(|\arg z|)|v_n(F(z)) - v_{n-1}(F(z))|, \\ \quad \text{if } \pi/2 < |\arg z| < \pi. \end{cases}$$

3.3.3. Adjusted Stieltjes Fractions $\mathcal{F}^{ASt(z)}$.

Let $z \in \mathbf{C}$ be given with $|\operatorname{Arg} z| < \pi/2$. We define

$$\mathcal{F}^{ASt(z)} := (\Omega, \{0\}_{n=1}^{\infty}), \tag{3.97}$$

where

$$\Omega = \{\Omega_n\}_{n=1}^{\infty} := \{\langle 1, [u \in \mathbf{C} : \arg u = \arg z], 0, 1\rangle\}_{n=1}^{\infty}. \tag{3.98}$$

Then

$$\mathcal{F}^{ASt(z)} = \left[\mathop{\mathrm{K}}_{j=1}^{\infty} \left(\frac{1}{b_j z} \right) : b_j > 0 \right] \tag{3.99}$$

is called the family of *Adjusted Stieltjes fractions.*

THEOREM 3.10. *Let* $\mathcal{F}^{ASt(z)}$ *be the family of LFASs defined by* (3.97)–(3.99). *Then*

(A) *The best sequence of value regions* $\{U_j(\mathcal{F}^{ASt(z)})\}_{n=0}^{\infty}$ *with respect to* $\mathcal{F}^{ASt(z)}$ *satisfies*

$$\begin{aligned} c(U_j(\mathcal{F}^{ASt(z)})) &:= c(U(\mathcal{F}^{ASt(z)})) \\ &= V := [u \in \mathbf{C} : 0 < |\operatorname{Arg} u| \leq |\operatorname{Arg} z|], \end{aligned} \tag{3.100}$$

$j = 0, 1, 2, \ldots.$

(B) *If* $F = F(z) \in \mathcal{F}^{ASt(z)}$ *converges to a finite value* $v(F(z)) \in \mathbf{C}$ *and if* n *is a positive integer, then the best truncation error bound for* $v_n(F(z))$ *with respect to* $\mathcal{F}^{ASt(z)}$ *is given by*

$$\beta_n(F, \mathcal{F}^{ASt(z)}) = K_n(F(z))|v_n(F) - v_{n-1}(F)|, \tag{3.101a}$$

where

$$K_n(F(z)) = \begin{cases} 1, \quad \text{if } |\arg z| \leq \left|\arg\left(\frac{-1}{h_n(F)}\right)\right| - \frac{\pi}{2}, \\ \csc\left(\left|\arg\left(\frac{-1}{h_n(F)}\right)\right| - |\arg z|\right), \\ \quad \text{if } \left|\arg\left(\frac{-1}{h_n(F)}\right)\right| - \frac{\pi}{2} < |\arg z|. \end{cases} \tag{3.101b}$$

A proof of this theorem can be given that is very similar to that of Theorem 3.8 and hence it is omitted.

3.3.4. Positive T-Fractions: $\mathcal{F}^{T(z)}$.

Let $z \in S_\pi := [u \in \mathbf{C} : 0 \leq |\arg z| < \pi]$ be given. We define

$$\mathcal{F}^{T(z)} := \mathcal{F}(\Omega, \{0\}_{n=0}^{\infty}), \tag{3.102}$$

where

$$\Omega = \{\Omega_n\}_{n=1}^{\infty}, \quad \Omega_n := \langle [F_n z : F_n > 0], [1+G_n z : G_n > 0], 0, 1, \rangle \tag{3.103}$$

for $n = 1, 2, 3, \ldots$. Then

$$\mathcal{F}^{T(z)} = \Bigg[F : F = \frac{F_1 z}{1 + G_1 z} + \frac{F_2 z}{1 + G_2 z} + \cdots, \quad F_n, G_n > 0, \quad n = 1, 2, 3, \ldots\Bigg] \tag{3.104}$$

is called the family of *Positive T-fractions.*

THEOREM 3.11. *Let $\mathcal{F}^{T(z)}$ be the family of LFASs defined by* (3.102)–(3.104). *Then:*

(A) *The best sequence of value regions $\{U_n(\mathcal{F}^{T(z)})\}_{n=0}^{\infty}$ with respect to $\mathcal{F}^{F(z)}$ satisfies*

$$U_n(\mathcal{F}^{T(z)}) := U(\mathcal{F}^{T(z)}) := \begin{cases} [u : 0 < \arg u < \arg z] \text{ if } \arg z > 0, \\ [u : \arg z < \arg u < 0] \text{ if } \arg z < 0, \\ [u : \arg u = 0], \text{ if } \arg z = 0. \end{cases} \tag{3.105}$$

(B) *If $F(z) \in \mathcal{F}^{T(z)}$ converges to a finite value $v(F(z)) \in \mathbf{C}$ and if n is a positive integer, then the best truncation error bound for $v_n(F(z))$ with respect to $\mathcal{F}^{T(z)}$ is given by*

$$\beta_n(F, \mathcal{F}^{T(z)}) = K_n(F(z))|v_n(F) - v_{n-1}(F)|, \tag{3.106a}$$

where $K_n(F(z))$ is defined as follows:

Case (a) $(0 \leq |\arg z| \leq \pi/2)$

$$K_n(F(z)) := 1, \quad \text{if } 0 \leq |\arg z| \leq \tfrac{\pi}{2}. \tag{3.106b}$$

Case (b) *Suppose that* $\pi/2 < |\arg z| < \pi$.

(b1)

$$K_n(F(z)) := \frac{|h_n(z)|}{|\operatorname{Im} h_n(F)|}, \quad \text{if } 0 < \left|\arg\left(\tfrac{-1}{h_n(F)}\right)\right| \leq \tfrac{\pi}{2}. \tag{3.106c}$$

(b2)

$$K_n(F(z)) := 1, \quad \text{if } \tfrac{\pi}{2} < \left|\arg\left(\tfrac{-1}{h_n(F)}\right)\right| \leq \tfrac{3\pi}{2} - |\arg z|. \tag{3.106d}$$

(b₃)

$$K_n(F(z)) := |\csc\beta|, \quad \text{if } \tfrac{3\pi}{2} - |\arg z| < \left|\arg\left(\tfrac{-1}{h_n(F)}\right)\right|, \tag{3.106e}$$

where

$$\beta := 2\pi - |\arg z| - \left|\arg\left(\frac{-1}{h_n(F)}\right)\right|. \tag{3.106f}$$

Proof. (A): Suppose $0 < \arg z < \pi$. Let $\{V_n\}$ denote the (constant) sequence of sets

$$V_n := V := [u : 0 < \arg u < \arg z], \quad n = 0, 1, 2, \ldots.$$

Suppose $F(z) \in \mathcal{F}^{T(z)}$. Then

$$t_n^{F(z)}(V) = \frac{F_n z}{1 + G_1 z + V}$$

for some $F_n > 0$. Let

$$H_1 := [u : \operatorname{Im} u > 0] \quad \text{and} \quad H_2 := [u : \arg z - \pi < \arg u < \arg z].$$

One can show that

$$t_n^{F(z)}(H_1) \subseteq H_1 \quad \text{and} \quad t_n^{F(z)}(H_2) \subseteq H_2.$$

It follows that

$$t_n^{F(z)}(V) = t_n^{F(z)}(H_1 \cap H_2) = T_n^{F(z)}(H_1) \cap T_n^{F(z)}(H_2) \subseteq H_1 \cap H_2 = V.$$

Furthermore, every point in V can be expressed as $F_1 z/(1 + G_1 z)$ for some $F_1, G_1 > 0$. Therefore $V \subseteq c(U_n(\mathcal{F}^{T(z)}))$ and by (2.4) $U_n(\mathcal{F}^{T(z)}) \subseteq V$. A similar argument holds when $-\pi < \arg z < 0$.

(B) Let $k \geq 1$ be given. Define $\{G_j\}_{j=1}^{\infty}$ by

$$G_j := \frac{F_1 z}{1 + G_1 z} + \frac{F_2 z}{1 + G_2 z} + \cdots + \frac{F_{n+k} z}{1 + G_{n+k} z} + \frac{z}{(1/j) + z} + \frac{z}{1 + z} + \frac{z}{1 + z} + \frac{z}{1 + z} + \cdots.$$

An argument analogous to one in the proof of Theorem 3.7 shows $\{G_j\}_{j=1}^{\infty}$ satisfies the hypothesis of Theorem 2.5. By (3.84)

$$h_n(F(z)) = 1 + G_n z + \frac{F_n z}{1 + G_{n-1} z} + \cdots + \frac{F_2 z}{1 + G_1 z}, \quad n = 2, 3, 4, \ldots.$$

By (2.5b), $(F_n z)/(1+G_{n-1}z+U(\mathcal{F}^{T(z)})) \subseteq U(\mathcal{F}^{T(z)})$, whenever $F_n, G_{n-1} > 0$ and thus $h_n(F(z)) \in 1 + G_n z + U(\mathcal{F}^{T(z)})$ and hence

$$\frac{-1}{h_n(F(z))} \in \frac{1}{-1 - G_n z - U(\mathcal{F}^{T(z)})}.$$

The remainder of our proof is similar to that given for Theorem 3.8(B) and hence is omitted. Q.E.D.

We conclude this section by stating the following result. Jefferson [30] gave this result for the special case with $G_n = 1$, $n \geq 1$. The general form given here was proved by Gragg [18].

THEOREM 3.12. *If a positive T-fraction*

$$F = \underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n z}{1+G_n z}\right) = \frac{F_1 z}{1+G_1 z} + \frac{F_2 z}{1+G_2 z} + \frac{F_3 z}{1+G_3 z} + \cdots,$$
$$F_n, G_n > 0$$

converges to a finite value $v(F)$, *then for* $n \geq 2$,

$$|v(F) - v_n(F)| \leq K_n(F(z))|v_n(F) - v_{n-1}(F)|,$$

where

$$K_n(F(z)) = \begin{cases} 1, & \text{if } 0 \leq |\arg z| \leq \pi/2, \\ |\csc(\arg z)|, & \text{if } \pi/2 < |\arg z| < \pi. \end{cases}$$

4. Best Truncation Error Bounds for Limit k-Periodic MCFs

Many modified continued fraction (MCF) expansions of special functions have the form

$$\underset{j=1}{\overset{\infty}{\mathrm{K}}} (a_j(z), b_j(z), w_j(z)),$$

where the elements $a_j(z), b_j(z)$ and converging factors $w_j(z)$ are complex-valued functions of a complex variable z. The MCF is called *periodic with period* k if $a_{rk+m}(z) = a_m(z)$, $b_{rk+m}(z) = b_m(z)$, $w_{rk+m}(z) = w_m(z)$ for $m \geq 1$ and $r \geq 0$. The MCF is called *limit k-periodic* if, for $m = 1, 2, 3, \ldots$,

$$\lim_{r\to\infty} a_{rk+m}(z) = \alpha_m(z), \quad \lim_{r\to\infty} b_{rk+m}(z) = \beta_m(z)$$

and

$$\lim_{r\to\infty} w_{rk+m}(z) = \omega_m(z).$$

An MCF is called *limit periodic* if it is limit 1-periodic. Section 4.1 deals with limit periodic MCFs

$$\mathop{\mathrm{K}}_{j=1}^{\infty} (a_j(z), 1, w_j(z)), \quad \text{where} \quad \lim_{j\to\infty} a_j(z) = \alpha(z), \quad \lim_{j\to\infty} w_j(z) = \omega(z),$$

where $\alpha(z) \in \mathbf{C} - (-\infty, -1/4]$ in Section 4.1.1 and $\alpha(z) = \infty$ in Section 4.1.2. Section 4.2 deals with MCFs

$$\mathop{\mathrm{K}}_{j=1}^{\infty} (1, b_j(z), w_j(z)),$$

where $\lim_{j\to\infty} b_j(z) = \infty$ in Section 4.2.1 and where the MCF is limit 4-periodic in Section 4.2.2.

4.1. LIMIT PERIODIC CFS $K(a_j/1)$ AND MCFS $K(a_j, 1, w_j)$

Our interest in this section is in best truncation error bounds for continued fractions (CFs)

$$\mathop{\mathrm{K}}_{j=1}^{\infty} \left(\frac{a_j}{1}\right) = \frac{a_1}{1} \underset{+}{} \frac{a_2}{1} \underset{+}{} \frac{a_3}{1} \underset{+}{} \cdots \tag{4.1}$$

and modified continued fractions (MCFs) $K(a_j, 1, w_j)$ whose elements a_j satisfy a limit-periodic condition of the form

$$\lim_{j\to\infty} a_j = a \in \widehat{\mathbf{C}} = \mathbf{C} \cup [\infty]. \tag{4.2}$$

Most of the results in this section apply to the case in which

$$\lim_{j\to\infty} a_j = a \in \mathbf{C} - (-\infty, -1/4]. \tag{4.3}$$

By using the parabola theorem of [53] one can readily prove:

THEOREM 4.1. *If the elements a_j of a CF $F = K(a_j/1)$ satisfy (4.3), then F converges to a value*

$$v(F) = \lim_{n\to\infty} v_n(F) \in \widehat{\mathbf{C}} = \mathbf{C} \cup [\infty]. \tag{4.4}$$

If a number a satisfies (4.3), then the fixed points x_1 and x_2 of the transformation

$$t(w) = \frac{a}{1+w} \tag{4.5}$$

are given by

$$x_1 = \sqrt{a + \frac{1}{4}} - \frac{1}{2}, \quad x_2 = -\sqrt{a + \frac{1}{4}} - \frac{1}{2}, \quad (\mathrm{Re}\sqrt{} > 0), \tag{4.6a}$$

and they satisfy

$$|x_1| < |x_2|, \quad x_2 = -(x_1 + 1), \quad a = -x_1 x_2. \tag{4.6b}$$

The periodic CF $K(a/1)$ converges to the attractive fixed point

$$x_1 = v(K(a/1)), \quad \text{where} \quad K\left(\frac{a}{1}\right) = \frac{a}{1} \, \frac{}{+} \, \frac{a}{1} \, \frac{}{+} \, \frac{a}{1} \, \frac{}{+} \cdots, \tag{4.7}$$

(see, e.g., [37], Theorem 3.2). If $F = K(a_j/1)$ is a limit-periodic CF satisfying (4.3), then for the nth tail $F^{(n)}$ of F, we have

$$\lim_{n\to\infty} v(F^{(n)}) = v(K(a/1)) = v(F) = x_1. \tag{4.8}$$

(see, e.g., [37], pp. 113–114). With $F = K(a_j/1)$ and $\{T_n(F,w)\}$ defined by (1.1e) and with generating sequence $\{t_j^F(w)\}$ given by

$$t_0^F(w) := w, \quad t_j^F(w) := \frac{a_j}{1+w}, \quad j = 1, 2, 3, \ldots, \tag{4.9}$$

we obtain (see (1.38))

$$v(F) = T_n(F, v(F^{(n)})), \quad n = 1, 2, 3, \ldots. \tag{4.10}$$

Equations (4.8) and (4.10) provide motivation for considering MCFs $K(a_j, 1, w_j)$ with converging factors

$$w_j = x_1 = v(K(a/1)), \quad j = 1, 2, 3, \ldots. \tag{4.11}$$

THEOREM 4.2. (Lemma 2.1 in [5]). *If the elements a_j of $F = K(a_j/1)$ satisfy (4.3), then the critical tail sequence $\{-h_n(F)\}$ (see (1.42)) satisfies*

$$\lim_{n\to\infty} (-h_n(F)) = x_1 \quad \textit{if}\; v(F) = \infty \tag{4.12a}$$

and

$$\lim_{n\to\infty} (-h_n(F)) = x_2 = -(x_1 + 1), \quad \textit{if}\; v(F) \in \mathbf{C}. \tag{4.12b}$$

We now consider families $\mathcal{F} = \mathcal{F}(\Omega, W)$ of LFASs defined in the following:

4.1.1. $K(a_j, 1, x_1)$, $a_j \to a \in \mathbf{C} - (-\infty, -1/4]$.

Let a, k, $\{a_j\}_{j=1}^{k}$ and $\{\rho_j\}_{j=k}^{\infty}$ satisfy

$$\begin{aligned} &a \in \mathbf{C} - (-\infty, -1/4], \quad 0 \le k \in \mathbf{Z}, \\ &0 \ne a_j \in \mathbf{C}, \quad j = 1, 2, \ldots, k, \end{aligned} \tag{4.13a}$$

$$0 < \rho_j < |x_2|, \quad \text{for } j = k, k+1, k+2, \ldots, \tag{4.13b}$$

and

$$\rho_j \rho_{j-1} < \rho_{j-1}|x_2| - \rho_j|x_1|, \quad \text{for } j = k+1, k+2, k+3, \ldots, \tag{4.13c}$$

where x_1 and x_2 are defined by (4.6a). Let $\Omega = \{\Omega_j\}$ and $W = \{w_j\}$ be defined by

$$\Omega_0 := [\langle 0, 1, 1, 0 \rangle], \tag{4.14a}$$

$$\Omega_j := \langle E_j, 1, 0, 1 \rangle \quad \text{and} \quad w_j = x_1, \quad j = 1, 2, 3, \ldots, \tag{4.14b}$$

where

$$E_j := \begin{cases} [a_j], \quad j = 1, 2, \ldots, k, \\ [u \in \mathbf{C} : \ |u(1 + \bar{x}_1) - x_1(|x_2|^2 - \rho_j^2)| + \rho_j |u| \\ \leq \rho_{j-1}(|x_2|^2 - \rho_j^2)], \quad j \geq k+1. \end{cases} \tag{4.14c}$$

We define a family $\mathcal{F} := \mathcal{F}(\Omega, W)$ of LFASs by

$$\begin{aligned} \mathcal{F} &:= \mathcal{F}(\Omega, W) := \mathcal{F}(a, 1, k, \{a_j\}_1^k, \{\rho_j\}_k^\infty) \\ &:= [F = K(a_j, 1, x_1) : \ \{a_j\} \text{ satisfies (4.3) and} \\ &\qquad\qquad 0 \neq a_j \in E_j, \quad j \geq 1]. \end{aligned} \tag{4.15}$$

Condition (4.13c) implies that

$$a \in E_j \quad \text{and hence} \quad x_1 = \frac{a}{1 + x_1} \in U_j(\mathcal{F}) \quad \text{for } j \geq k+1.$$

If $a \neq 0$, the element set E_j, for $j \geq k+1$, in (4.14c) is a closed, bounded, convex subset of $\mathbf{C}$ with an axis of symmetry given by the line passing through the ray $\arg u = \arg a$. The boundary ∂E_j of E_j (for $j \geq k+1$) is called a *Cartesian oval.*

If $a = 0$, then $x_1 = 0$, $x_2 = -1$ and E_j (for $j \geq k+1$) in (4.14c) reduces to the circular region

$$E_j = [u \in \mathbf{C} : \ |u| \leq \rho_{j-1}(1 - \rho_j)], \quad j \geq k+1. \tag{4.16}$$

A sequence of value regions $\{V_n\}$ with respect to $\mathcal{F}(\Omega, W)$ is given by

$$V_n := \begin{cases} [u \in \mathbf{C} : \ |u - x_1| \leq \rho_n], \quad n = k, k+1, k+2, \ldots, \\ \frac{a_{n+1}}{1 + (V_{n+1} \cup [x_1])}, \quad n = k-1, k-2, \ldots, 1, 0. \end{cases} \tag{4.17}$$

For $n \geq k$, the value-region-defining conditions (2.5) can be verified by using the $VW\Omega$-method described in Section 3.1. For $0 \leq n \leq k-1$, conditions (2.5) follow directly from (4.17). The following result was proven in [5], Theorems 2.2 and 2.4.

THEOREM 4.3. *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a family of LFASs of the form* (4.15) *and let $F = K(a_j, 1, x_1) \in \mathcal{F}$ be given. Then:*

(A) *$F = K(a_j, 1, x_1)$ and $K(a_j/1)$ converge to the same value $v(F) \in \widehat{\mathbf{C}} = \mathbf{C} \cup [\infty]$.*

(B) *If there exists an integer $k_0 \geq k$ such that*

$$|h_{k_0}(F) + x_1| > \rho_{k_0}, \tag{4.18}$$

then

$$|h_n(F) + x_1| > \rho_n, \quad n = k_0 + 1, k_0 + 2, k_0 + 3, \ldots, \tag{4.19}$$

and $F = K(a_j, 1, x_1)$ and $K(a_j/1)$ converge to the same finite value $v(F) \in \mathbf{C}$.

(C) *If $\lim_{j \to \infty} \rho_j = 0$ and $K(a_j/1)$ converges to a finite value f, then there exists an integer $k_0 \geq k$ such that* (4.18) *holds. Hence $f = v(F) \in \mathbf{C}$.*

The results in our next theorem were proven in [5], Theorems 3.1 and 4.1.

THEOREM 4.4. *Let $\mathcal{F} = \mathcal{F}(\Omega, W) = \mathcal{F}(a, 1, k, \{a_j\}_1^k, \{\rho_j\}_k^\infty)$ be a family of LFASs* (4.15) *and let $F = K(a_j, 1, x_1) \in \mathcal{F}$ be given. Then:*

(A) *If there exists an integer $k_0 \geq k$ such that*

$$|h_{k_0}(F) + x_1| > \rho_{k_0} \quad (i.e.\ , -h_{k_0}(F) \notin V_{k_0}), \tag{4.20}$$

then F converges to a finite value $v(F) \in \mathbf{C}$ and, for all $n \geq k_0 + 1$,

$$\begin{aligned} &|v(F) - v_n(F)| \\ &\leq \frac{\rho_n \prod_{j=1}^n |a_j(F)|}{|B_{n-1}(F)|^2 |h_n(F) + x_1|(|h_n(F) + x_1| - \rho_n)} \\ &= \frac{\rho_n |h_n(F)|}{|h_n(F) + x_1|(|h_n(F) + x_1| - \rho_n)} \cdot |v_n(F) - v_{n-1}(F)|. \end{aligned} \tag{4.21}$$

(B) *If $K(a_j/1)$ converges to a finite value $v(K(a_j/1))$ and if $\lim_{j \to \infty} \rho_j = 0$, then there exists an integer $k_0 \geq k$ such that* (4.20) *holds and hence* (4.21) *holds for $n \geq k_0 + 1$:*

(C) *Let $\{\rho_j\}$ satisfy the following additional conditions for all $j \geq k + 1$:*

(a) *If $a \in \mathbf{C} - (-\infty, 0]$ and $\alpha := \arg a$, then*

$$\rho_{j-1}|x_2| - \rho_j|x_1| < \sqrt{|a|}\cos(\alpha/2), \tag{4.22a}$$

and

$$\rho_{j-1} \leq \frac{1}{2}\cos\left(\frac{\alpha}{2}\right) + \operatorname{Re}(x_1 e^{-i\alpha/2}) = \operatorname{Re}\left(\sqrt{a + \frac{1}{4}e^{-i\alpha/2}}\right). \tag{4.22b}$$

(b) *If* $-\frac{1}{4} < a \leq 0$, *then*

$$(\rho_{j-1} + |x_1|)(|x_2| - \rho_j) \leq \frac{1}{4} \quad and \quad \rho_{j-1} \leq \sqrt{a + \frac{1}{4}} = x_1 + \frac{1}{2}. \tag{4.23}$$

If there exists an integer $k_0 \geq k$ *such that* (4.20) *holds, then the truncation error bound in* (4.21) *is the best bound* $\beta_n(F, \mathcal{F})$ *for* $v_n(F)$ *with respect to* $\mathcal{F}$ *for* $n \geq k_0 + 1$.

Proof. (A): We make use of Theorem 2.6 with $w_m = x_1$. Condition (4.20) implies that $-h_{k_0}(F) \notin V_{k_0}$. Therefore since $T_{k_0}(F, -h_{k_0}(F)) = \infty$, the set $T_{k_0}(F, V_{k_0})$ is a closed, bounded circular disk. Hence the nestedness of the sequence $\{T_n(F, V_n)\}_{k_0}^{\infty}$ implies that $T_n(F, V_n)$ is a closed, bounded disk for all $n \geq k_0$. By Theorem 4.1, F converges to a finite value $v(F)$. Assertion (4.21) is then an immediate consequence of Theorem 2.6(A).

(B): If $K(a_j/1)$ converges to a finite value $v(K(a_j/1))$, it follows from Lemma 4.2 that

$$\lim_{n\to\infty} h_n(F) = -x_2 = x_1 + 1.$$

Thus if $\lim_{j\to\infty} \rho_j = 0$, there exists an integer $k_0 \geq k$ such that (4.20) holds, and hence by (A), (4.21) holds for $n \geq k_0 + 1$.

(C): It was shown in [4], Theorem 3.1. that, subject to the additional conditions (4.22) and/or (4.23),

$$c(U_n(\mathcal{F})) = V_n := [u \in \mathbf{C} : |u| \leq \rho_n], \quad n = k, k+1, k+2, \ldots. \tag{4.24}$$

Now suppose $a \neq 0$. Then $a \in E_n$ for all $n \geq k_0 + 1$ and so

$$x_1 = \frac{a}{1 + x_1} \in U_n(\mathcal{F}), \quad \text{for } n = k_0 + 1, k_0 + 2, k_0 + 3, \ldots. \tag{4.25}$$

Hence assertion (C) follows from Theorem 2.6(B) since (4.25) implies condition (a) of Theorem 2.5. *On the other hand, if* $a = 0$, then assertion (C) follows from Theorem 2.6(B), since condition (b) of Theorem 2.5 holds. Q.E.D.

REMARK . If V_n satisfies (4.24) for $n \geq k$, then the V_n defined by (4.17), for $0 \leq n \leq k - 1$, also satisfies

$$V_n = c(U_n(\mathcal{F})), \quad 0 \leq n \leq k - 1. \tag{4.26}$$

We state as a corollary of Theorem 4.4 the result obtained when the parameter $a = 0$ and the element sets E_j are circular disks given by (4.16).

THEOREM 4.5. *Let* $\mathcal{F} = \mathcal{F}(\Omega, W) = \mathcal{F}(0, 1, k, \{a_j\}_1^k, \{\rho_j\}_k^{\infty})$ *be a family of LFASs* (4.15), *with* $a = 0$. *Let* $F = K(a_j, 1, 0) \in \mathcal{F}$ *be given. Let* $\{\rho_j\}_{j=k}^{\infty}$ *satisfy*

$$0 < \rho_j \leq \frac{1}{2}, \quad j = k, k+1, k+2, \ldots. \tag{4.27}$$

Then: (A) *If there exists an integer* $k_0 \geq k$ *such that*

$$|h_{k_0}(F)| > \rho_{k_0}, \quad (i.e.\ , h_{k_0}(F) \notin V_{k_0}), \tag{4.28}$$

then F *converges to a finite value* $v(F) \in \mathbf{C}$ *and, for all* $n \geq k_0 + 1$, *the best truncation error bound for* $v_n(F)$ *with respect to* $\mathcal{F}$ *is given by*

$$\begin{aligned} \beta_n(F, \mathcal{F}) &= \frac{\rho_n \prod_{j=1}^{n} a_j(F)}{|B_{n-1}(F)|^2 |h_n(F)|(|h_n(F)| - \rho_n)} \\ &= \frac{\rho_n}{|h_n(F)| - \rho_n} \cdot |v_n(F) - v_{n-1}(F)|. \end{aligned} \tag{4.29}$$

(B) *If* $K(a_j/1)$ *converges to a finite value* $v(K(a_j/1)) = v(F)$ *and if* $\lim_{j\to\infty} \rho_j = 0$, *then there exists an integer* $k_0 \geq k$ *such that* (4.28) *holds and hence* (4.29) *holds for all* $n \geq k_0 + 1$.

REMARK . Corollary 4.1 is an improvement of [3, Theorem 3.2].

Proof. It follows from $a = x_1 = 0, x_2 = -1$ and (4.27) that conditions (4.13b,c) and (4.23) hold. The corollary is therefore an immediate consequence of Theorem 4.4. Q.E.D.

4.1.2. CFs $K(a_j/1)$ *and MCFs* $K(a_j, 1, w_j)$ *with* $\lim_{j\to\infty} a_j = \infty$.

We conclude this section by stating a result (Theorem 4.6) for CFs $K(a_j/1)$ and MCFs $K(a_j, 1, w_j)$ for which

$$\lim_{j\to\infty} a_j = \infty. \tag{4.30}$$

A proof of this result can be found in [24] (see, also [23]). Use is made of the following terminology.

$$P_\alpha := [u \in \mathbf{C} : |u| - \text{Re}\,(ue^{-i2\alpha}) \leq \frac{1}{2}\cos^2\alpha], \quad \text{for } -\tfrac{\pi}{2} < \alpha < \tfrac{\pi}{2}. \tag{4.31}$$

P_α is a region bounded by a parabola ∂P_α with focus at the origin $u = 0$, axis of symmetry along the ray $\arg u = 2\alpha$, and ∂P_α passes through $u = -1/4$. For a sequence $\{c_n\}$ and ρ satisfying

$$\begin{aligned} &c_n \in \mathbf{C}, \quad |c_{n-1}| \leq |1 + c_n|, \quad 0 < \rho < |1 + c_n|, \\ &\quad \text{for all } n = 1, 2, 3, \ldots, \end{aligned} \tag{4.32a}$$

we define $E_n(\{c_j\}, \rho)$, for $n \geq 1$, by

$$\begin{aligned} E_n(\{c_j\}, \rho) := [u \in \mathbf{C} : &|u(1 + \bar{c}_n) - c_{n-1}(|1 + c_n|^2 - \rho^2)| + \\ &\rho|u| \leq \rho(|1 + c_n|^2 - \rho^2)]. \end{aligned} \tag{4.32b}$$

The boundary $\partial E_n(\{c_j\}, \rho)$ is a Cartesian oval (see remark following (4.15)).

THEOREM 4.6. *Let* α, ρ *and* R *satisfy*

$$-\frac{\pi}{2} < \alpha < \frac{\pi}{2} \quad \text{and} \quad 0 < R < \rho \cos \alpha. \tag{4.33}$$

Let $G = K(a_j/1)$ *be a CF whose elements* a_j *satisfy the conditions* (4.30),

$$a_j \in P_\alpha, \quad j = 1, 2, 3, \ldots, \tag{4.34}$$

and the limit points of $\{a_{j+1} - a_j\}$ *all lie in the disk*

$$D(\alpha, \rho, R) := [u \in \mathbf{C} : |u - 2\rho^2 e^{i2\alpha}| \leq 2R]. \tag{4.35}$$

Then: (A) $G = K(a_j/1)$ *converges to a value*

$$v(G) = \lim_{n\to\infty} v_n(G) \in \hat{\mathbf{C}} = \mathbf{C} \cup [\infty]. \tag{4.36}$$

(B) *Let* $F = K(a_j, 1, w_j)$ *be the MCF whose converging factors* w_j *are given by*

$$w_j := \sqrt{a_{j+1} + \frac{1}{4}} - \frac{1}{2}, \quad j = 1, 2, 3, \ldots, \quad (\text{Re}\sqrt{} > 0). \tag{4.37}$$

If $v(G) \in \mathbf{C}$, *then*

$$\lim_{n\to\infty} \left| \frac{v(G) - v_n(F)}{v(G) - v_n(G)} \right| = 0 \tag{4.38}$$

and hence $v(G) = v(F) = \lim_{n\to\infty} v_n(F) \in \mathbf{C}$

(C) *If* $v(G) \in \mathbf{C}$ *and*

$$a_m \in E_m(\{w_j\}, \rho) \quad \text{and} \quad \rho < |1 + w_m|, \quad \text{for } m = 1, 2, 3, \ldots,$$

then

$$|v(F) - v_n(F)| \leq 2\rho \prod_{j=1}^{n} \frac{|a_j|}{(|1 + w_j| - \rho)^2}, \quad n = 1, 2, 3, \ldots. \tag{4.39}$$

4.2. LIMIT k-PERIOD CFS $K(1/b_j)$ AND MCFS $K(1, b_j; w_j)$.

In this section we consider best truncation error bounds for continued fractions

$$\operatorname*{K}_{j=1}^{\infty} \left(\frac{1}{b_j}\right) = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots \tag{4.40}$$

and modified continued fractions $K(1, b_j; w_j)$ whose elements b_j satisfy a limit 4-periodic condition $\lim_{n\to\infty} b_n = \infty$ or $\lim_{n\to\infty} b_{4n+i} = \beta_i$ where $1 \leq i \leq 4$. The results in this section are restricted to the case in which $|b_n| \geq 2$ for all sufficiently large n. This condition ensures that $F = K(1/b_j)$ converges to $v(F)$ in the extended complex plane (Theorem 4.35 in [37]).

4.2.1. $K(1/b_j)$, $b_j \to \infty$.

Let $k \geq 0$ be a given non-negative integer; let $\{b_j\}_{j=1}^{k}$ be a given sequence of complex numbers; and let $\{\rho_n\}_{n=k}^{\infty}$ be a sequence of positive numbers such that

$$\lim_{n\to\infty} \rho_n = 0 \quad \text{and} \quad \rho_n + \frac{1}{\rho_{n-1}} \geq 2, \quad \text{for } n = k+1, k+2, \ldots. \tag{4.41}$$

Let $W := \{0\}$ and let $\Omega = \{\Omega_j\}$ be defined by

$$\Omega_j := \langle 1, E_j, 0, 1 \rangle, \quad j = 1, 2, 3, \ldots, \tag{4.42}$$

where

$$E_j := \begin{cases} [b_j], & j = 1, 2, \ldots, k \\ \left[u \in \mathbf{C} : \ |u| \geq \rho_j + \dfrac{1}{\rho_{j-1}}\right], & j \geq k+1. \end{cases} \tag{4.43}$$

We define a family $\mathcal{F}$ of LFASs by

$$\begin{aligned} \mathcal{F} &:= \mathcal{F}(\Omega, W) := \mathcal{F}(1, \infty, k, \{b_j\}_1^k, \{\rho_j\}_k^\infty) \\ &:= \left[F = \mathop{\mathrm{K}}_{j=1}^{\infty} (1, b_j, 0) : \ b_j \in E_j \quad \text{for } j \geq 1\right]. \end{aligned} \tag{4.44}$$

We recall that $\mathrm{K}_{j=1}^{\infty}(1/b_j) = \mathrm{K}_{j=1}^{\infty}(1, b_j, 0)$. A sequence of value regions $\{V_n\}$ with respect to $\mathcal{F}(\Omega, W)$ is given by

$$V_n := \begin{cases} [u : \ |u| \leq \rho_n], & n = k, k+1, k+2, \ldots, \\ \frac{1}{b_{n+1} + V_{n+1}} & n = k-1, k-2, \ldots, 1, 0. \end{cases} \tag{4.45}$$

THEOREM 4.7. (Theorem 2.2 in [7]). *If the elements b_j of $F = K(1/b_j)$ satisfy* $\lim_{n\to\infty} b_j = \infty$, *then for the critical tail sequence $\{h_n(F)\}$ we have*

$$\lim_{n\to\infty} h_n(F) = 0 \quad \text{if } v(F) = \infty \tag{4.46}$$

and

$$\lim_{n\to\infty} h_n(F) = \infty \quad \text{if } v(F) \neq \infty. \tag{4.47}$$

The following result is subsequently used.

THEOREM 4.8. (Theorem 3.1 in [7]). *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a family of LFASs of the form* (4.44) *and let $F = K(1/b_j) \in \mathcal{F}$ be given. Then:*

(A) *If there exists an integer* $k_0 \geq k$ *such that* $|h_{k_0}(F)| > \rho_{k_0}$, *then*

$$|h_n(F)| > \rho_n, \quad n = k_0, k_0 + 1, \ldots, \tag{4.48}$$

and

$$K(1/b_j) \quad \textit{converges to a finite value } v(F). \tag{4.49}$$

(B) *If* $F = K(1/b_j)$ *converges to a finite value* $v(F)$, *then there exists a* $k_0 \geq k$ *such that* $|h_{k_0}(F)| > \rho_{k_0}$, *and hence the assertions of (A) hold and* $v(F) \in \mathbf{C}$.

The following truncation error bounds were obtained in Theorems 3.2 and 4.2 in [7].

THEOREM 4.9. *Let* $\mathcal{F} = (\mathcal{F}(\Omega, W)) = \mathcal{F}(1, \infty, k, \{b_j\}_1^k, \{\rho_j\}_k^\infty)$ *be a family of LFASs* (4.44) *and let* $F = K(1/b_j) \in \mathcal{F}$ *be given. Then*

(A) *If there exists an integer* $k_0 \geq k$ *such that*

$$|h_{k_0}(F)| > \rho_{k_0}, \tag{4.50}$$

then F *converges to a finite value* $v(F) \in \mathbf{C}$ *and, for all* $n \geq k_0 + 1$

$$\begin{aligned} |v(F) - v_n(F)| &\leq \frac{\rho_n}{|B_{n-1}(F)|^2 |h_n(F)|(|h_n(F)| - \rho_n)} \\ &= \frac{\rho_n |v_n(F) - v_{n-1}(F)|}{(|h_n(F)| - \rho_n)}. \end{aligned} \tag{4.51}$$

(B) *If* $K(1/b_j)$ *converges to a finite value* $v(F)$, *then there exists a* $k_0 \geq k$ *such that* $|h_{k_0}(F)| > \rho_{k_0}$ *and hence* (4.51) *holds for* $n \geq k_0 + 1$.

(C) *If (4.50) holds for some integer* $k_0 \geq k$, *then, for* $n \geq k_0 + 1$, *the truncation error bound in* (4.51) *is the best bound* $\beta_n(F, \mathcal{F})$ *for* $v_n(F)$ *with respect to* $\mathcal{F}$.

4.2.2. $K(1, b_j, w_j)$, $b_{4j+i} \to \beta_i$ *as* $j \to \infty$ *and* $w_{4n+i} = 1/\beta_{i+1}$, $i = 0, 1, 2, 3$, *and* $m \geq 0$.

We now consider CFs $K(1/b_j)$ and MCFs $K(1, b_j, w_j)$ for which the elements b_j are complex numbers that satisfy limit 4-periodic properties.

Let $k \geq 0$ be a given non-negative integer; let $\{\beta_j\}_1^4$ satisfy

$$\beta_2 = \beta_4 = \infty, \quad \beta_1, \beta_3 \in [u \in \mathbf{C} : \ |u| > 2]; \tag{4.52}$$

and let $\{\rho_j\}_k^\infty$ be a sequence of positive numbers satisfying

$$\lim_{j \to \infty} \rho_j = 0, \tag{4.53}$$

$$\rho_{4j+i} < \left|\frac{1}{\beta_{i+1}}\right|, \quad \text{for } i = 0, 2 \quad \text{and} \quad 4j + i \geq k, \tag{4.54}$$

and

$$2 - \frac{|\beta_i|}{|1 + |\beta_i|\rho_{4j+i-1}} \leq \rho_{4j+i} \leq \frac{|\beta_i|^2 \rho_{4j+i-1}}{1 + |\beta_i|\rho_{4j+i-1}}, \tag{4.55}$$

for $i = 1, 3$ and $4j + i \geq k$.

Let $\{E_j\}$ be a sequence of subsets of $\mathbf{C}$ defined by

$$E_j := [b_j], \quad j = 1, 2, \ldots, k, \tag{4.56}$$

$$E_{4j+i} := \left[b \in \mathbf{C}: \left|b + \frac{1}{\beta_{i+1}}\right| \geq \rho_{4j+i} + \frac{1}{\rho_{4j+i-1}}\right], \quad i = 0, 2, \quad 4j + i \geq k + 1 \tag{4.57}$$

$$E_{4j+i} := \left[b \in \mathbf{C}: \left|\frac{\beta_i}{1 - (|\beta_i|\rho_{4j+i-1})^2}\right| \leq -\rho_{4j+i} + \frac{|\beta_i|^2 \rho_{4j+i-1}}{1 - (|\beta_i|\rho_{4j+i-1})^2}\right], \quad i = 1, 3, \quad 4j + i \geq k + 1. \tag{4.58}$$

Then $\Omega = \{\Omega_j\}$ is defined by

$$\Omega_j := \langle 1, E_j, 0, 1 \rangle, \quad j = 1, 2, 3, \ldots. \tag{4.59}$$

It follows from (4.57) and (4.58) that if

$$b_j \in E_j, \quad j = k + 1, k + 2, k + 3, \ldots, \tag{4.60}$$

then

$$\lim_{j\to\infty} b_{4j+1} = \beta_1, \quad \lim_{j\to\infty} b_{4j+3} = \beta_3, \quad \lim_{j\to\infty} b_{4j+2} = \lim_{j\to\infty} b_{4j+4} = \infty. \tag{4.61}$$

We define a sequence of converging factors $W = \{w_j\}$, for $j = 0, 1, 2, \ldots$, by

$$w_{4m+i} := \lim_{j\to\infty} \frac{1}{b_{4j+i+1}} = \begin{cases} 0, & i = 1, 3, \\ 1/\beta_1, & i = 4, \\ 1/\beta_3, & i = 2. \end{cases} \tag{4.62}$$

A family $\mathcal{F}$ of LFASs is then defined by

$$\mathcal{F} := \mathcal{F}(\Omega, W) := \mathcal{F}(1, \{\beta_j\}_1^4, k, \{\beta_j\}_1^k, \{\rho_j\}_k^\infty) := [F = K(1, b_j, w_j) : b_j \in E_j \quad \text{for } j \geq 1]. \tag{4.63}$$

The generating sequence $\{t_j^F(w)\}$ for $F \in \mathcal{F}$ is defined by

$$t_0^F(w) := w, \quad t_j^F(w) := \frac{1}{b_j(F) + w}, \quad j = 1, 2, 3, \ldots. \tag{4.64}$$

With $\{T_n(F, w)\}$ defined by (1.1e) we have

$$v_n(F) := T_n(F, w_n), \quad n = 1, 2, 3, \ldots \tag{4.65}$$

and

$$v(F) := \lim_{n \to \infty} T_n(F, w_n) = T_n(F, v(F^{(n)})). \tag{4.66}$$

REMARKS . (1) Our choice of $\{w_j\}$ given by (4.62) is motivated by (1.38), (4.61) and

$$\lim_{m \to \infty} v(F^{(4m+i)}) = \lim_{m \to \infty} \frac{1}{b_{4m+i+1}} =: w_{4m+i}, \quad m \geq 0, \quad i = 0, 1, 2, 3. \tag{4.67}$$

(2) Condition (4.54) ensures $0 \notin V_{4j+i}$ and hence $E_{4j+i+1} \neq \emptyset$, for $i = 0, 2$ and $4j + i \geq k$. Condition (4.55) ensures that $\beta_i \in E_{4j+i}$, for $i = 1, 3, 4j + i \geq k$, and that $E_j \cap [u \in \mathbf{C} : |u| < 2] = \emptyset$ for $j = k, k+1, \ldots$.

(3) If $K(1/b_j)$ is a continued fraction satisfying (4.61) and $|b_j| \geq 2$ for $j \geq k$ for some positive integer k, then there exists a sequence of positive numbers $\{\rho_j\}_{j=0}^{\infty}$ satisfying (4.53), (4.54) and (4.55) such that

$$K(1, b_j, w_j) \in \mathcal{F}(1, \{\beta_j\}_1^4, k, \{b_j\}_1^k, \{\rho_j\}_k^{\infty})$$

where $\{w_j\}_{j=1}^{\infty}$ is defined by (4.62).

THEOREM 4.10 (Lemma 2.2 in [8]). *If the elements b_j of $F = K(1/b_j)$ satisfy (4.61), $|b_j| \geq 2$ for $j \geq k$ for some positive integer k, and $b_j \neq 0$ for $j = 1, 2, 3, \ldots$, then the critical tail sequence $\{-h_n(F)\}$ satisfies*

$$\lim_{n \to \infty} h_{4n+i}(F) = \begin{cases} 0, & i = 1, 3 \\ -1/\beta_3, & i = 2 \\ -1/\beta_1, & i = 4, \end{cases} \quad \text{if } v(F) = \infty \tag{4.68}$$

and

$$\lim_{n \to \infty} h_{4n+i}(F) = -\beta_i, \quad \text{if } v(F) \neq \infty. \tag{4.69}$$

We use the following result to obtain truncation error bounds.

THEOREM 4.11 (Theorem 3.2 in [8]). *Let $\mathcal{F} = \mathcal{F}(\Omega, W)$ be a family of LFASs of the form (4.63) and let $F = K(1, b_j; w_j) \in \mathcal{F}$ be given. Then:*

(A) *If there exists an integer $k_0 \geq k$ such that*

$$\left| h_{k_0}(F) + \frac{1}{\beta_{(k_o \bmod 4)+1}} \right| > \rho_{k_0} \tag{4.70}$$

then

$$\left| h_n(F) + \frac{1}{\beta_{(n \bmod 4)+1}} \right| > \rho_n \quad \text{for } n = k_0, k_0+1, \ldots, \tag{4.71}$$

and $K(1/b_j)$ and $K(1, b_j, w_j)$ converge to the same finite value $v(F) \in \mathbf{C}$.

(B) *If $K(1/b_j)$ converges to a finite value f, then there exists a $k_0 \geq k$ such that* (4.70) *holds. Hence $f = v(F) \in \mathbf{C}$.*

The following theorem was proven in Theorems 3.3 and 4.1 in [8].

THEOREM 4.12. *Let $\mathcal{F} = (\mathcal{F}(\Omega, W)) = \mathcal{F}(1, \{\beta_i\}_1^4, k, \{b_j\}_1^k, \{\rho_j\}_k^\infty)$ be a family of LFASs* (4.63) *and let $F = K(1, b_j, w_j) \in \mathcal{F}$ be given. Then:*

(A) *If there exists an integer $k_0 \geq k$ such that* (4.70) *holds, then $K(1/b_j)$ and F both converge to the same finite value $v(F) \in \mathbf{C}$ and for $n \geq k_0$*

$$\begin{aligned} |v(F) - v_n(F)| &\leq \frac{\rho_n}{|B_{n-1}(F)|^2 |w_n + h_n(F)|(|w_n + h_n(F)| - \rho_n)} \\ &= \frac{\rho_n |h_n(F)||v_n(F) - v_{n-1}(F)|}{|w_n + h_n(F)|(|w_n + h_n(F)| - \rho_n)}. \end{aligned} \tag{4.72}$$

(B) *If $K(1/b_j)$ converges to a finite value $v(K(1/b_j))$, then there exists a $k_0 \geq k$ such that* (4.70) *holds and hence* (4.72) *holds for $n \geq k_0$.*

(C) *If there exists an integer $k_0 \geq k$ such that* (4.70) *holds, then the truncation error bound in* (4.72) *is the best bound $\beta_n(F, \mathcal{F})$ for $v_n(F)$ with respect to $\mathcal{F}$ for $n \geq k_0$.*

5. Asymptotically Best Truncation Error Bounds for LFASs

An approach to truncation error estimates for limit periodic LFT algorithms, differing from the one treated in detail in this article, was explored by one of the authors [59]. Earlier work pointing in the direction can be found in [63], [57] and [60]. An LFT algorithm is the special case of an LFAS (see (1.1a)) where $w_j = 0, j \geq 1$.

One starts with formulas based on the invariance of the cross ratio under ℓ.f.t. (see [56]), that is,

$$\left(\frac{S(u) - S(v)}{S(u) - S(w)}\right)\left(\frac{S(w) - S(z)}{S(v) - S(z)}\right) = \left(\frac{u - v}{u - w}\right)\left(\frac{w - z}{v - z}\right). \tag{5.1}$$

In (5.1) we replace S by T_n and set

$$-z = h_n := -T_n^{-1}(\infty). \tag{5.2}$$

This leads to the simplification

$$\frac{T_n(u_n) - T_n(v_n)}{T_n(u_n) - T_n(w_n)} = \left(\frac{u_n - v_n}{u_n - w_n}\right)\left(\frac{w_n + h_n}{v_n + h_n}\right). \tag{5.3}$$

Making suitable substitutions for u_n, v_n, w_n one arrives at

$$\frac{T_n(0) - T_{n+m}(0)}{T_n(0) - T_{n-1}(0)} = \frac{T_m^{(n)}(0)}{-a_n}\left(\frac{-a_n + c_n h_n}{T_m^{(n)}(0) + h_n}\right) \tag{5.4}$$

and

$$\frac{T_{k+1}(0) - T_k(0)}{T_k(0) - T_{k-1}(0)} = \frac{a_{k+1}}{a_k}\left(\frac{a_k - c_k h_k}{a_{k+1} + b_{k+1} h_k}\right). \tag{5.5}$$

Here $\{T_n\}$ is defined in terms of $\{t_n(w)\}$ as in $(1.1b, c, d)$ except that we have dropped the F superscript. $T_m^{(n)}$ is defined as

$$T_m^{(n)}(w) = t_{n+1} \circ \cdots \circ t_{n+m}(w).$$

Combining (5.4) and (5.5) one obtains

$$\begin{aligned} T_{n+m}(0) - T_n(0) = & \frac{T_m^{(n)}(0)(a_{n+1} + b_{n+1}h_n)(a_0 b_1 - b_0 a_1)}{(T_m^{(n)}(0) + h_n) b_0 b_1} \times \\ & \prod_{k=1}^{n}\left(\frac{a_k - c_k h_k}{a_{k+1} + b_{k+1} h_k}\right). \end{aligned} \tag{5.6}$$

This formula is valid for general $\{T_n\}$ provided the denominator of the right side of (5.6) does not vanish.

The formula (5.6) becomes particularly useful for *limit periodic* LFT algorithms. From now on we shall restrict ourselves to such sequences $\{T_n\}$.

Set

$$\lim_{n\to\infty} t_n(w) =: t(w) =: \frac{a + cw}{b + dw}, \tag{5.7}$$

where we shall assume that

$$a := \lim_{n\to\infty} a_n, \quad b := \lim_{n\to\infty} b_n, \quad c := \lim_{n\to\infty} c_n, \quad d := \lim_{n\to\infty} d_n, \tag{5.8}$$

$a, b, c, d \in \mathbf{C}$. We exclude the cases where $t(w)$ is the identity or parabolic or elliptic. Then, if $t(w)$ is not singular, it has exactly two distinct fixed points x_1 and

x_2. If $t(w)$ is singular, we shall denote by x_2 its fixed point and by x_1 the point $-a/b$ for which $t(w)$ is not defined. We also shall assume that both x_1 and x_2 are finite. Next, we introduce

$$r := \frac{dx_1 + b}{dx_2 + b} \tag{5.9}$$

and choose the subscripts so that $|r| < 1$. This can be done since $t(w)$ is assumed not to be elliptic.

It can be shown (the proof is quite intricate) that

$$\lim_{n\to\infty} h_n = -x_1, \tag{5.10}$$

provided $t_n(x_2) \neq x_2$ for all $n > n_0$. It further is true that

$$\lim_{k\to\infty} \frac{a_k - c_k h_k}{a_{k+1} + b_{k+1} h_k} = -r, \tag{5.11}$$

and that there exists a constant M such that

$$\left| \frac{T_m^{(n)}(0)}{T_m^{(n)}(0) + h_n} \right| < M, \tag{5.12}$$

for all $m > 0$ and all $n > n_0$.

In general we only know that such an M exists. However if more information is available about the sequence $\{t_n\}$, then an explicit bound on (5.12) may be obtainable. This is illustrated by our discussion of $K(a_n/1)$ later in this section. The remaining quantities in (5.6) can be easily calculated on the basis of the available information.

For any r' such that

$$|r| < |r'| < 1$$

and $n > n_2 > \max(n_0, n_1)$, the formula (5.6) can be recast into the inequality

$$|T_{n+m}(0) - T_n(0)| < K(r')|r'|^n. \tag{5.13}$$

In general $\lim_{r' \to r} K(r') = \infty$; but if

$$\sum_{n=1}^{\infty} \Delta_n < \infty,$$

where

$$\Delta_n := \max(|a - a_n|, |b - b_n|, |c - c_n|, |d - d_n|), \tag{5.14}$$

then the stronger statement

$$|T_{n+m}(0) - T_n(0)| < K(r)|r|^n \tag{5.15}$$

is valid. Here $K(r) < \infty$.

For pure periodic sequences $\{\overset{p}{T_n}\}$ the truncation error is known to be

$$|\overset{p}{T_n}(0) - w_2| = \left| r^n \frac{w_2}{w_1} (\overset{p}{T_n}(0) - w_1) \right|. \tag{5.16}$$

It follows that the estimates (5.13) and (5.15) can be said to be *asymptotically best*.

The formula

$$\frac{f - T_n(x_2)}{f - T_n(0)} = \frac{f^{(n)} - x_2}{f^{(n)}} \quad \frac{h_n}{h_n + x_2} \tag{5.17}$$

is an easy consequence of (5.3). Here we have set

$$f := \lim_{n\to\infty} T_n(0), \quad f^{(n)} := \lim_{m\to\infty} T^{(n)}_{n+m}(0).$$

(5.17) was initially proved for $K(a_n/1)$ in [61]. Since $f^{(n)} - x_2 \to 0$, it follows from (5.17) that $\{T_n(x_2)\}$ converges to f much faster than $\{T_n(0)\}$ does.

Further analysis shows that $K(r')$ depends on the behavior of $\{\Delta_n\}$, while, clearly, r is completely determined by a, b, c, w in $t(w)$. It can also be shown (the argument is delicate) that $f^{(n)} - x_2$ is roughly proportional to Δ_n.

If $t(w)$ is singular, then $r = 0$ and it follows that the convergence of $\{T_n(0)\}$ is extremely fast. For $K(a_n/1)$, with $a_n \to 0$, this was first observed in [3]. For Schur algorithms with $\gamma_n \to e^{i\theta}$, see [57].

For the special case $K(a_n/1)$, which was analyzed in [63], (5.6) becomes

$$S_{n+m}(0) - S_n(0) = \frac{S^{(n)}_m(0)(a_{n+1} + h_n)(-a_1)}{S^{(n)}_m(0) + h_n} \prod_{k=1}^{n} \left(\frac{a_k}{a_{k+1} + h_k} \right). \tag{5.18}$$

For r we obtain

$$r = \frac{x_1 + 1}{x_2 + 1} = \frac{-x_2}{-x_1} = \frac{x_2}{x_1}. \tag{5.19}$$

Using the fact that

$$\frac{h_k - 1}{h_k} = \frac{a_k}{a_{k-1}} \cdot \frac{a_{k-1}}{a_k + h_{k-1}}$$

we can show that (5.18) is equivalent to (3.3) in [63].

If we assume that for all $n \geq 1$ and some θ, $0 < \theta < 1$

$$a_n \in P(\alpha, \theta) = [w : |w| - \operatorname{Re} we^{-i2\alpha} \leq (\cos\alpha)^2(1 - \theta^2)/2] \tag{5.20}$$

where $2\alpha = \arg(\lim a_n)$, then we can conclude that

$$\operatorname{Re}\left(e^{-i\alpha} S_m^{(n)}(0)\right) \geq (\cos\alpha)(1-\theta)/2. \tag{5.21}$$

Hence an explicit estimate for M in (5.12) can be obtained, since h_n can be computed from the given data.

Using a mixture of methods, which are brought together in [65], one can establish the following explicit results for $K(a_n/1)$:

(A) If $|a_m| \leq \min(1/6, \alpha n^{-\rho})$, $\alpha > 0$, $\rho > 0$, $m \geq n \geq n_1$, then there exist constants $K_1 > 0$, $M_1 > 0$ such that for $n > n_1$, $k > 0$

$$|f_{n+k} - f_n| \leq K_1 \left(\frac{M_1}{n}\right)^{\rho(n+\frac{3}{2})}.$$

(B) If $\lim a_n = a \in \mathbf{C} - (-\infty, -1/4]$, then for every q satisfying

$$\left|\frac{-1+\sqrt{1+4a}}{+1+\sqrt{1+4a}}\right| < q < 1$$

there exists a $K_2 = K_2(q, n_2) > 0$ such that for $n \geq n_2$, $k > 0$

$$|f_{n+k} - f_n| < K_2 q^n.$$

(C) If $a_n \in P(\alpha, \theta)$ (see (5.20)) for some $0 < \theta \leq 1$ and $a_n = O(n^\beta)$ for some β, $0 < \beta \leq 1$, then there exist $K_3 > 0$, $M_3 > 0$, $E_3 > 0$ and $L_3 > 1$ such that for $n \geq n_3$, $k > 0$

$$|f_{n+k} - f_n| < \begin{cases} a\dfrac{K_3}{n^{E_3}}, & \text{for } \beta = 1, \\[2ex] \dfrac{M_3}{L^{n^{1-\beta}}}, & \text{for } 0 < \beta < 1. \end{cases}$$

(D) For the S-fraction $K(a_n z/1)$ with $a_n > 0$, $|\arg z| < \pi$, let $a_n = O(n^\alpha)$, $0 < \alpha \leq 2$. Then there exist constants $K_4 > 0$, $M_4 > 0$, $E_4 > 0$ and $L_4 > 1$ such that for $n \geq n_4$, $k > 0$

$$|f_{n+k}(z) - f_n(z)| < \begin{cases} \dfrac{K_4}{n^{E_4\sqrt{z}}} & \text{for } \alpha = 2 \\[2ex] \dfrac{M_4}{L_4^{n\delta}}, \quad \delta := \dfrac{2-\alpha}{2\sqrt{z}}, & \text{for } 0 < \alpha < 2. \end{cases}$$

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