The Discrete Korteweg–de Vries Equation

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Abstract. We review the different aspects of integrable discretizations in space and time of the Korteweg–de Vries equation, including Miura transformations to related integrable difference equations, connections to integrable mappings, similarity reductions and discrete versions of Painlevé equations as well as connections to Volterra systems.

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1. Introduction

When Korteweg and de Vries published their famous equation in the *Philosophical Magazine* [1], exactly one hundred years ago on the occasion of this conference, they were probably far from being aware of the enormous ramifications this nonlinear partial differential equation would have, and whose revived interest was marked at the end of the sixties by the celebrated paper by Gardner, Green, Kruskal and Miura [2]. What is now called the KdV equation was the drosophila of a class of nonlinear evolution equations, which triggered the development of new methods and structures (inverse scattering transform, Bäcklund transformations, bi-Hamiltonian structures, master-symmetries, etc.), applicable not only to the KdV equation itself, but also to a host of other systems as well (cf., e.g., [3, 4]). Independently, and curiously enough at roughly the same time as Korteweg and de Vries were studying models for shallow water waves, there was an important development in geometry, marked by the classic works of Bäcklund [5], Darboux [6], and Bianchi [7]. It was only in this century that people realized how relevant the works of these classical geometers were to the study of the much more applied problem of the behaviour of water in canals. It is not surprising from a modern perspective why it took almost eighty years before this connection was fully understood: the road from geometry to the analysis of water waves goes via algebra, and a kind of algebra that was invented only

sixty years later, namely the type of infinite-dimensional Lie algebras that are nowadays known as Kac–Moody algebras. In this talk we do not want to go into even more recent and more exotic connection, such as with knot theory, quantum groups or moduli spaces of curves. Who could guess that (via string theory and conformal field theory) the small and beautiful equation that Korteweg and de Vries wrote down in their paper [1] would stand at the very foundations of present-day descriptions of the universe?

Let us now come to the other theme of our talk: difference or discrete systems. Korteweg and de Vries lived in an era that had no knowledge of computers. In their days, nonetheless, from the point of view of classical education in mathematics, differential analysis started with difference equations (cf., e.g., [8]). And it was thought to be very natural always to link differential and difference equations together. It was the time when another famous Dutch mathematician, Thomas Stieltjes, developed theories of continued fractions, recursion relations and orthogonal polynomials, thus standing in a long and rich tradition of 18th- and 19th-century classical analysis. Little did Korteweg and de Vries realize that also these subjects were intimately linked to their water-wave equation. The study of linear difference equations was being developed in the beginning of this century, mainly by the propagents of the Birkhoff school (cf., e.g., [9]). Unfortunately, the good tradition of treating differential equations always alongside their difference counterparts, and vice versa, seemed to have got lost during the second half of this century, especially after the second world war.

Today, the advent of computers provides new motivations to look at difference equations. Neural networks, cellular automata, self-organizing phenomena and complexity are some of the key words of modern developments inspired by the birth and growth of newer and newer generations of computers. On the other hand, numerical analysis, and its applications to all branches of sciences where computer calculations play a role, relies heavily on discreteness in all its aspects: interpolations, convergence algorithms, and their analytical bases: orthogonal polynomials, recurrence relations. It is unlikely that Korteweg and de Vries could envisage how important their equation was to these aspects of modern-day research. Little also do the people who use these tools and methods, know that the small and elegant equation that Korteweg and de Vries wrote down in order to capture phenomena of wave motion is of relevance to them.

In this paper we want to tie some of these different threads together. We will look into exact discretizations of the KdV equation and highlight a few important aspects of them. A starting point is an integrable lattice version of the KdV equation, which is a nonlinear partial *difference* equation, i.e. a system in which both the spatial- as well as the time-variable is discrete. Such systems were systematically studied in a number of papers, e.g., in [10–18] (cf., also [19–23]). The lattice model is, in a sense, more fundamental than the original continuous equation, as one can always retrieve the latter by applying an appropriate continuum limit on the lattice system. However, the reverse is not true: to obtain a

discretization of the KdV equation which retains its essential integrability characteristics, is a highly nontrivial undertaking. Once having it at its disposal, one can use it as a universal model to study a number of features that, as we will see, become more transparent on the level of the difference system.

The outline of this paper is as follows. In Section 2 we will exhibit the lattice version of the KdV equation, or rather of its potential (i.e. integrated) version. We will show what continuum limits to apply in order to get back to the continuous KdV equation and what are its relations to other integrable lattice systems. In Section 3, we will list some other integrable lattice equations that are less directly related to the KdV equation, but still are of importance. Next, we will look into special solutions of the KdV equation. First, in Section 4, solutions arising from periodic initial value problems in the lattice will be considered, leading to a reduction to integrable finite-dimensional mappings. Second, in Section 5, we will exhibit similarity type of solutions which lead to lattice and difference versions of the second Painlevé transcendent. Finally, we give some insight into the algebraic background of these lattice equations, deriving them from the so-called direct linearization approach method. In this approach, briefly reviewed in Section 6, the relevant equations are embedded in an infinite matrix structure, which gives a powerful handle on establishing the interconnection of the different (Miura-) related systems. For example, a discrete-time version of the Volterra model arises naturally in this way, thus leading possibly to connections with matrix models in two-dimensional quantum gravity and string theory, but also with numerical schemes, Padé approximants and convergence algorithms.

2. The KdV Lattice and Related Lattice Equations

The lattice version of the KdV equation that we want to investigate in this paper is the following nonlinear partial difference equation

$$\left(p - q + u_{n,m+1} - u_{n+1,m}\right)\left(p + q - u_{n+1,m+1} + u_{n,m}\right) \;=\; p^2 - q^2. \quad (2.1)$$

To explain the notations in Equation (2.1), we mention that $u = u_{n,m}$ is the dynamical (field) variable at site (n, m), $n, m \in \mathbf{Z}$, $p, q \in \mathbf{C}$ are lattice parameters. Equation (2.1) was derived in [10] from the direct linearization approach (see Section 5 below), and further studied in a number of papers [11–14].

We mention first a few important aspects of Equation (2.1). First, we note that, in contrast to the continuum KdV equation, the lattice equation is covariant with respect to the interchange of the two discrete variables, n and m, interchanging also the lattice parameters p and q. This might come as a surprise, but as we shall see below, we must conclude that the noncovariance of the continuous KdV might be considered to be an artefact of the continuum limit. Actually, the discrete variables being on the same footing, is a very nice feature of the equation which allows us to play some games on the lattice (like making particular choices of

initial value- and boundary-value problems) which would be quite awkward in the continuum.

Another feature of the equation is that it arises from a discrete action principle. The action for the KdV lattice (2.1) reads

$$
S = \sum_{n,m \in \mathbf{Z}} \left[u_{n,m+1} \left(u_{n+1,m+1} - u_{n,m} \right) + \right.
$$

$$
\left. + \varepsilon \delta \ln \left(\varepsilon + u_{n,m} - u_{n+1,m+1} \right) \right], \tag{2.2}
$$

where $\delta = p - q$, $\varepsilon = p + q$. The Euler–Lagrange equations for (2.2), which are obtained by variation of S with respect to the variables $u_{n,m}$, i.e.

$$
\frac{\delta S}{\delta u_{n,m}} = 0,
$$

leads to the equation

$$
u_{n+1,m} - u_{n,m+1} + u_{n-1,m} - u_{n,m-1} +
$$

$$
+ \frac{\varepsilon \delta}{\varepsilon + u_{n,m} - u_{n+1,m+1}} - \frac{\varepsilon \delta}{\varepsilon + u_{n-1,m-1} - u_{n,m}} = 0, \tag{2.3}
$$

which is a consequence of Equation (2.1). In fact, as we shall demonstrate below, Equation (2.1) goes to the potential (i.e. integrated) KdV equation, whereas Equation (2.3) goes to the KdV equation itself. However, by abuse of terminology, we will refer to (2.1) as the lattice KdV equation, because we will be mainly concerned with that equation.

A third important aspect of the lattice Equation (2.1), directly related to the integrability of ths system, is that it arises as the compatibility condition of an overdetermined linear system (Lax or Zakharov–Shabat pair). In fact, the Lax pair takes the form of a set of two equations for the translations of a 'wavevector' ϕ_k in the n- and m-directions,

$$
(p - k)\phi_k(n + 1, m) = L_k \cdot \phi_k(n, m), \tag{2.4a}
$$

$$
(q - k)\phi_k(n, m + 1) = M_k \cdot \phi_k(n, m), \tag{2.4b}
$$

where L_k is given by

$$
L_k = \begin{pmatrix} p - u_{n+1,m} & 1 \\ k^2 - p^2 + (p + u_{n,m})(p - u_{n+1,m}) & p + u_{n,m} \end{pmatrix}, \tag{2.5}
$$

and where M_k is given by a similar matrix obtained from (2.5) by making the replacements $p \to q$ and $(n + 1, m) \to (n, m + 1)$. The subscript k in (2.4) and (2.5) denotes the dependence on the spectral parameter. Establishing a Lax pair and an action for the KdV system ensures us that, in principle, we can expect many of the integrability characteristics to hold for this system, such as the

existence of an infinite number of integrals of the discrete motion, of an inverse scattering scheme and of a canonical formalism. However, to make these things precise and to be able to solve suitable initial-value problems on the lattice, we should supply the KdV system with appropriate boundary conditions. Below, we shall investigate solution and initial value problems in two specific situation: (i) periodic initial data on well-chosen configurations of lattice points ('staircases' in the lattice). This will lead to exactly integrable *mappings*, i.e. discrete-time systems with a finite number of degrees of freedom; (ii) localized initial data under the condition of scaling invariance leading to lattice and discrete versions of the second Painlevé transcendent.

Now why is it justified to refer to (2.1) as the lattice KdV equation? Well, let us investigate what happens under a continuum limit, bringing us back to the continuum situation. Since we have two discrete variables in the lattice equation, namely n and m, we have to perform the continuum limit in two steps: one letting the variable m become continuous, reducing our equation to a *differential-difference equation*, i.e. an equation with one discrete and one continuous variable, and a second step in which the remaining discrete variable will become continuous. Both steps are achieved by shrinking the corresponding lattice step (encoded in the parameters p and q) to zero. The most convenient way of doing this is first, on the lattice, to do a change of discrete variables, namely $u_{n,m} =: u_{n'}(m)$, and then doing the limit by taking

$$\delta \equiv p - q \longmapsto 0, \quad m \mapsto \infty, \quad \delta m \mapsto \tau, \tag{2.6}$$

where $n' = n + m$ is to remain fixed. This limit is motivated from the behaviour of discrete plane-wave factors

$$\rho_k(n, m) = \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m, \tag{2.7}$$

which, in fact, are related to solutions of (2.4) for the linearized problem, namely

$$\left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m \longmapsto \left(\frac{p+k}{p-k}\right)^{n'} \exp\left\{\frac{2k\tau}{p^2 - k^2}\right\} \tag{2.8}$$

(cf. [10, 11, 14]). By this limit, the lattice KdV equation (2.1) goes over into the following differential-difference equation (we omit the prime of the n' variable)

$$1 + \partial_\tau u_n = \frac{2p}{2p - u_{n+1} + u_{n-1}}. \tag{2.9}$$

This differential-difference equation is related to the Kac–van Moerbeke–Volterra equation [24]. Next, the second continuum limit is performed by taking

$$p \longrightarrow \infty, \quad n \longrightarrow \infty, \quad \tau \longrightarrow \infty, \tag{2.10a}$$

such that

$$2\frac{n}{p} + 2\frac{\tau}{p^2} \longmapsto x, \qquad \frac{2}{3}\frac{n}{p^3} + 2\frac{\tau}{p^4} \longmapsto t, \qquad (2.10b)$$

in which case (2.9) goes over into the potential KdV equation

$$u_t = u_{xxx} + 3u_x^2, \qquad (2.11)$$

which is the integrated version of the KdV equation.

Before looking into special reductions of the lattice KdV equation, let us focus for a moment on a very important aspect of such integrable systems, namely that they are not isolated objects: they have a number of companion systems which are related to the original equation by some kind of implicit transformation, usually referred to as Miura transformation. This holds true for the original KdV equation, but also for its lattice version (2.1). First, there is the discrete analogue of the *modified* Korteweg–de Vries (MKdV) equation, which reads

$$(p - r)\frac{v_{n,m+1}}{v_{n+1,m+1}} - (q - r)\frac{v_{n+1,m}}{v_{n+1,m+1}}$$
$$= (p + r)\frac{v_{n+1,m}}{v_{n,m}} - (q + r)\frac{v_{n,m+1}}{v_{n,m}}, \qquad (2.12)$$

which we will refer to as the lattice modified KdV equation.

The Miura transformation between the two Equations (2.12) and (2.1) can be obtained from the relations

$$(p - r)v_{n,m} - (p + u_{n,m})v_{n+1,m} = t_{n+1,m}, \qquad (2.13a)$$

$$(p - u_{n+1,m})v_{n,m} - (p + r)v_{n+1,m} = t_{n,m}, \qquad (2.13b)$$

in which $t_{n,m}$ and $v_{n,m}$ are some new functions of the lattice sites. By the general covariance of the lattice systems, Equations (2.13) hold also for the other lattice direction, i.e. with the replacements: $(n + 1, m) \longmapsto (n, m + 1)$ and $p \longmapsto q$. Thus, by combining these relations we can eliminate, for instance, $t_{n,m}$ entirely, to obtain the Miura transformation between (2.1) and (2.12) consisiting of the relations

$$p - q + u_{n,m+1} - u_{n+1,m}$$
$$= (p - r)\frac{v_{n,m+1}}{v_{n+1,m+1}} - (q - r)\frac{v_{n+1,m}}{v_{n+1,m+1}}, \qquad (2.14a)$$

$$p + q + u_{n,m} - u_{n+1,m+1}$$
$$= (p - r)\frac{v_{n,m}}{v_{n+1,m}} + (q + r)\frac{v_{n+1,m+1}}{v_{n+1,m}}. \qquad (2.14b)$$

The lattice MKdV Equation (2.12) also carries a Lax pair which is again of the form (2.4), together with

$$
L_k = \begin{pmatrix} p - r & v_{n+1,m} \\ (k^2 - r^2)\dfrac{1}{v_{n,m}} & (p + r)\dfrac{v_{n+1,m}}{v_{n,m}} \end{pmatrix},
\tag{2.15}
$$

instead of (2.5), and where again M_k is given by a similar matrix obtained from (2.15) by making the same replacements $p \to q$ and $(n + 1, m) \to (n, m + 1)$. Again, there is an action principle, now not directly in terms of the variable $v_{n,m}$, but in terms of its logarithm. Thus writing $v_{n,m} = e^{y_{n,m}}$, we have the action

$$
\mathcal{S} = \sum_{n,m \in \mathbf{Z}} \Big[y_{n,m+1} \big(y_{n+1,m+1} - y_{n,m} \big) + F \big(y_{n,m} - y_{n+1,m+1} + \sigma \big) -
$$

$$
- F \big(y_{n,m} - y_{n+1,m+1} + \rho \big) \Big],
\tag{2.16}
$$

in which the function F is given by

$$
F(x) \equiv \int_{-\infty}^{x} d\xi \, \log(1 + e^{\xi}),
\tag{2.17}
$$

which, in fact, is directly related to the Euler dilogarithm function, and where σ and ρ are parameters related to the lattice parameters p and q, namely by

$$
e^{\sigma} = \frac{q - r}{p + r}, \qquad e^{\rho} = \frac{p - r}{q + r}.
$$

So, everything that holds true for the lattice KdV equation holds true also for the lattice MKdV equation. Performing the continuum limit (2.6) also to this case, we obtain in the first step

$$
p \, \partial_{\tau} y_n = \tanh \left[\tfrac{1}{2}(y_{n+1} - y_{n-1}) \right], \quad y_n \equiv \ln v_n,
\tag{2.18}
$$

which is a differential-difference version of the MKdV equation, and in the next step we get

$$
y_t = y_{xxx} + 2y_x^3,
\tag{2.19}
$$

which is the potential MKdV equation.

There is, however, an even more fundamental lattice equation, related to both the lattice KdV and MKdV equations, which we write in terms of a new object $s_{n,m}$, related to the MKdV field by

$$
1 - (p + r')s_{n+1,m} + (p - r)s_{n,m} = v_{n+1,m} w_{n,m},
\tag{2.20}
$$

in which $w_{n,m}$ is the MKdV field for the choice r' as the fixed parameter. For $s_{n,m}$ we have the lattice equation

$$\frac{1 - (p + r')s_{n+1,m} + (p - r)s_{n,m}}{1 - (q + r')s_{n,m+1} + (q - r)s_{n,m}}$$

$$= \frac{1 - (q + r)s_{n+1,m+1} + (q - r')s_{n+1,m}}{1 - (p + r)s_{n+1,m+1} + (p - r')s_{n,m+1}}. \qquad (2.21)$$

We will refer to Equation (2.21) as the lattice Krichever–Novikov equation. It was first derived in [10], where it was shown that, in fact, the other lattice equations are included in this more general equation for special choices of the parameters r and r'. Furthermore, continuum limits yield the various differential-difference counterparts of the KdV equation, as well as of the MKdV equation, as was shown in [11]. Finally, one can derive a Lax pair for Equation (2.21) which is of the form

$$L_k = \begin{pmatrix} p - r & P'_{n+1,m} \\ (k^2 - r^2)\dfrac{1}{P_{n,m}} & (p + r)\dfrac{P'_{n,m}}{P_{n,m}} \end{pmatrix}, \qquad (2.22)$$

with

$$P_{n,m} = 1 - (p + r)s_{n+1,m} + (p - r')s_{n,m},$$
$$P'_{n,m} = 1 - (p + r')s_{n+1,m} + (p - r)s_{n,m}.$$

The equation for s is a very rich equation, and contains many parameter-subcases. In the continuum limit we obtain

$$\partial_\tau s_n = \frac{s_{n-1} - s_{n+1} - 2p(s_n^2 + s_{n+1}s_{n-1}) + (r + r')s_n(s_{n+1} - s_{n-1})}{2p + (p - r')(p - r)s_{n-1} - (p + r')(p + r)s_{n+1}}, \qquad (2.23)$$

respectively,

$$s_t = s_{xxx} + 3\frac{(s_{xx} + r's_x)(s_{xx} + rs_x)}{1 - (r + r')s - 2s_x}. \qquad (2.24)$$

in the two subsequent steps. In the special case that $r = r' = 0$, Equation (2.24) reduces to another very important equation in the KdV family: the Schwarzian KdV equation, or Krichever–Novikov equation (cf. [25]),

$$\psi_t = \psi_x S(\psi), \quad S(\psi) \equiv \frac{\psi_{xxx}}{\psi_x} - \frac{3}{2}\frac{\psi_{xx}^2}{\psi_x^2}, \qquad (2.25)$$

which is an equation invariant under Möbius transformations

$$\psi \longmapsto \frac{a\psi + b}{c\psi + d}, \qquad (2.26)$$

i.e. the group $SL(2, \mathbf{R})$. In that special case the lattice Equation (2.21) can be easily seen to reduce to an even more simple equation, namely

$$\frac{(z_{n,m} - z_{n+1,m})(z_{n,m+1} - z_{n+1,m+1})}{(z_{n,m} - z_{n,m+1})(z_{n+1,m} - z_{n+1,m+1})} = \frac{q^2}{p^2}, \tag{2.27}$$

in which

$$s_{n,m} = z_{n,m} + \frac{n}{p} + \frac{m}{q}.$$

The left-hand side of Equation (2.27) is the conformally invariant cross-ratio of four point in the complex plane, which – as was noted before in, e.g., [26] (cf. also [27]) – can be considered to be the discrete analogue of the Schwarzian derivative S. It is remarkable that the equation: cross-ratio equal to constant can be considered to be an integrable lattice equation carrying a Lax pair.

3. Other Lattice Equations

Of course, the lattice KdV is not the only equation for which all of this holds: there are many other integrable lattice equations known, and they were constructed in a number of papers [10–17], as well as in the literature [19–23]. We cannot go here into the details of all these equations, as the focuspoint in this celebration paper is the Korteweg–de Vries example. However, to demonstrate the wide variety of equations, let us just list a few of them.

Probably the best known example of an integrable lattice equation is the lattice analogue of the sine-Gordon equation,

$$\sin \left(\theta_{n,m} + \theta_{n+1,m} + \theta_{n,m+1} + \theta_{n+1,m+1}\right)$$
$$= \frac{p}{q} \sin \left(\theta_{n,m} - \theta_{n+1,m} - \theta_{n,m+1} + \theta_{n+1,m+1}\right), \tag{3.1}$$

is directly related to the lattice MKdV for fixed parameter $r = 0$ by a simple transformation of the form

$$v_{n,m} \longmapsto \exp \left\{2i(-1)^m \theta_{n,m}\right\}.$$

The lattice sine-Gordon equation was first presented by Hirota [20], and therefore sometimes referred to as the Hirota equation. Actually, it was already present in the work of Bianchi [7], appearing as permutability condition of Bäcklund transformations for the usual sine-Gordon equation, which has a great significance in the classical differential geometry of surfaces in $\mathbf{R}^3$.

Another important class of lattice equations is obtained by extending the lattice KdV equation to higher-order partial difference systems. This is the so-called lattice Gel'fand–Dikii hierarchy, presented first in [17], which is the discrete analogue of the integrable family of partial differential equations associated with the higher-order differential spectral problem (cf., e.g., [28]). The lattice KdV

equation is naturally embedded as the lowest member in this class of equations, which is labelled by the roots of unity, $\omega \equiv \exp(2\pi i/N)$, $N \in \mathbf{Z}$. Let us mention only the first member after the KdV lattice (which is the case $N = 2$), leading to a lattice analogue of the Boussinesq (BSQ) equation ($N = 3$). The linear system in this case is given by

$$(p + \omega k)\phi_k(n + 1, m) = L_k \cdot \phi_k(n, m), \tag{3.2a}$$

$$(q + \omega k)\phi_k(n, m + 1) = M_k \cdot \phi_k(n, m), \tag{3.2b}$$

in which

$$L_k = \begin{pmatrix} p - u_{n+1,m} & 1 & 0 \\ -v_{n+1,m} & p & 1 \\ k^3 + p^3 + * & w_{n,m} & p + u_{n,m} \end{pmatrix}, \tag{3.3}$$

and again M_k is obtained by replacing $p \to q$ and $(n + 1, m) \to (n, m + 1)$. In (3.3) the v and w are auxiliary fields (not to be confused with the variables appearing in the previous section), and the term $*$ in the left-lower corner of the matrix L_k is determined by the condition that the determinant $\det(L_k) = p^3 + k^3$.

The compatibility relations of (3.3) lead to the relations

$$
\begin{aligned}
v_{n,m+1} - v_{n+1,m} &= u_{n+1,m+1}\left(p - q + u_{n,m+1} - u_{n+1,m}\right) + \\
&\quad + + q\, u_{n+1,m} - p\, u_{n,m+1}, \\
w_{n,m+1} - w_{n+1,m} &= -u_{n,m}\left(p - q + u_{n,m+1} - u_{n+1,m}\right) + \\
&\quad + p\, u_{n+1,m} - q\, u_{n,m+1}, \\
v_{n+1,m+1} - w_{n,m} &= pq - (p + q + u_{n,m})(p + q - u_{n+1,m+1}) + \\
&\quad + \frac{p^3 - q^3}{p - q + u_{n,m+1} - u_{n+1,m}},
\end{aligned}
\tag{3.4}
$$

from which one obtains the lattice BSQ equation

$$
\begin{aligned}
&\frac{p^3 - q^3}{p - q + u_{n+1,m+1} - u_{n+2,m}} - \frac{p^3 - q^3}{p - q + u_{n,m+2} - u_{n+1,m+1}} - \\
&\quad - u_{n,m+1}u_{n+1,m+2} + u_{n+1,m}u_{n+2,m+1} + \\
&\quad + u_{n+2,m+2}\left(p - q + u_{n+1,m+2} - u_{n+2,m+1}\right) + \\
&\quad + u_{n,m}\left(p - q + u_{n,m+1} - u_{n+1,m}\right) \\
&= (2p + q)\left(u_{n+1,m} + u_{n+1,m+2}\right) - \\
&\quad - (p + 2q)(u_{n,m+1} + u_{n+2,m+1}).
\end{aligned}
\tag{3.5}
$$

From (3.5) by appropriate continuum limits, we recover the continuum BSQ equation. An intermediate continuum limit of the lattice BSQ yields the following

differential-difference equation

$$3p^2\partial_\tau \left(\frac{1}{1+\dot{u}_n}\right) = (1+\dot{u}_{n+1})(u_{n+2} - u_{n-1} - 3p) -$$
$$-(1+\dot{u}_{n-1})(u_{n+1} - u_{n-2} - 3p), \qquad (3.6)$$

whereas the second limit yields the (potential) BSQ equation. There is also an action for the BSQ lattice, given by (here $\varepsilon \equiv p^2 + pq + q^2$, δ as before)

$$S = \sum_{n,m\in\mathbf{Z}} \Big[\varepsilon\delta \ln\big(\delta + u_{n,m+1} - u_{n+1,m}\big) + q\,u_{n,m}\,u_{n,m+1} - p\,u_{n,m}\,u_{n+1,m} -$$

$$-(p + q + u_{n,m})\,(p + q - u_{n+1,m+1})\,(\delta + u_{n,m+1} - u_{n+1,m})\Big]. \qquad (3.7)$$

So, in principle, the situation for the lattice BSQ equation is similar as for the KdV lattice, except that here we are dealing with a system in which there are not only nearest-neighbour terms in the equation, but also next-nearest-neighbour terms. Similar to the KdV situation, there is also here a Miura transformation to a modified equation, which is the discrete analogue of the modified BSQ equation (the continuum modified BSQ was first derived in [29]), and it reads

$$\frac{(p^2 + pr + r^2)\,v_{n+1,m+1} - (q^2 + qr + r^2)\,v_{n,m+2}}{(p-r)\,v_{n,m+2} - (q-r)\,v_{n+1,m+1}}\,\frac{v_{n+1,m+2}}{v_{n,m+1}} -$$

$$-\frac{(p^2 + pr + r^2)\,v_{n+2,m} - (q^2 + qr + r^2)\,v_{n+1,m+1}}{(p-r)\,v_{n+1,m+1} - (q-r)\,v_{n+2,m}}\,\frac{v_{n+2,m+1}}{v_{n+1,m}}$$

$$= (p-r)\left(\frac{v_{n,m}}{v_{n+1,m}} - \frac{v_{n+1,m+2}}{v_{n+2,m+2}}\right) -$$

$$-(q-r)\left(\frac{v_{n,m}}{v_{n,m+1}} - \frac{v_{n+2,m+1}}{v_{n+2,m+2}}\right). \qquad (3.8)$$

Apart from the above examples of lattice equations related in some way to our original lattice Equation (2.1), there are also lattice equations belonging to other types of spectral problems. Lattice systems related to the AKNS scheme [4], for PDE's include a complicated partial difference version of the nonlinear Schrödinger equation (NLS) given in [19]. However, a more elegant version was derived in [11] which reads

$$|\alpha|^2 + |\beta|^2 + 2\mathrm{Re}\left(\theta\alpha\beta\,\phi_{n,m}\phi^*_{n,m+1}\right)$$

$$= |\alpha|^2\big(1 + |\phi_{n,m+1}|^2\big)\frac{\beta^*\phi_{n+1,m+1} - \theta\beta\phi_{n,m}}{\theta\alpha\phi_{n+1,m} - \alpha^*\phi_{n,m+1}} +$$

$$+|\beta|^2\big(1 + |\phi_{n,m}|^2\big)\frac{\theta\alpha\phi_{n+1,m} - \alpha^*\phi_{n,m+1}}{\beta^*\phi_{n+1,m+1} - \theta\beta\phi_{n,m}}, \qquad (3.9)$$

in which α and β are parameters with $\mathrm{Re}(\alpha) = \mathrm{Re}(\beta)$, depending on the lattice parameters, θ is a parameter with modulus 1, $|\theta| = 1$, and the $*$ denotes complex

conjugation. This equation is Miura-related to a lattice version of the isotropic Heisenberg spin chain (IHSC) equation, also derived in [11], reads

$$\mathbf{S}_{n+1,m+1} - \mathbf{S}_{n+1,m} + \mathbf{S}_{n,m+1} - \mathbf{S}_{n,m}$$

$$= \Delta \left(\frac{\gamma(\mathbf{S}_{n,m})(\mathbf{S}_{n,m} + \mathbf{S}_{n,m+1}) + \alpha \mathbf{S}_{n,m+1}\mathbf{S}_{n,m}}{1 + \mathbf{S}_{n,m+1} \cdot \mathbf{S}_{n,m}} \right),$$

$$\gamma(\mathbf{S}_{n,m})^2 = (\mathbf{S}_{n,m+1} \cdot \mathbf{S}_{n,m})^2 + \beta \mathbf{S}_{n,m+1} \cdot \mathbf{S}_{n,m} + \beta - \alpha^2 - 1,$$

$$|\mathbf{S}_{n,m}|^2 = 1, \tag{3.10}$$

where $(\Delta f)_{n,m} \equiv f_{n+1,m} - f_{n,m}$, and α, β are arbitrary real constants. Another lattice version of the IHSC was presented in [23], but that model seem to carry complex spins. An anisotropic version of (3.10), i.e. the lattice Landau–Lifschitz equation, was presented in [15].

Furthermore, there are lattice equations in $(2+1)$ dimensions, which involve three lattice directions associated with three lattice parameters p, q, and r. Specific examples are the lattice KP equation [12, 13],

$$\frac{p - r + u_{n,m+1,k+1} - u_{n+1,m+1,k}}{p - r + u_{n,m,k+1} - u_{n+1,m,k}}$$

$$= \frac{q - r + u_{n+1,m,k+1} - u_{n+1,m+1,k}}{q - r + u_{n,m,k+1} - u_{n,m+1,k}}, \tag{3.11}$$

and the lattice sine-Gordon equation in $(2+1)$ dimensions [20], which is a coupled system, namely

$$r \left(v_{n,m+1,k+1} w_{n,m+1,k} - w_{n+1,m,k} v_{n+1,m,k+1} \right) +$$

$$+ p \left(\frac{v_{n+1,m,k+1}}{v_{n,m,k+1}} - \frac{w_{n,m+1,k}}{w_{n+1,m+1,k}} \right) + q \left(\frac{w_{n+1,m,k}}{w_{n+1,m+1,k}} - \frac{v_{n,m+1,k+1}}{v_{n,m,k+1}} \right) = 0,$$

$$p \left(\frac{v_{n+1,m,k}}{v_{n,m,k}} - \frac{w_{n,m+1,k}}{w_{n+1,m+1,k}} \right) + q \left(\frac{w_{n+1,m,k}}{w_{n+1,m+1,k}} - \frac{v_{n,m+1,k}}{v_{n,m,k}} \right) = 0, \tag{3.12}$$

and the lattice modified KP (MKP) equation [13],

$$(p + s) \left(\frac{v_{n+1,m,k+1}}{v_{n,m,k+1}} - \frac{v_{n+1,m+1,k}}{v_{n,m+1,k}} \right) +$$

$$+ (q + s) \left(\frac{v_{n+1,m+1,k}}{v_{n+1,m,k}} - \frac{v_{n,m+1,k+1}}{v_{n,m,k+1}} \right) +$$

$$+ (r + s) \left(\frac{v_{n,m+1,k+1}}{v_{n,m+1,k}} - \frac{v_{n+1,m,k+1}}{v_{n+1,m,k}} \right) = 0, \tag{3.13}$$

in which s is an additional fixed parameter. For the special choice $s = -r$ this equation reduces to what is referred to in [23] as the discrete KP equation, and

which is related to the DAGTE (discrete analogue of generalized Toda equation) of [20]. We can of course add other equations to this small list, such as the IHSC in $(2+1)$ dimensions, the lattice Davey–Stewartson equation and generalizations to matrix systems. A systematic construction of lattice equations in $(2+1)$ dimensions was presented in [16]. Although these lattice equations share with their continuum counterparts many of the characteristics of integrable systems, such as the existence of Lax pairs and discrete inverse scattering scheme, many of the features of these systems, still have to be investigated. There is to date little systematic study about solutions of such equations. Since a few years a number of people are engaged in developing the theory of integrable lattice equations further, i.e. the search for analytic solutions of initial and boundary problems, similarity reductions and the problem of quantization. In this contribution, we will highlight for the lattice counterparts of the KdV equation what has been done so far. In particular, we will exhibit two special classes of solutions (reductions): solutions of periodic initial-value problems leading to integrable mappings, the lattice analogue of similarity solutions leading to lattice versions of the second Painlevé equation. A third class of solutions of the lattice equations, in particular of the lattice KP equation, consists of pole-solutions (cf. [30]), of rational, trigonometric (hyperbolic) or elliptic type. As was shown in a number of recent papers, [31–33], these lead to integrable discrete-time many-body problems of (relativistic or nonrelativistic) Calogero–Moser type. We will not have the opportunity of displaying these solutions here.

4. Mappings of KdV Type

A first type of solutions of the lattice KdV equation (2.1) that we will discuss here, are solutions arising from a periodic initial-value problem on the lattice. We will show here, that these solutions give a rise of a reduction to integrable dynamical *mappings* with $2P$ degrees of freedom $(P = 1, 2, \ldots)$. An important feature of the lattice KdV equation (2.1) is that the variables in the equation are arranged along an elementary square or 'plaquette' in the lattice, i.e. involving the lattice points (n, m), $(n+1, m)$, $(n, m+1)$, and $(n+1, m+1)$ only. Thus, a 'local' initial-value problem for the lattice KdV can be given by specifying values $a_0, a_1, a_2, a_3, \ldots$ of the field u on a staircase consisting of alternating horizontal and vertical steps (cf. [34]), see Figure 1.

Choosing such initial data for the variables along such a staircase on the lattice, namely

$$u_{j,j} =: a_{2j}, \qquad u_{j+1,j} =: a_{2j+1}, \quad j \in \mathbf{Z},$$

we perform iterations by updating the lattice variables u along a vertical shift in the m-direction, i.e.

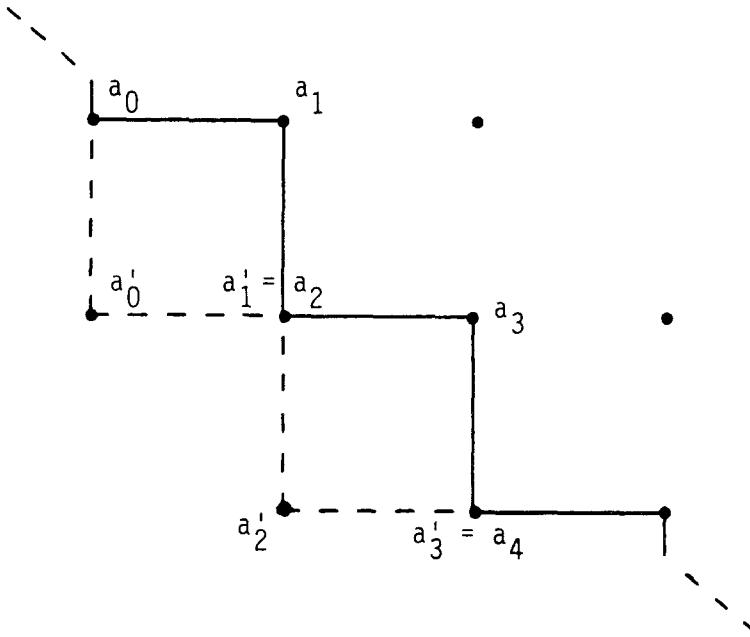$$u_{j,j+1} = a'_{2j}, \qquad u_{j+1,j+1} = a'_{2j+1},$$

Fig. 1.   Configuration of periodic initial data on the lattice.

using the lattice KdV equation (2.1). In this way we obtained the following mapping

$$a'_{2j} = a_{2j+1} - \delta + \frac{\varepsilon\delta}{\varepsilon - a_{2j+2} + a_{2j}}, \qquad a'_{2j+1} = a_{2j+2}, \tag{4.1}$$

where $\delta = p - q$, $\varepsilon = p + q$, which by imposing periodic initial conditions on the staircase, i.e.

$$a_{2(j+P)} = a_{2j}, \qquad a_{2(j+P)+1} = a_{2j+1},$$

reduces to a finite-dimensional mapping of dimension $2P$. Introducing the variables for the differences on odd and even sites of the staircase, i.e.

$$x_j \equiv \varepsilon + a_{2j-1} - a_{2j+1}, \qquad y_j \equiv \varepsilon + a_{2j} - a_{2j+2}, \tag{4.2}$$

the mapping can be even further reduced to a $(2P - 2)$-dimensional one which reads

$$\begin{cases} x'_j = y_j, \\ y'_j = x_{j+1} - \dfrac{\varepsilon\delta}{y_{j+1}} + \dfrac{\varepsilon\delta}{y_j}, \end{cases} \qquad j = 1, \dots, P, \tag{4.3}$$

with the Casimirs

$$\sum_{j=1}^{P} x_j = C_1, \qquad \sum_{j=0}^{P-1} y_j = C_2,$$

C_1 and C_2 being constant equal to $P\varepsilon$.

The mapping (4.3) is a multidimensional generalization of the McMillan mapping [35], and it arises as the compatibility condition of a linear (Zakharov–Shabat type of) problem, which is obtained using a special property of the matrices L_k and M_k of (2.4), (2.5) and performing at each site of the staircase a gauge transformation (cf. [34, 36]). The compatibility equation for the mapping (4.3) in terms of the reduced variables x_j and y_j is

$$L'_j(k) \cdot M_j(k) = M_{j+1}(k) \cdot L_j(k), \tag{4.4}$$

where

$$L_j(k) = \begin{pmatrix} 0 & 1 \\ ld + \varepsilon\delta & x_{j+1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ ld & y_j \end{pmatrix},$$

$$M_j(k) = \begin{pmatrix} -\varepsilon\delta/y_j & 1 \\ ld & y_j \end{pmatrix}, \tag{4.5}$$

in which $\lambda = k^2 - p^2$. From this reduced linear system one can construct a monodromy matrix $T(k)$ by gluing the elementary translation matrices L_j along the staircase over one period P, leading to

$$T(k) = \prod_{j=0}^{P-1} L_j(k). \tag{4.6}$$

The trace of the monodromy matrix (4.6) is invariant under the mapping as a consequence of the periodicity. Thus, by expanding the trace in powers of k^2 we obtain $P - 1$ nontrivial independent invariants.

The generalized McMillan mappings, viewed as periodic reductions of the lattice KdV equation (2.1), can be integrated using the finite-gap integation technique (cf. [37]). For this purpose, it is useful to reformulate the linear system in a different way, namely in terms of $2P \times 2P$ matrices as follows

$$\mathbf{L} \cdot \varphi = \lambda\varphi, \quad \mathbf{L} = \begin{pmatrix} 0 & -y_0 & 1 & & & 0 \\ 0 & \varepsilon\delta & -x_1 & 1 & & 0 \\ \vdots & & 0 & -y_1 & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 1 \\ \eta & 0 & & \cdots & 0 & -y_{P-1} \\ -\eta x_0 & \eta & \cdots & \cdots & 0 & \varepsilon\delta \end{pmatrix}, \tag{4.7a}$$

and

$$\varphi' = \mathbf{M} \cdot \varphi, \quad \mathbf{M} = \begin{pmatrix} -\varepsilon\delta/y_0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ \vdots & & 0 & -\varepsilon\delta/y_1 & 1 & \\ \vdots & & & \ddots & \ddots & \ddots \\ 0 & & \cdots & & 0 & -\varepsilon\delta/y_{P-1} & 1 \\ \eta & 0 & \cdots & & & 0 & 0 \end{pmatrix}. \tag{4.7b}$$

It is may be one of the advantages of the lattice systems that one can take advantage of these kind of reformulations. The compatibility of the system (4.7)

$$\mathbf{L}' = \mathbf{M} \cdot \mathbf{L} \cdot \mathbf{M}^{-1}, \tag{4.8}$$

again yields the mapping (4.3). The Floquet parameter η, arising from the quasi-periodicity of the eigenfunctions ϕ_k of (4.7), in conjunction with the spectral parameter λ, form the local coordinates on the following invariant spectral curve

$$\det(\mathbf{L} - \lambda\mathbf{1}) = \lambda^P (\lambda + \varepsilon\delta)^P + \eta^2 - \eta\operatorname{tr}(T(k)). \tag{4.9}$$

The explicit solutions of the mappings (4.3) can be expressed in terms of the theta-functions associated with the hyperelliptic curve (4.9). Furthermore, the invariants, arising from the coefficients of (4.9) are in involution with respect to the Poisson brackets

$$\{x_j, y_{j'}\} = \delta_{j,j'} - \delta_{j,j'+1}, \qquad \{x_j, x_{j'}\} = \{y_j, y_{j'}\} = 0. \tag{4.10}$$

The involution property of the invariants which was proven in [36], follows also from the Poisson bracket

$$\{\operatorname{tr} T(k), \operatorname{tr} T(k')\} = 0, \tag{4.11}$$

which also follows as a consequence of a nonultralocal Yang–Baxter equation for the mappings established in [38]. The mapping (4.3) is a canonical transformation and can be inferred also from a generating function which we obtained in [36] from the action of the lattice KdV equation. This is another important advantage of the discrete situation: starting from the generalized McMillan mappings, one has an unambiguous quantization procedure, leading to what we have called a theory of integrable quantum mappings (cf. [39–44]). In fact, as was exhibited in [41, 42], there is a very interesting new type of quantum Yang–Baxter structure underlying these discrete-time systems. As we can associate with any periodic solution of the lattice KdV equation a McMillan mapping of arbitrary dimension, and as the lattice KdV equation can be considered to be a lattice regularization of the continuum KdV equation, we can adopt the point of view that this quanization procedure of the corresponding McMillan mappings amounts to a quantization of the KdV system. We will not go into any more details on this issue here, but refer to the paper by Prof. Faddeev in this symposium for a treatment of similar issues (cf., also [45]).

As we have demonstrated above, the lattice KdV equation yields integrable mappings of arbitrary dimension as special reductions. This is very useful for nonlinear dynamics, as such maps can be used as a starting point for the study of near-integrable phenomena (e.g., by studying small perturbations around the integrable situation). It is only relatively recently that integrable maps have been

systematically studied (cf. [46–51]), for other treatments in the literature. Possibly, this offers a hand to people working in the arena of chaotic phenomena, who now have at their disposal a large number of exactly solvable, yet highly nontrivial, discrete dynamical systems (other than the simple linear ones that were are traditionally used), to study the more analytical aspects of transitions from integrability to chaos.

5. Similarity Reductions of Integrable Lattices

Similarity solutions of the lattice equations provide another class of solutions that we want to consider in this talk. As is well-known the similarity solutions of integrable nonlinear partial differential equations (PDEs) give rise to the Painlevé transcendents. For instance, the Painlevé II equation

$$y'' = 2\,y^3 + \xi\,y + \mu, \tag{5.1}$$

in which ξ is the independent variable, is related to the KdV equation as well as to the MKdV equation by similarity reduction (for the connection between Painlevé equations and soliton equations, see [52–54] and references therein). The Painlevé equations were discovered in the beginning of this century in connection with the problem of classifying all second-order ordinary differential equations without movable singularities other than poles (cf. [55–57]). They were encountered in physics from the seventies onwards in many places, e.g., in the correlation functions of quantum exactly solvable (spin) models and, more recently, in connection with 2D quantum gravity and random matrix models. The issue of constructing difference analogues of the Painlevé equations has been outstanding for a long time, but in the last few years major progress in this direction has been achieved (see [58] for a review). One of the methods that has been applied to obtain such discrete Painlevé equations was the method of similarity reduction, extended to the lattice situation in [59].

   The similarity reduction in the lattice case exhibits some new features with respect to the continuum case. Let us illustrate our idea by the analogy to that case. For example, the reduction to similarity solutions for the potential KdV can be formulated in terms of a system consisting of the equation itself and an integrable constraint, namely

$$0 = u + xu_x + 3tu_t, \tag{5.2}$$

from which one derives the similarity variable $\xi \equiv x/3(t^{1/3})$, i.e. $u(x,t) = t^{-1/3} = u(\xi)$. As was shown in [59], in the lattice case we supplement the original lattice equation, which is the lattice (potential) KdV Equation (2.1), with an integrable constraint

$$0 = u + pn\,\frac{u_{n+1,m} - u_{n-1,m}}{2p - u_{n+1,m} + u_{n-1,m}} + qm\,\frac{u_{n,m+1} - u_{n,m-1}}{2q - u_{n,m+1} + u_{n,m-1}}, \tag{5.3}$$

which, in contrast to the continuous case, is nonlinear. Although it might not be possible to solve explicitly for a similarity variable, nevertheless it provides us with a *system* of difference equations, which carries an associated isomonodromic deformation problem. In fact, the similarity reduction can be obtained from the Lax pair of the lattice KdV equation (2.4), together with

$$
k \frac{d}{dk} \phi_k = S_- \cdot \phi_k + pn \left[-F + \frac{1}{R_p} \left(\frac{V_p}{p-k} + \frac{V_p'}{p+k} \right) \right] \cdot \phi_k +
$$
$$
+ qm \left[-F + \frac{1}{R_q} \left(\frac{V_q}{q-k} + \frac{V_q'}{q+k} \right) \right] \cdot \phi_k. \tag{5.4}
$$

In Equation (5.4) we used the following notations:

$$
R_p = 2p - u_{n+1,m} + u_{n-1,m}, \qquad R_q = 2q - u_{n,m+1} + u_{n,m-1},
$$

$$
V_p = \begin{pmatrix} p - u_{n+1,m} & 1 \\ * & p + u_{n-1,m} \end{pmatrix},
$$

$$
V_p' = \begin{pmatrix} -(p + u_{n-1,m}) & 1 \\ * & -(p - u_{n+1,m}) \end{pmatrix}, \tag{5.5}
$$

and V_q, resp., V_q' denote the matrices obtained from (5.5) by interchanging $p \leftrightarrow q$ and $(n \pm 1, m) \leftrightarrow (n, m \pm 1)$. $*$ denotes in (5.5) the product of the diagonal entries, and finally the matrices F and S_- are given by

$$
F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad S_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Clearly, by imposing the constraint (5.3) arising from the compatibility of the linear system (2.4) of the lattice KdV equation and the isomonodromy problem (5.4), we have an integrable reduction corresponding to the lattice analogue of a similarity reduction. In order to give an interpretation of this reduction from the point of view of the initial-value problem, let us note that global solutions can be obtained by imposing initial data on localized configurations of sites ('embryons') on the lattice (cf. [59]). This contrasts the situation of the mappings treated in the previous sections, where we have a periodic initial-value problem on an infinite path in the two-dimensional lattice. The similarity reduction is completely analogous to the continuous case, in which the PDE allows for similarity solutions that are obtained from an ordinary differential equation that requires initial data at a single point in spacetime.

So far, we have discussed the similarity reduction of the lattice KdV equation. However, to get a more direct connection to the Painlevé II equation, it is more convenient to work with the lattice MKdV equation, this providing an alternative way of obtaining a lattice discretization thereof. In that case, the compatibility

condition for (2.4) and (5.4) will again lead to an integrable constraint which is

$$n \, \frac{v_{n+1,m} - v_{n-1,m}}{v_{n+1,m} + v_{n_1,m}} + m \, \frac{v_{n,m+1} - v_{n,m-1}}{v_{n,m} + v_{n,m-1}} = 0. \tag{5.6}$$

Again, the system consisting of the lattice MKdV Equation (2.12) together with the similarity constraint (5.6) can be viewed as a lattice analogue of the Painlevé II equation. It is this system that we shall refer to as the lattice P-II system.

We shall now discuss the continuum limits that enables one to recover Painlevé II from the lattice P-II system. In Section 2, we already discussed the continuum limit that leads from the lattice equation to the (potential) KdV equation. As the lattice P-II is written as a system containing the original equation plus a similarity constraint, it suffices to look at what happens under this limit with the similarity constraint. Again, we have to perform the limit in two steps: the first leading to the differential-difference equation, and then the next step to the partial differential equation. Performing the first step, (2.6), we obtain from (5.3) the following nonautonomous differential-difference equation

$$0 = u_n + \frac{(pn - \tau)\,(u_{n+1} - u_{n-1}) + p\tau\,\partial_\tau(u_{n+1} + u_{n-1})}{2p - u_{n+1} + u_{n-1}} +$$
$$+ \frac{2p\tau\,(u_{n+1} - u_{n-1})}{(2p - u_{n+1} + u_{n-1})^2}. \tag{5.7}$$

The subsequent next step, performing the limit (2.10) leads to the similarity constraint (5.2). Thus, by two subsequent limits, the lattice system goes over to the system containing the potential KdV equation and the *linear* similarity constraint, which is then related to the Painlevé II equation by a Miura transformation.

To get the more direct connection to Painlevé II, we can also use the lattice MKdV rather than the lattice KdV. In that case, the similarity constraint (5.6) has a limit

$$0 = 2\tau\,\partial_\tau(y_{n+1} + y_{n-1}) + n\left(e^{y_{n+1} - y_{n-1}} - e^{y_{n-1} - y_{n+1}}\right), \tag{5.8}$$

and the second continuum limit leads to the MKdV and similarity constraint

$$x y_x + 3t y_t = 0, \tag{5.9}$$

which immediately leads to the Painlevé II equation (for special value of the parameter $\mu = 0$) after solving the linear constraint in terms of a similarity variable.

A more direct way of obtaining Painlevé type equations on the differential-difference level, is by eliminating from (5.7) and (5.8) the derivatives with respect to τ by using Equations (2.9), resp., (2.18), which in the case of the MKdV equation, leads to an ordinary difference equation, namely

$$z_{n+1} + z_{n-1} + \frac{2p}{\tau} \, \frac{n \, z_n}{1 - z_n^2} = 0, \qquad z_n \equiv \tanh\left[\tfrac{1}{2}\left(y_{n+1} - y_{n-1}\right)\right], \tag{5.10}$$

which under the above limit also leads to Painlevé II. It is this equation that we would like to call nowadays the *discrete Painlevé II equation*. Other discrete Painlevé equations, i.e. nonautonomous ordinary difference equations that tend to the Painlevé transcendents in a proper continuum limit and that share many of the integrability characteristics with the original Painlevé equations, have been derived and studied in the recent literature (cf. [60–63]). This is an area of intensive research and many new and exciting results are to be expected here.

6. Direct Linearization

Let us now explain how the lattice equations arise from the direct linearization approach. This generalization of the inverse scattering approach, first introduced by Fokas and Ablowitz in [71] and generalized in [72–74] (see also [18] for a review), employs singular linear integral equations with general integration measure and contour. The integral equation introduced in [71] is of the form

$$u_k + \rho_k \int_C d\lambda(\ell) \, \frac{u_\ell}{k + \ell} = \rho_k. \tag{6.1}$$

Here u_k is a wave function to be solved from the integral equation depending on a complex spectral parameter k and on the coordinates of the system. As we shall note below, these coordinates can be chosen to be discrete as well as continuous, and it is the freedom in this choice that makes integral equations of the type (6.1) a convenient tool to develop discrete integrable systems. Furthermore, in Equation (6.1) C is a contour in the complex k-plane and $d\lambda(k)$ is a suitably chosen integration measure, whereas ρ_k is a free-wave function depending in a given way on k and on the coordinates of the system. The contour C and measure $d\lambda(k)$ need to be chosen to be such that the solution u_k of the integral equation for given ρ_k is unique.

Consider now, for example, the free-wave function

$$\rho_k = e^{kx + k^3 t}, \tag{6.2}$$

and let u_k be the solution of (6.1) for some choice of C and $d\lambda(k)$, then the potential

$$u(x, t) = \int_C d\lambda(k) \, u_k(x, t) \tag{6.3}$$

with the same C and $d\lambda(k)$ satisfies

$$(\partial_t - \partial_x^3)u - 3(\partial_x u)^2 = 0, \tag{6.4}$$

i.e. $v = \partial_x u$ satisfies the Korteweg–de Vries equation.

The integral Equation (6.1) is particularly useful to obtain the lattice discretization of the KdV equation. For this we need to see how Bäcklund transformations

(BTs) arise in the direct linearization approach. The BTs are generated by singular transformations of the measure, or equivalently by a transformation of the free-wave function

$$\rho_k \longrightarrow \tilde{\rho}_k = \frac{p+k}{p-k}\,\rho_k. \tag{6.5}$$

In fact, it can be shown that $\tilde{u} = \int_C d\lambda(k)\,\tilde{u}_k$ with $\tilde{u}_k$ being the solution of (6.1) with ρ_k replaced by $\tilde{\rho}_k$, is again a solution of (6.4). Considering now two different BTs, one given by $\tilde{}$ and p as paremeter as in Equation (6.5), and the other one being given by replacing the $\tilde{}$ and p replaced by $\hat{}$ and q, we obtain by combining the two BTs

$$(p + q - \hat{\tilde{u}} + u)(p - q + \hat{u} - \tilde{u}) = p^2 - q^2, \tag{6.6}$$

which is the so-called Bianchi identity expressing the commutativity of the two BTs. The connection with the lattice KdV is now clear: the two BTs can be independently iterated retaining the commutativity, leading to a *lattice* of Bäcklund transformed fields u, and the Bianchi identity (6.6) can be interpreted as a consistency condition on a lattice in the form of a partial *difference* equation. Thus, associating the two BTs $\tilde{}$ and $\hat{}$ with two basic translations on a two-dimensional lattice, one finds the lattice KdV equation (2.1). Thus, this partial difference equation is integrable in the sense that solutions can be obtained solving a linear integral equation.

Now, to see the interconnection between the various different lattice equations, we introduce a slight generalization of the integral Equation (6.1), namely

$$\mathbf{u}_k + \rho_k \int_C d\lambda(\ell)\,\frac{\mathbf{u}_\ell}{k+\ell} = \rho_k \mathbf{c}_k, \tag{6.7}$$

where $\mathbf{u}_k$ is now a vector solution, each entry being associated with the corresponding entry of the vector $\mathbf{c}_k$ in the inhomogeneous term, which is given by $(\mathbf{c}_k)_i = k^i$, $i \in \mathbf{Z}$. Thus, we get an infinite vector of solutions, and for this 'wave'-vector we can derive an infinite-matrix system of equations. In fact, introducing the infinite-component 'potential' matrix $\mathbf{U}$, with entries $u_{i,j}$, $i,j \in \mathbf{Z}$, defined as

$$\mathbf{U} = \int_C d\lambda(\ell)\,\mathbf{u}_\ell\,{}^t\mathbf{c}_\ell, \tag{6.8}$$

in which ${}^t\mathbf{c}_\ell$ denotes the adjoint vector to $\mathbf{c}_\ell$ and considering again a BT generated by (2.7), we can derive the following relations for the transformed wave-vectors $\tilde{\mathbf{u}}_k$

$$(p - k)\tilde{\mathbf{u}}_k = (p + \mathbf{\Lambda} - \tilde{\mathbf{U}} \cdot \mathbf{O}) \cdot \mathbf{u}_k,$$

$$(p + k)\mathbf{u}_k = (p - \mathbf{\Lambda} + \mathbf{U} \cdot \mathbf{O}) \cdot \tilde{\mathbf{u}}_k. \tag{6.9a}$$

The matrix $\boldsymbol{\Lambda}$ and its transposed ${}^t\boldsymbol{\Lambda}$ that we have introduced in Equation (6.9) are index-raising operators acting from the left, respectively, from the right, i.e. it acts on, e.g., $\mathbf{u}_k$ by $(\boldsymbol{\Lambda} \cdot \mathbf{u}_k)^{(i)} = u_k^{(i+1)}$ and $\mathbf{O}$ is a projection matrix singling out the central element of the infinite-component matrix, i.e. for example $(\mathbf{O} \cdot \mathbf{u}_k)^{(i)} = u_k^{(0)} \delta_{i,0}$. By integrating Equations (6.9) over the region C with the same measure $d\lambda(\ell)$, we arrive at a system of equations for $\mathbf{U}$

$$\widetilde{\mathbf{U}} \cdot (p - {}^t\boldsymbol{\Lambda}) = (p + \boldsymbol{\Lambda} - \widetilde{\mathbf{U}} \cdot \mathbf{O}) \cdot \mathbf{U}. \tag{6.9b}$$

The system of Equations (6.9b) is an infinitely coupled system, and from it one can derive an ininitely coupled system of *nonlinear* equations for the matrix $\mathbf{U}$. However, it can be shown that by properly combining (6.9b) for different lattice directions and different lattice parameters $p, q, \ldots$, we can get rid of the index-raising operators $\boldsymbol{\Lambda}$ and find closed partial difference equations, together with their Lax pairs, by using Equations (6.9) to obtain finite-dimensional matrix systems for suitably chosen subsets of components of the vector $\mathbf{u}_k$. Thus, we can obtain from (6.9) the basic set of Equations (2.13), as well as (2.14) and (2.20) of Section 2, by making the following identifications

$$u \equiv U_{0,0}, \qquad v \equiv 1 - \left(\frac{1}{r + \boldsymbol{\Lambda}} \cdot \mathbf{U}\right)_{0,0},$$

$$w \equiv 1 - \left(\mathbf{U} \cdot \frac{1}{r' + {}^t\boldsymbol{\Lambda}}\right)_{0,0}, \tag{6.10a}$$

$$s \equiv \left(\frac{1}{r + \boldsymbol{\Lambda}} \cdot \mathbf{U} \cdot \frac{1}{r' + {}^t\boldsymbol{\Lambda}}\right)_{0,0}, \qquad t \equiv \left(\frac{1}{r + \boldsymbol{\Lambda}} \cdot \mathbf{U} \cdot {}^t\boldsymbol{\Lambda}\right)_{0,0}. \tag{6.10b}$$

Considering these objects as functions of the lattice sites (n, m), we obtain the corresponding lattice variables, respectively $u_{n,m}$, $v_{n,m}$, $w_{n,m}$, $s_{n,m}$, and $t_{n,m}$ of Section 2. Introducing, furthermore, the variables depending on the spectral parameter k

$$v_k \equiv \left(\frac{1}{r + \boldsymbol{\Lambda}} \cdot \mathbf{u}_k\right)^{(0)}, \qquad u_k \equiv (\mathbf{u}_k)^{(0)}, \tag{6.11}$$

we can easily derive from having the general system (6.9) the set of equations

$$(p - k)\tilde{v}_k = (p - r)v_k + \tilde{v}u_k, \tag{6.12a}$$

$$(p + k)v_k = (p + r)\tilde{v}_k - v\tilde{u}_k, \tag{6.12b}$$

from which one can also derive the Lax representations presented in Section 2. As a particular interesting special example included in this entire parameter family of related objects, we look into the variables for $r = -p$, $r' = -p$. In

that case, we have a very simple situation and, in fact, the system (6.12) leads to the following linear system

$$\kappa P_{n,m}(\kappa) = P_{n+1,m}(\kappa) + R_{n,m} P_{n-1,m}(\kappa), \tag{6.13a}$$

$$P_{n,m+1}(\kappa) = P_{n,m}(\kappa) + \frac{1}{\kappa} Q_{n,m} P_{n-1,m}(\kappa), \tag{6.13b}$$

in terms of the objects

$$P_{n,m}(\kappa) = \left(\frac{p+k}{p-k}\right)^{-n/2} \left(\frac{q+p}{q-k}\right)^{-m/2} v_k(n,m),$$

$$R_{n,m} \equiv \frac{h_{n,m}}{h_{n-1,m}}, \qquad Q_{n,m} \equiv -\frac{2p}{p+q} \frac{h_{n,m+1}}{h_{n-1,m}}, \tag{6.14}$$

where $h_{n,m} = 1 + 2ps_{n,m}$, and where $\kappa = 2p\left(p^2 - k^2\right)^{-1/2}$ is the 'effective' spectral parameter. The compatibility conditions of the system (6.13) leads to the following discrete equation for the variable $Q_{n,m}$

$$\frac{Q_{n,m+1}}{Q_{n+1,m}} = \frac{\alpha Q_{n,m} - 1}{\alpha Q_{n+1,m+1} - 1}, \tag{6.15}$$

in which $\alpha = (q^2 - p^2)/(4p^2)$ arises as an integration constant. Equation (6.15) can be considered to be a discrete version of the Volterra–Kac–van Moerbeke equation (cf. [24]), which is recovered after a suitable continuum limit. In [63] another lattice discretization of the Volterra equation was discussed in connection with a q-deformation of the discrete Painlevé I equation, arising in connection with orthogonal polynomials. We mention that the continuous-time Volterra system plays a role in matrix models for two-dimensional quantum gravity (cf., e.g., [60, 64, 65]). We expect, that the discrete Volterra Equation (6.15) would play a similar role in a discrete variant of that theory. Furthermore, such discrete lattice equations are also occurring in connection with convergence acceleration algorithms in numerical analysis (cf., e.g., [66]) and also [67], as well as in the QD-algorithm and the theory of Padé approximants (cf. [68–70]). This shows the ubiquitousness of the KdV equation in its discrete form: one hundred years after its appearance we have not yet ceased to marvel at its great variety of disguises.

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