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Exact solutions of Rayleigh's equation and sufficient conditions for inviscid instability of parallel, bounded shear flows

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1. Introduction

The two-dimensional inviscid linearized stability of parallel shear flows, defined on the domain $-y_0 \le y \le y_0$, $-\infty < x < \infty$, is governed by the well-known Rayleigh's equation (Rayleigh, 1880):

$$\phi'' - \left[\alpha^2 + \frac{U''}{U-c}\right]\phi = 0, \tag{1a}$$

subject to the boundary conditions

$$\phi(\pm y_0) = 0. \tag{1b}$$

Here, the real part of $\{\phi(y)\exp[i\alpha(x-ct)]\}$ defines the stream function $\psi(x, y, t)$ for two-dimensional disturbances $(u(x, y, t), v(x, y, t))$ to the basic parallel shear flow $U(y)$, with $u \equiv \psi_y$, $v \equiv -\psi_x$. The primes in (1a) stand for differentiation with respect to y, and α and c are the real wavenumber and (possibly complex) eigenvalue respectively. The flow $U(y)$ is unstable if there are non-trivial solutions ϕ to (1a,b) for complex values of the eigenvalue c.

For smooth basic velocity profiles $U(y)$, equations 1(a,b) have, in general, not been very amenable to theoretical analysis, although series solutions and competent numerical algorithms are available. For details of the available theoretical results on Rayleigh's equation, as well as for a review of the voluminous literature on hydrodynamic stability, the reader is referred to the text books of C. C. Lin (1955), S. Chandrasekhar (1961) and Drazin and Reid (1981), and to the review articles of Drazin and Howard (1966) and Bayly, Orszag and Herbert (1988). Here we confine ourselves to stating some of the relevant inviscid stability theorems.

I. *Rayleigh's inflexion point theorem (Rayleigh, (1880))*

A necessary condition for instability is that the basic velocity profile $U(y)$ should have an inflexion point on $y \in (-y_0, y_0)$.

II. *Fjørtoft's theorem (Fjørtoft, (1950))*

A necessary condition for instability is that $U''(y)[U(y) - U(y_s)] < 0$ somewhere in the field of flow, where y_s is a point at which $U'' = 0$.

This version of Fjørtoft's theorem is stated in Drazin and Reid (1981).

III. *Howard's semicircle theorem (Howard, (1961))*

For unstable velocity profiles $U(y)$, c must lie in the following semicircle in the complex plane:

$$\{c_r - \tfrac{1}{2}(U_{\max} + U_{\min})\}^2 + c_i^2 \leq \{\tfrac{1}{2}(U_{\max} - U_{\min})\}^2,$$

where $c_i > 0$, c_r and c_i are the real and imaginary parts of c and $U_{\max}$ and $U_{\min}$ are the maximum and minimum values of $U(y)$ in the flow domain.

Recently, Barston (1991) has provided extensions of Fjørtoft's theorem for flows with multiple inflexion points. Note that the above theorems provide necessary, but not sufficient conditions for instability.

The only notable class of exact solutions of (1a) for arbitrary (possibly complex) c and arbitrary real α is for the case of piecewise linear velocity profiles $U(y)$; here, the eigenvalue c can be determined such that the eigenfunction ϕ satisfies appropriate jump conditions at the points of discontinuity of U or U' (Rayleigh (1880); see also Drazin and Howard (1966) and Drazin and Reid (1981)). For the special case of $U(y) = 1 - \exp(-y)$, $(y \geq 0)$, (1a) can be converted to the hypergeometric equation and hence solved in closed form for arbitrary complex c and real α [see Lin (1955, p. 90) and Drazin and Howard (1966, p. 42)]. Apart from the above cases, only neutrally stable exact solutions of (1a,b) are available, for specific velocity profiles like $U(y) = \sin(y)$, $U(y) = \operatorname{sech}^m(y)\tanh(y)$, $U(y) = \tanh(y)$ and $U(y) = \operatorname{sech}^2(y)$; see Drazin and Howard (1966).

The main objective of the present paper, addressed in Sections 3–5, is to prescribe sufficient conditions for the inviscid instability of a general class of smooth velocity profiles $U(y)$ which are odd (satisfying $U(y) = -U(-y)$). Thus this paper is primarily concerned with *constructing a class of basic profiles $U(y)$ that is guaranteed to be unstable*; we are *not* concerned here with the instability or otherwise of a *given* arbitrary $U(y)$, which can be tackled by previously known methods—either numerical or series methods, or for example, the criteria of Rosenbluth and Simon (1964; see their equations (2) and (4)). The method used here is novel, in that the basic velocity profile $U(y)$ is not specified a priori. Rather, the quantities specified are the real wave number α, the complex eigenvalue c and the real part $h(y)$ of the adjoint eigenfunction. Rayleigh's equation can then be converted into a nonlinear integral equation for $U(y)$, along with an appropriate linear integral equation as the boundary condition. The solution $U(y)$ of these integral equations, if it exists, is clearly guaranteed to be unstable. Sections

3–5 are therefore concerned with specifying conditions on $h(y)$ that ensure the existence and uniqueness of solutions to these integral equations. The main message of these sections is that the nonlinear integral formulation can be handled by elementary techniques and provides substantial new insight into the global properties of unstable velocity profiles and the corresponding eigenfunctions. Indeed, it is clear that only a global (or integral) analysis can provide sufficient conditions for instability; local properties of velocity profiles, such as the existence of inflexion points, cannot do so. Previously available series solutions, though globally valid, do not provide any insight into the global properties of unstable velocity profiles; that is, given these series solutions, we still do not know how to prescribe *a class of* profiles that are guaranteed to be unstable.

In Section 2, a method is provided for obtaining new, exact neutrally stable solutions of Rayleigh's equation. The basic idea here is to specify a functional relationship $\Phi(U)$ between the eigenfunction $\Phi(U(y))$ and $U(y)$; the resulting nonlinear ordinary differential equation for $y = y(U)$ can be solved in closed form. The problem then reduces to specifying real c and α such that the homogeneous boundary conditions are satisfied; a specific example of exact neutrally stable solutions for a class of jet-like velocity profiles is given.

2. Exact solutions of (1a,b) for real c

We let the velocity U be the independent variable in (1a), and seek $y = y(U)$ when $\phi(y) = \Phi(U(y))$, where $\Phi(U)$ is a known function of U. This transformation, when applied to (1a), leads to the following differential equation for $y(U)$:

$$y'' - F'y' + G(y')^3 = 0, \tag{2}$$

where the primes refer to differentiation with respect to U, and F and G are defined by

$$F(U) = \int^U \frac{\Phi''(s)}{\Phi'(s) - \dfrac{\Phi(s)}{s-c}}\, ds,$$

$$G(U) = \frac{\alpha^2\Phi(U)}{\Phi'(U) - \dfrac{\Phi(U)}{U-c}}.$$

Equation (2) is a Bernoulli's equation for y', and has the following closed-form solution:

$$(y')^{-2} \equiv (dU/dy)^2 = \left[B + 2\int_0^U G(s)\exp[2F(s)]\, ds\right]\exp[-2F(U)], \tag{3}$$

where B is an arbitrary real constant. Equation (3) can be solved for y as a function of U, given an appropriate function $\Phi(U)$. In principle, one can then obtain $U(y)$, and finally $\phi(y) = \Phi(U(y))$. In general, a solution $\phi_0(y)$ of (1a) generated by the procedure described above will not satisfy the homogeneous boundary conditions (1b). However, given one solution ϕ_0 of (1a), a second linearly independent solution ϕ_1 can be generated explicitly, and these two solutions combined appropriately to satisfy (1b) for specific values of c and α. In this manner, one can obtain classes of exact neutrally stable solutions of (1a,b) which were not previously known; an example is given below. We let

$$\Phi(U) = U^{b+(1/2)}, \qquad -\infty < y < \infty, \tag{3a}$$

where b is any real constant satisfying $-1/2 < b < \infty$. If $b = 1/2$, it is easy to show that (3a) and (3) lead to the well-known exact solution for the Bickley jet (see Drazin and Howard (1966) and Drazin and Reid (1981)). Therefore we shall assume $b \neq 1/2$ in what follows. Equation (3) can then be reduced to (taking $B = 0$)

$$\frac{dU}{dy} = \operatorname{sgn}(\alpha y\{2b-1\}) \left(\frac{2\alpha U}{U - \dfrac{2b+1}{2b-1} c} \right) [\xi(U)]^{1/2}, \tag{3b}$$

where $\xi(U)$ is defined by

$$\xi(U) \equiv \left(\frac{U^2}{2b+3} - \frac{2bcU}{(2b-1)(b+1)} + \frac{c^2}{2b-1} \right) (2b-1)^{-1}. \tag{3c}$$

Without loss of generality, we may take c to be an arbitrary positive constant in what follows. It can be shown from an elementary analysis that equations (3b,c) define a symmetric, positive smooth velocity profile satisfying $U \to 0$ as $|\alpha y| \to \infty$. The maximum velocity can be made to occur at $y = 0$ and is given by $U_{\max} = \beta$, where β is the smallest positive root of $\xi(U) = 0$; further it can be demonstrated (for example, by a simple computation) that β satisfies

$$\beta/c > 1, \qquad \text{for all } -1/2 < b < \infty, b \neq 1/2,$$

and in particular,

$$1 < \beta/c < (2b+1)/(2b-1) \qquad \text{for } b > 1/2.$$

Hence there is no singularity in (dU/dy) for all $-1/2 < b < \infty, b \neq 1/2$, as can be seen from (3b). Note that $\beta/c > 1$ implies that $U(y) - c$ vanishes for some $y = y_c$; since we have shown that $U(y)$ is indeed smooth for $-\infty < y < \infty$, it follows that $U''(y_c)$ must vanish as required by (1a) (note from (3a) that $\phi(y_c)$ does not vanish). That $U''(y_c)$ vanishes can be explicitly demonstrated; first note that $U'(y_c)$ is given by (3b) with

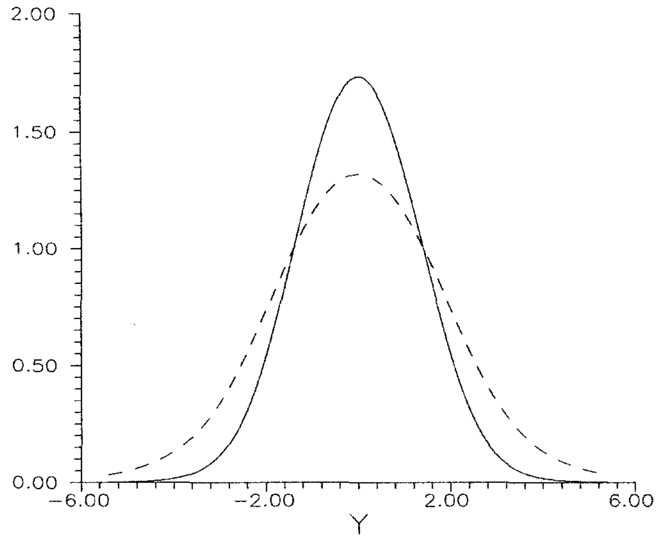

Figure 1
A plot of $U(y)$ and $\phi(y)$ for the example of equation (3d) and $\alpha = c = 1$; the solid and dashed lines represent $U(y)$ and $\phi(y)$ $[=\{U(y)\}^{1/2}]$ respectively.

$U(y_c) = c$ on the right hand side; upon differentiating (3b) and replacing $U'(y_c)$ and $U(y_c)$ by the appropriate quantities, we find that $U''(y_c) = 0$, as expected. The decay of U as $|\alpha y| \to \infty$ is easily demonstrated by a simple asymptotic analysis. Letting $U \to 0$ in (3b), we obtain

$$dU/dy \approx -\text{sgn}(\alpha y)(2\alpha U)/(2b+1),$$

the solution of which decays exponentially as $|\alpha y| \to \infty$. In particular, (3b) can be integrated explicitly for the special case $b = 0$ and the result is

$$\sin^{-1}(3^{-1/2}U/c) - 3^{-1/2}\ln((3^{1/2}c + [3c^2 - U^2]^{1/2})/U)$$
$$= (\pi/2) - (2/3^{1/2})|\alpha y|. \tag{3d}$$

It is easy to verify that the above equation does indeed define a symmetric, positive smooth velocity profile for arbitrary real α and positive c and with a maximum velocity of $(3^{1/2}c)$ at $y = 0$; further, $U(y)$ decays exponentially as $|\alpha y| \to \infty$. Plots of $U(y)$ [as given by (3d)] and $\phi(y)$ $[=[U(y)]^{1/2}$, as in (3a) with $b = 0]$ are given in Fig. 1; here we have taken $\alpha = c = 1$. Note that $U(y_c) = 1$ at $y_c \sim \pm 1.4004$, and that $U''(y_c)$ does indeed vanish, as can be established by differentiation of (3d) $[U''(y) = -y''(U)/y'^3]$ and as is clear from Fig. 1 (here y_c occurs at the points where the solid and the dashed lines cross).

We have demonstrated that equation (3) leads to at least one class of smooth velocity profiles on an infinite domain, namely that defined by (3b,c), for which the eigenfunction $\phi(y) = \Phi(U(y))$ (given by (3a) in this case) satisfies the homogeneous boundary conditions. The problem that is worthy of further investigation is to identify the most general class of functions $\Phi(U)$ that form neutrally stable eigenfunctions for smooth velocity profiles $U(y)$ given by equation (3), both on bounded and unbounded domains.

The solution procedure described in this section is useful only for real values of c. Although (3) is still valid for complex c, one now needs to guess a complex function $\Phi(U)$ such that (3) leads to a real velocity profile $U(y)$, and this turns out to be a formidable task.

3. Integral formulation of (1a,b) for complex c

Let c be a given complex constant $(c_r + ic_i)$, where the subscripts r and i stand for real and imaginary parts, and $c_i \neq 0$. Further, let $\alpha \neq 0$ be a given real constant. We make the following change of variables in 1(a,b):

$$y = y_*/\alpha, \qquad y_0 = y_{0*}/\alpha, \qquad \phi(y) = \phi_r(y_*) + i\phi_i(y_*),$$
$$U(y) = c_r - c_i w(y_*).$$

Separating out the real and imaginary parts in (1a,b), we obtain the following pair of equations for ϕ_r and ϕ_i:

$$\phi_r'' - \left(1 + \frac{w''w}{w^2+1}\right)\phi_r - \left(\frac{w''}{w^2+1}\right)\phi_i = 0 \tag{4a}$$

$$\phi_i'' - \left(1 + \frac{w''w}{w^2+1}\right)\phi_i + \left(\frac{w''}{w^2+1}\right)\phi_r = 0 \tag{4b}$$

$$\phi_r(\pm y_{0*}) = \phi_i(\pm y_{0*}) = 0. \tag{4c}$$

In what follows, we let $\alpha = c_i = 1$, so that we may set $y_* = y, y_{0*} = y_0$, without loss of generality. Assume that $w(y)$ is an odd, twice-differentiable function, so that

$$w(y) = -w(-y). \tag{5a}$$

Then it is easy to verify that there is at least one solution of (4a,b) satisfying (see Drazin and Howard (1966))

$$\phi_i(y) = \phi_r(-y). \tag{5b}$$

In fact, a second linearly independent solution of (1a) can also be constructed to satisfy (5b), so that the general solution of (1a) is of the type (5b) to within an arbitrary multiplicative constant. We now define two new independent variables $h(y)$ and $g(y)$ to satisfy the following equations:

$$\phi_r(y) = [1 + w(y)]h(y) - [1 - w(y)]g(y), \tag{6a}$$

$$\phi_i(y) = [1 - w(y)]h(y) + [1 + w(y)]g(y). \tag{6b}$$

From 5(a,b) it follows that $h(y)$ is even and $g(y)$ is odd. Upon inserting (6a,b) into (4a) (or (4b)) and separating out the odd and even parts, we obtain the following pair of ordinary differential equations:

$$h'' - h + w(g'' - g) + 2w'g' = 0, \tag{7a}$$

$$g'' - g - w(h'' - h) - 2w'h' = 0, \tag{7b}$$

and the boundary conditions reduce to

$$h(y_0) = g(y_0) = h'(0) = g(0) = 0. \tag{8}$$

In (8), we have reduced the domain to $(0 \le y \le y_0)$, since h is even and g is odd. It should be pointed out here that there is nothing new in the transformations (6a,b); it can be shown that $(h + ig)$ is merely the adjoint eigenfunction given to within a multiplicative constant by $\phi/(U - c)$ (see Drazin and Reid (1981), Sec. 21).

We may eliminate g from (7a,b) by simple manipulations. Letting

$$\theta \equiv \left(\frac{w^2 + 1}{2w'}\right)(h'' - h) + wh' = -g', \tag{9}$$

equations (7a,b) and (8) reduce to

$$\theta'' - \theta = -\frac{d}{dy}[2w'h' + w(h'' - h)], \tag{10a}$$

subject to

$$h'(0) = h'''(0) = h(y_0) = g(y_0) = 0, \tag{10b}$$

where

$$g(y) = -\theta' - w(h'' - h) - 2w'h'. \tag{10c}$$

Note that the additional requirement $h'''(0) = 0$ in (10b) ensures that $g(0) = 0$ is satisfied.

We now treat (10a) as an equation to be solved for $w(y)$, given an even function $h(y)$ satisfying $h'(0) = h'''(0) = h(y_0) = 0$. Noting from (5a) that $w(y)$ is odd, the boundary conditions on w reduce to

$$w(0) = w''(0) = g(y_0) = 0, \tag{11}$$

where g is defined by (10c). Clearly, real velocity profiles $w(y)$ which solve (10a) subject to (11) are unstable. Therefore the conditions on $h(y)$ such that (10a) and (11) can be solved for real $w(y)$ form sufficient conditions for the instability of such profiles. It is surprising that the questions of existence and uniqueness of solutions to this nonlinear variable-coefficient boundary-value problem for $w(y)$ can be addressed theoretically for a fairly general class of functions h.

We first solve (10a) for θ, treating the right-hand side as known, to obtain the following integro-differential equation:

$$\theta = K\cosh(y) + \int_0^y \frac{d}{ds}[2w'h' + w(h'' - h)]\sinh(s - y)\,ds, \tag{12}$$

where K is a real constant of integration. After a couple of integrations by parts on the integral in (12), using $w(0) = h'(0) = 0$, we may eliminate all derivatives of w from the integrand; substituting for θ from (9) in the left hand side of (12), we obtain

$$w' = \frac{(w^2 + 1)(h'' - h)}{H(y;K)}, \tag{13}$$

where

$$H(y;K) \equiv 2K\cosh(y) - 6w(y)h'(y) + 2\int_0^y w(s)\lambda(s,y)\,ds, \tag{14a}$$

and λ is defined by

$$\lambda(s,y) \equiv [h''(s) + h(s)]\cosh(s - y) + 2h'(s)\sinh(s - y). \tag{14b}$$

Finally, using $w(0) = 0$, we convert (13) to the following nonlinear integral equation for w:

$$w(y) = \tan\left[\int_0^y \frac{[h''(r) - h(r)]}{H(r;K)}\,dr\right]. \tag{15a}$$

To obtain an integral form of the boundary condition $g(y_0) = 0$, we replace θ in (10c) by the right hand side of (12); after a couple of integrations by parts, we obtain

$$g(y_0) = 0, \tag{15b}$$

where

$$g(y) \equiv -K\sinh(y) - \int_0^y w(s)\lambda_y(s,y)\,ds. \tag{16}$$

The problem thus reduces to solving (15a) and specifying K such that (15b) holds (if this is possible).

4. Existence, uniqueness and properties of solutions to equations (15a,b)

In the rest of this paper, we consider a class of non-negative functions h that decrease monotonically on the interval $(0, y_0)$; our goal is to theoretically state sufficient conditions for the existence and uniqueness of solutions to (15a,b). To be specific let $y_1 \in (0, y_0)$ be a constant, and let $h(y)$ satisfy the following restrictions

$$h(y) \text{ is analytic on } y \in [0, y_0], \tag{17a}$$

$$h'(0) = h'''(0) = h(y_0) = 0, \tag{17b}$$

$$h''(y) - h(y) < 0 \text{ on } y \in [0, y_1) \quad \text{and} \quad h''(y) - h(y) > 0 \text{ on } y \in (y_1, y_0), \tag{17c}$$

$$h''(y_1) - h(y_1) = 0 \quad \text{and} \quad h'''(y_1) - h'(y_1) > 0, \tag{17d}$$

$$h''(y) + h(y) \geq 0 \text{ on } y \in [0, y_0] \tag{17e}$$

and

$$h'(y) < 0 \text{ on } y \in (0, y_0]. \tag{17f}$$

It is clear that this class of functions h is non-negative and decreases monotonically from $h(0)$ to zero. Note from (17e,f) that $\lambda(s, y)$ and $\lambda_y(s, y)$ are non-negative functions on $y \in [0, y_0]$, $s \in [0, y]$. An example of a function satisfying (17) is $h(y) = (1/2) + \cos(y)$, with $y_0 = 2\pi/3$, and $y_1 = \cos^{-1}(-1/4)$. It is sufficient to consider $K < 0$ in (15a,b) and (16); clearly, for a given solution $W(y)$ of (15a,b) with $K = K_0$, $-W(y)$ is also a solution with $K = -K_0$. The following lemma is useful in proving the uniqueness and existence of solutions to (15a,b).

Lemma 1. A smooth solution $w(y)$ of (15a), if it exists for a given $K < 0$ and for functions h satisfying (17), must satisfy the following restrictions:

$$\left| \int_0^y \frac{[h''(r) - h(r)]}{H(r; K)}\, dr \right| < \pi/2 \qquad \text{on } y \in [0, y_0], \tag{18a}$$

and either of (18b) *or* (18c), which are given by

$$H(y; K) < 0 \text{ (resp. } >0) \qquad \text{for } y < y_1 \text{ (resp. } >y_1),$$

$$H(y_1; K) = 0, \qquad H'(y_1; K) > 0 \quad \text{and} \quad w(y) > 0 \qquad \text{on } y \in (0, y_0], \tag{18b}$$

or

$$H(y; K) < 0 \text{ on } y \in [0, y_0) \quad \text{and} \quad w(y) > 0 \text{ on } y \in (0, y_1]. \tag{18c}$$

The proof of Lemma 1 is as follows; equation (18a) is obviously necessary for $w(y)$ to be nonsingular. It is clear that $H(0; K) = 2K < 0$. Equations (18b,c) essentially state that if at all $H(y; K)$ vanishes, it must do so at $y = y_1$, and further, $H(y; K)$ must change sign as y crosses y_1 with $H'(y_1; K) > 0$. The first part of this assertion follows from (13) and (17c), for w' would become singular at any $y \neq y_1$ where H vanishes. Further, it is also obvious from (13) and (17d) that w' would become singular at $y = y_1$ if both H and H' vanish there; hence the requirement in (18b) that $H'(y_1; K)$ be non-vanishing. In fact $H'(y_1; K)$ must be positive if $H(y_1; K)$ vanishes, for its negativeness implies that H also vanishes somewhere on $(0, y_1)$ and this is not permitted. The statements in (18b,c) on the sign of $w(y)$ now follow, and the proof of Lemma 1 is complete.

The "converse" of Lemma 1 is also important for our purposes, and may be stated as follows.

Lemma 2. Suppose $h(y)$ satisfies (17); suppose further that for some $K < 0$ and some $y_2 \in (0, y_0)$ a smooth solution $w(y)$ of (15a) exists for all $y \in [0, y_2)$, but either $w(y)$ or some higher derivative becomes singular at $y = y_2$. Then we must necessarily have $H(y_2; K) = 0$ and also $y_2 \neq y_1$.

Remark. From equations (13) and (14a), we conclude that $w(y; K)$ must be finite as y approaches y_2 from below. To see this, assume the contrary; then from (13), (14a) and (17e,f) we find that

$$w' \sim -w(h'' - h)/[6h'], \qquad \text{as } y \to y_2^- .$$

But the above equation implies that both w and w' are of the same order of magnitude as $y \to y_2^-$, and this establishes the desired contradiction. Note that $w'(y; K)$ does become unbounded as y approaches y_2 from below. Hence w' (given by (13)) must have an integrable singularity at $y = y_2$.

Proof of Lemma 2. Suppose first that $H(y_2; K) \neq 0$. Define $w(y_2; K)$ as the limit (from below) of $w(y; K)$. By repeated differentiation of (13) we may demonstrate recursively that all (one-sided) derivatives of w exist and are bounded at $y = y_2$. This implies that w does not have a pole or a branch point at $y = y_2$. From (15a) and (17), we conclude that w cannot have a jump discontinuity (either finite or infinite) as y crosses y_2; for example, suppose that w has a finite jump discontinuity as y crosses y_2. But then the right hand side of (15a) would be continuous at $y = y_2$ while the left hand side is not, and we have the desired contradiction. This assertion is also true of all derivatives of w as may be demonstrated inductively by repeated differentiation of (13). Therefore $w(y_2; K)$ must be infinitely differentiable at $y = y_2$, and we have arrived at a contradiction.

Suppose next that $y_2 = y_1$ and $H(y_1; K) = 0$; if $H'(y_1; K) \neq 0$, the preceding arguments again apply and therefore there can be no singularity at $y = y_1$. In fact, $H'(y_1; K)$ cannot vanish if $H(y_1; K) = 0$; to see this, we differentiate (14a) to find

$$H' = -2g - 6w'h' - 6wh'' + 2w(h'' + h), \tag{19}$$

where g is given by (16). Now if H and H' were to vanish at $y = y_1$, we find from (13) and (17) that $w' \to +\infty$ as $y \to y_1$ (from below); further, from the remark below Lemma 2, $w(y)$ must be well defined and finite as y approaches y_1 from below. But then we find from (17f) and (19) that $H' \to +\infty$ as $y \to y_1^-$, which is a contradiction. Therefore we have established that $y_2 \neq y_1$, and the proof of Lemma 2 is complete.

An immediate consequence of Lemmas 1 and 2 is the following.

Lemma 3. Suppose $h(y)$ satisfies (17) and suppose that for some $K < 0$, a smooth solution $w(y)$ of (15a) exists for all $y \in [0, y_2]$ (so that Lemma 1 applies), where $y_2 \in (0, y_0]$. Then $w(y)$ must be infinitely differentiable on $y \in [0, y_2]$.

The proof of Lemma 3 is contained in that of Lemma 2. Note that infinite differentiability at $y = y_2$ implies that the solution $w(y)$ can be extended to some $y > y_2$, at least for sufficiently small $(y - y_2)$. We next deduce an important requirement of solutions of (15a,b). Note that the function $g(y)$ is smooth and vanishes at $y = 0$ and $y = y_0$. Therefore it is obvious that g' must change sign at least once on $y \in (0, y_0)$. But from (9), this means that both $(h'' - h)/w'$ and wh' (or equivalently, $H(y; K)$ and wh') cannot have the same sign for all $y \in (0, y_0)$. This requirement is indeed satisfied by solutions $w(y)$ of (15a) for which (18b) is true, as can be seen from Lemma 1. But if $w(y)$ satisfies (18c), it can be shown that g' cannot change sign on $y \in (0, y_0)$. Hence in this paper, we shall only be concerned with solutions w that satisfy (18a,b).

Theorem 1. *Uniqueness of solutions to (15a,b).* Let a given function $h(y)$ satisfy (17) and let K be restricted to negative values. Then there can exist at most one smooth solution $w(y)$ to (15a,b) that satisfies (18a,b).

Proof. See Appendix A.

Remark. Note that we are not interested in the possible existence of non-smooth solutions to (15a,b); these must necessarily have an infinite derivative (and an infinite jump in the derivative) at some point, as seen from the remark below Lemma 2, and are therefore not physically significant.

We next address the issue of existence of solutions to (15a) that satisfy (18a,b). Our method here is to directly construct a convergent iterative solution on $y \in [0, y_1]$ and then to show that this solution can be extended to $y = y_0$ without running into any singularity. Consider the following iterative scheme.

$$w_n(y; K) \equiv \tan\left[\int_0^y \frac{h''(r) - h(r)}{H_n(r; K)}\,dr\right], \qquad \forall n \in N, \tag{20}$$

where the symbol $\forall$ should be read as "for all", N is defined to be the set of all non-negative integers, and H_n is defined by

$$H_0(y; K) \equiv 2K \cosh(y), \tag{21a}$$

and

$$H_{n+1}(y; K) \equiv 2K \cosh(y) - 6w_n(y; K)h'(y) + 2\int_0^y w_n(s; K)\lambda(s, y)\, ds, \qquad \forall n \in N. \tag{21b}$$

Theorem 2. *Existence of solutions to (15a).* Suppose $h(y)$ satisfies (17). There exists a negative sequence $\{K_n\}$, $1 \le n \in N$, converging to the finite limit K_∞, such that

$$H_n(y_1; K_n) = 0, \qquad \forall \text{ positive } n \in N, \tag{22}$$

and further, $\{K_n\}$ satisfies the following ordering:

$$0 > K_n > K_{n+1}, \qquad \forall \text{ positive } n \in N. \tag{23}$$

The sequence $\{w_n(y; K_\infty)\}$ converges uniformly on $y \in [0, y_1]$ to the bounded, smooth limit function $w_\infty(y; K_\infty)$ which is the exact solution of (15a). Further, for $K = K_\infty$, there exists a smooth solution of (15a) on $y \in [0, y_0]$ that coincides with $w_\infty(y; K_\infty)$ on $y \in [0, y_1]$ and satisfies (18a,b).

Proof. See Appendix B.

Remark. In proving this result in Appendix B, we have frequently appealed to a well-known theorem from elementary calculus, which is stated below for completeness.

Intermediate Value Theorem

If $f(y)$ is a continuous, real function on $y \in [a, b]$ and if $f(a) = A$, and $f(b) = B > A$, then for every $C \in [A, B]$, there must exist at least one $c \in [a, b]$ for which $f(c) = C$.

5. Sufficient conditions for instability

In this section, we study the following question. Is there any function $h(y)$ satisfying (17) for which the solution $w(y)$ of (15a), guaranteed to exist by Theorem 2 for $K = K_\infty$, also satisfies (15b)? We can go surprisingly far in answering this rather difficult question. The key result that we shall use is that if h_0 and h_1 are two functions that satisfy (17), then any linear combination $Ah_0 + Bh_1$, where A and B are non-negative constants, must also satisfy (17) as can be easily verified.

In particular, let h_0 and h_1 satisfy (17) with $y_1 = y_1^{(0)}$ and $y_1 = y_1^{(1)}$ respectively; let us require the further restrictions that both $(h_1''' - h_1')$ and $(h_0''' - h_0')$ be positive for all y between $y_1^{(0)}$ and $y_1^{(1)}$. Let h_ε be defined by

$$h_\varepsilon(y) \equiv \varepsilon h_1(y) + (1-\varepsilon)h_0(y), \qquad \forall \varepsilon \in [0, 1]. \tag{24}$$

Since h_0 and h_1 satisfy (17), the Intermediate value theorem ensures that there exists a y_ε between $y_1^{(0)}$ and $y_1^{(1)}$ such that h_ε satisfies (17) with $y_1 = y_\varepsilon$; further, y_ε depends continuously on $\varepsilon \in [0, 1]$. Note that the additional restrictions on $(h_1''' - h_1')$ and $(h_0''' - h_0')$ mentioned above ensure that there is one and only one such y_ε, so that $h_\varepsilon(y)$ in (24) is indeed of the class given by (17). Now from Theorem 2, there must exist a smooth solution $w_\varepsilon(y)$ of (15a) (with $h = h_\varepsilon(y)$) that satisfies (18a,b) for some $K = K_\infty(\varepsilon) < 0$. A crucial result is the fact that $K_\infty(\varepsilon)$ depends continuously on $\varepsilon \in [0, 1]$; this is a straightforward consequence of the iterative procedure used to construct K_∞ in Theorem 2 and will not be proved here. But it then follows from (15a), (24), (17) and Lemmas 1–3 that $w_\varepsilon(y)$ (with $K = K_\infty(\varepsilon)$) also depends continuously on $\varepsilon \in [0, 1]$ *uniformly* in $y \in [0, y_0]$; in fact for $y \in [0, y_\varepsilon]$, this is a straightforward consequence of the iterative procedure used to construct w_ε. We are now ready to state the main result of this paper.

Define $g_\varepsilon(y)$ by equation (16) with $h(y)$, $w(y)$ and K replaced by $h_\varepsilon(y)$, $w_\varepsilon(y)$ and $K_\infty(\varepsilon)$ respectively. Consider the even extension of h_ε and the odd extensions of w_ε and g_ε to the domain $y \in [-y_0, y_0]$.

Theorem 3. *Sufficient conditions for instability.* Suppose $h_0(y)$ and $h_1(y)$ both satisfy (17), with $y_1 = y_1^{(0)}$ and $y_1 = y_1^{(1)}$ respectively; let us require the further restrictions that both $(h_1''' - h_1')$ and $(h_0''' - h_0')$ be positive for all y between $y_1^{(0)}$ and $y_1^{(1)}$. Suppose further that we are able to find h_0 and h_1 such that $g_\varepsilon(y_0) \leq 0$ for $\varepsilon = 0$ and $g_\varepsilon(y_0) \geq 0$ for $\varepsilon = 1$. Then there must exist at least one $\varepsilon_0 \in [0, 1]$ such that $g_\varepsilon(y_0) = 0$. For $\varepsilon = \varepsilon_0$ denote $g_\varepsilon(y)$, $w_\varepsilon(y)$ and $h_\varepsilon(y)$ by $g(y; \varepsilon_0)$, $w(y; \varepsilon_0)$ and $h(y; \varepsilon_0)$ respectively. The class of velocity profiles defined by $U(y) = -w(y; \varepsilon_0)$ is unstable on the domain $y \in [-y_0, y_0]$ with the wavenumber and eigenvalue given by $\alpha = 1$ and $c = i$. The real and imaginary parts of the eigenfunction ϕ in (1a,b) are given by equations (6a,b) with $w(y)$ and $g(y)$ replaced by $w(y; \varepsilon_0)$ and $g(y; \varepsilon_0)$ respectively.

Proof. The function $g_\varepsilon(y)$ depends continuously on $\varepsilon \in [0, 1]$ *uniformly* in y; this follows from (16) and the corresponding continuous dependence of $h_\varepsilon(y)$, $w_\varepsilon(y)$ and $K_\infty(\varepsilon)$. Therefore the assertion regarding the existence of at least one $\varepsilon_0 \in [0, 1]$ follows from the Intermediate value theorem. The rest of Theorem 3 is a direct consequence of the results of Sections 3–5.

Remark 1. We may further relax the restrictions on h_1 stated in Theorem 3. For example, take $h_1(y)$ to satisfy (17a,b,e,f) with (17c,d) replaced by

$$h_1''(y) - h_1(y) < 0 \text{ on } y \in [0, y_0), \qquad h_1''(y_0) - h_1(y_0) = 0,$$

and

$$h_1'''(y) - h_1'(y) > 0 \text{ for all } y \in [y_1^{(0)}, y_0]$$

where $y_1^{(0)}$ is as defined in Theorem 3. This class of functions h_1 may be thought of as the limiting case of the class given by Eqs. (17) with $y_1 \to y_0$. A suitable example of such a h_1 is $h_1(y) = \cos(y)$, with $y_0 = \pi/2$. It is easy to see that Theorem 2 with y_1 replaced by y_0 and with the last sentence excluded, is valid for $h = h_1(y)$; the proof is identical to that given in Appendix B. The advantage of choosing $h_1(y)$ as in this Remark is that we know for sure that for all $h_1(y)$ in this class, we must have $g(y_0) > 0$, since from (9), $g'(y)$ is non-negative on $y \in [0, y_0]$ [note that when $h = h_1$, we have from Theorem 2 that $w' > 0$ and $w > 0$ in (9), for all $y \in (0, y_0)$]. Further, any linear combination $h_\varepsilon(y)$ as in (24) (for any h_0 satisfying (17) with $y_1 = y_1^{(0)}$ and with $[h_0''' - h_0']$ positive for all y between $y_1^{(0)}$ and y_0) must satisfy (17a–f) for all $\varepsilon \in [0, 1)$, as can be easily verified; here it is helpful to observe that $(h_0'' - h_0)$ is necessarily positive at $y = y_0$. Note that $K_\infty(\varepsilon)$ depends continuously on $\varepsilon \in [0, 1]$. Hence Theorem 3, with h_1 defined as in this Remark and with $y_1^{(1)}$ replaced by y_0, holds. The significance of this result is elaborated upon in Section 6.

Remark 2. By Theorem 1, the solution $w(y; \varepsilon_0)$ is the unique solution of (15a,b) with $h = h(y; \varepsilon_0)$ and K restricted to be negative. Note that we refer to a *class* of velocity profiles $U(y)$ in Theorem 3. What we mean here is that for *each* pair of functions $(h_0(y), h_1(y))$ satisfying the requirements of Theorem 3 (or with $h_1(y)$ as in Remark 1) there exists at least one unstable velocity profile $U(y)$. Further observe that by Howard's semicircle theorem (Theorem III of Section 1) we must have $|w(y; \varepsilon_0)| > 1$ for some $y \in [-y_0, y_0]$.

6. Concluding remarks

Theorem 3 is interesting in that we are able to linearly combine the real parts $\{h(y)\}$ of adjoint solutions for *two different velocity profiles*, and deduce that this linear combination is the corresponding real part of the adjoint eigenfunction for a third, unstable velocity profile $U(y)$.

Note that Theorem 3 requires the specification of two functions h_0 and h_1 with the desired properties. In Remark 1 below Theorem 3, we have explicitly specified a class of functions h_1 which are indeed suitable. The task of explicitly specifying the class of functions $h_0(y)$ satisfying the requirements of Theorem 3 remains; the problem here is that given any suitable candidate $h_0(y)$, the condition $g(y_0) \leq 0$ of Theorem 3 can be verified only by numerical solution of (15a,b). However, this is an extremely difficult (but not necessarily impossible) task, for the following reason. We are interested here in computing solutions $w(y)$ of (15a) that satisfy (18b); Theorem 2 shows that for a given $h(y)$ satisfying (17) there is only one value of K, namely $K = K_\infty$, for which a solution $w(y)$ of the desired type exists. Further, no solution exists in a neighbourhood of $K = K_\infty$. Therefore numerical solutions require that K_∞ and $w(y)$ be computed with great precision (in principle, infinite precision) so that the requirement $H(y_1; K) = 0$ of (18b) is satisfied "exactly"; only then can the solution be continued up to and beyond $y = y_1$. This presents a formidable challenge, and is an interesting research problem for the future.

What should be noted carefully is that once we find *one* suitable $h_0(y)$ satisfying the conditions of Theorem 3, such that $g(y_0) < 0$, then we may combine this h_0 with a suitable class of functions $h_1(y)$ (for example, that given in Remark 1 below Theorem 3) and apply Theorem 3 to obtain an *entire class of unstable profiles*. The significance of Remark 1 below Theorem 3 is now clear. Hence the main advantage of the integral equation approach of this paper is that we are able to prescribe a procedure by which *classes* of unstable velocity profiles can be specified. Note further that the eigenfunction is available in closed form, and Sections 4 and 5 provide us with a number of global properties of the eigenfunction and the corresponding unstable velocity profile. It should therefore be of great interest to study equations (15a,b) further, with the goal of generalizing Theorem 3 to a broader class of functions $h(y)$ than those satisfying (17).

We close by observing that although Theorem 3 is intended to be the main result of this paper, Section 2 contains some significant new exact neutrally stable solutions of Rayleigh's equation. A specific example of neutrally stable eigenfunctions for a *class* of jet-like velocity profiles on an infinite domain is given. The reader should therefore take note of the comment at the end of Section 2 that there is considerable scope for further generalization of these results.

Appendix A. Proof of Theorem 1

We first prove that there cannot be two different smooth solutions to (15a,b) for a given value of K. Suppose there are two solutions $W_1(y) \geq 0$ and $W_2(y) \geq 0$ to (15a,b), for some $K < 0$ and suppose that both W_1 and W_2 satisfy (18a,b). We assume that there is some sub-interval of $[0, y_0]$ on which W_1 and W_2 are not identically equal and then arrive at a contradiction. Define $H_1(y; K)$ and $H_2(y; K)$ by (14a) with w replaced by W_1 and W_2 respectively. From (18b), we must have $H_1(y_1; K) = H_2(y_1; K) = 0$; therefore it follows from (14a) that

$$\int_0^{y_1} [W_1(s) - W_2(s)]\lambda(s; y_1)\, ds = 3h'(y_1)[W_1(y_1) - W_2(y_1)]. \tag{A.1}$$

But from (17e,f) $\lambda(s; y_1)$ is non-negative and $h'(y_1)$ is negative. Hence (A.1) implies that if W_1 and W_2 are not identically equal on $y \in [0, y_1]$, then $(W_1 - W_2)$ must necessarily change sign on $y \in (0, y_1)$; suppose this change of sign does happen. Then there must necessarily exist a $y_3 \in (0, y_1)$ such that

$$W_1(y_3) = W_2(y_3), \tag{A.2a}$$

$$W_1(y) \geq W_2(y) \geq 0, \qquad \forall y \in [0, y_3], \tag{A.2b}$$

$$W_1(y) > W_2(y) > 0, \qquad \text{for } y \text{ on some sub-interval of } [0, y_3]. \tag{A.2c}$$

Note that in (A.2b,c), W_1 and W_2 can be interchanged without loss of generality. From (A.2b,c) and (17) it immediately follows that $0 > H_1 \geq H_2$ for all $y \in [0, y_3]$ with $H_1 > H_2$ for y on some sub-interval of $[0, y_3]$. But this in turn implies that $W_1(y_3) > W_2(y_3)$, as can be seen from (15a) and (17); this result contradicts (A.2a). It is clear that the source of this contradiction lies in the assumption that W_1 and W_2 are not identically equal on $y \in [0, y_1]$ and this assumption is therefore false. Next note that the boundary condition (15b) requires

$$\int_{y_1}^{y_0} [W_1(s) - W_2(s)]\lambda_y(s; y_0)\, ds = 0. \tag{A.3}$$

The lower limit of zero has been replaced by y_1 because we have just proved that W_1 and W_2 must be identically equal for $y \in [0, y_1]$. From (17e,f) λ_y is non-negative; it therefore follows from (A.3) that $(W_1 - W_2)$ must either be identically equal to zero on $y \in [y_1, y_0]$ or it must change sign there. We may eliminate the latter possibility by a similar argument to that given earlier. The conclusion is that there can be at most one smooth solution of (15a,b) satisfying (18a,b) for a given negative value of K.

Next suppose that there are two smooth solutions $W_1(y)$ and $W_2(y)$ of (15a) for $K = K_1$ and $K = K_2$ respectively, and that both W_1 and W_2 satisfy (18a,b). We will show that this assumption leads to a contradiction and must therefore be false. We take $K_1 < K_2 < 0$ without loss of generality. Define H_1 and H_2 as done earlier. Equation (18b) again requires that H_1 and H_2 both vanish at $y = y_1$ and this leads to

$$\int_0^{y_1} [W_1(s) - W_2(s)]\lambda(s; y_1)\, ds = 3h'(y_1)[W_1(y_1) - W_2(y_1)] + (K_2 - K_1)\cosh y_1. \tag{A.4}$$

From (15a) and (17), and the fact that $K_1 < K_2 < 0$, we conclude that $[W_1(s) - W_2(s)] < 0$ as $s \to 0^+$. But from (A.4) and (17e,f), we find that $[W_1(s) - W_2(s)]$ cannot remain negative for all $s \in [0, y_1]$ for then the left hand side of (A.4) would be negative and the right hand side positive. Hence it follows that $[W_1(s) - W_2(s)]$ must change sign at least once on $y \in (0, y_1)$. This implies the existence of a $y_4 \in (0, y_1)$ such that

$$W_1(y_4) = W_2(y_4) \tag{A.5}$$

and

$$0 < W_1(s) < W_2(s), \qquad \forall s \in (0, y_4). \tag{A.6}$$

But (A.6) and (14a), along with (17) imply that

$$H_1(s; K_1) < H_2(s; K_2) < 0, \qquad \forall s \in (0, y_4]. \tag{A.7}$$

Equation (A.7) together with (15a) and (17c) leads to the conclusion that $W_1(y_4) < W_2(y_4)$, which contradicts (A.5). Hence there can be at most one negative value of K for which both (15a) and (18b) are true. The proof of Theorem 1 is now complete.

Appendix B. Proof of Theorem 2

Define $A_n(y; K)$ to be the argument of the tan function in (20), as follows.

$$A_n(y; K) \equiv \int_0^y \frac{h''(r) - h(r)}{H_n(r; K)}\, dr, \qquad \text{for } y \in [0, y_1] \text{ and } \forall n \in N, \tag{B.1}$$

where, as in Section 4, N is defined to be the set of all non-negative integers. Further define

$$g_n(y; K) \equiv -K \sinh(y) - \int_0^y w_n(s; K)\lambda_y(s, y)\, ds. \tag{B.2}$$

In what follows in Appendix B, whenever the argument K is specified (without any suffix) for functions like w_n, H_n, g_n or A_n, we will implicitly assume that K is negative and such that the above functions are well defined. We now state and prove Lemmas 4 and 5 below, in preparation for proving Theorem 2.

Lemma 4. The functions defined by equations (20)–(21) and (B.1) – (B.2) must satisfy equations (B.3) below for any $n \in N$ and $\forall y \in (0, y_1)$:

$$H_n(y; K) < H_{n+1}(y; K) \quad \text{and} \quad \partial H_n / \partial K > 0, \tag{B.3a,b}$$

$$\pi/2 > A_{n+1}(y; K) > A_n(y; K) > 0 \quad \text{and} \quad \partial A_n / \partial K > 0, \tag{B.3c,d}$$

$$w_{n+1}(y; K) > w_n(y; K) > 0 \quad \text{and} \quad \partial w_n / \partial K > 0, \tag{B.3e,f}$$

and

$$g_n(y; K) > g_{n+1}(y; K) \quad \text{and} \quad \partial g_n / \partial K < 0. \tag{B.3g,h}$$

Proof. The upper bound of $\pi/2$ in (B.3c) and the indicated signs of A_n and w_n in (B.3c,e) are obviously required for these functions to be well defined for all $n \in N$. First take $n = 0$ in equations (B.3). Equation (B.3b) is obviously true and (B.3d,f) follow from (17c,d) and (20). From (17e,f) and the fact that w_0 is positive on $y \in (0, y_1]$, we find that (B.3a) holds; but this implies that $A_1(y; K) > A_0(y; K)$ for $y > 0$ (as can be verified from (B.1) and (17c,d)), and hence, from (20), equation (B.3e) follows. We have just demonstrated that (B.3a–f) hold for $n = 0$. These arguments can be repeated for $n = 1$; thus (B.3e) and (B.3f) for $n = 0$ imply that (B.3a) and (B.3b) respectively hold for $n = 1$, as can be seen from the relevant equations. But now (B.3c,d,e,f) follow for $n = 1$. By induction, it is clear that (B.3a–f) are true for any given $n \in N$, as claimed. Equations (B.3g,h) follow trivially from (B.2), (B.3e,f) and (17e,f). Lemma 4 is now proved.

Remark. A straightforward consequence of (B.3a–h) is that if the functions H_n, A_n, w_n or g_n exist for a given y at $K = K_* < 0$, then they must be continuously dependent on K for all $K \in (-\infty, K_*)$ at the same value of y. This fact will be used repeatedly in the rest of this Appendix without explicitly stating it each time.

Lemma 5. Existence of solutions to (15a) for large $|K|$.

Consider functions $h(y)$ satisfying (17). For $K < 0$ and for sufficiently large $|K|$, the sequence $\{w_n(y; K)\}$ defined by (20)–(21) converges uniformly on $y \in [0, y_1]$ to the bounded, smooth limit function $w_\infty(y; K)$, which is the exact solution of (15a). Further, w_∞ satisfies the following:

$$w_\infty(y; K) > w_n(y; K) > 0, \qquad \forall n \in N \text{ and } \forall y \in (0, y_1], \tag{B.4a}$$

and

$$w_\infty(y; K) \to 0^+ \qquad \text{as } K \to -\infty, \forall y \in (0, y_1]. \tag{B.4b}$$

For the proof of Lemma 5, see Appendix C, where the meaning of "sufficiently large $|K|$" is also precisely defined by an estimate. We now proceed with the proof of Theorem 2. In the rest of this Appendix, we will refer to the Intermediate value theorem as IVT.

We will first show that there exists a $K_1 < 0$ such that (22) holds for $n = 1$. Define $K_0 < 0$ by

$$K_0 \equiv \frac{1}{\pi} \int_0^{y_1} \frac{h''(r) - h(r)}{\cosh(r)}\, dr.$$

Obviously, $A_0(y_1; K_0) = \pi/2$; therefore $w_0(y_1; K)$ is continuous for $K \in (-\infty, K_0)$ and has a non-integrable singularity (a pole) at $K = K_0$. We therefore obtain from (21b) and (17):

$$H_1(y_1; K) \to +\infty \qquad \text{as } K \to K_0^-. \tag{B.5a}$$

It is also obvious that

$$H_1(y_1; K) \to -\infty \qquad \text{as } K \to -\infty. \tag{B.5b}$$

From (B.5a,b) and the IVT it follows that there exists a $K_1 \in (-\infty, K_0)$ such that (22) is true for $n = 1$; further, (B.3b) guarantees that K_1 is uniquely defined by (22).

We next prove (22) and (23) by induction. Assume that (22) holds for some $n = j \in N$ where $j \geq 1$ and for some $K_j < 0$. Then the function H_j satisfies

$$H_j(y_1; K_j) = 0, \tag{B.6}$$

and any one of equations (B.7a,b,c) below:

$$H_j(y; K_j) < 0 \; \forall y \in [0, y_1) \quad \text{and} \quad H_j'(y_1; K_j) > 0, \tag{B.7a}$$

or

$$H_j(y; K_j) < 0 \; \forall y \in [0, y_1) \quad \text{and} \quad H_j'(y_1; K_j) = 0, \tag{B.7b}$$

or

$$H_j(y_+(K_j); K_j) > 0, \tag{B.7c}$$

where we define $y_+(K)$ in (B.7c) to be that value of $y \in [0, y_1]$ that maximises $H_j(y; K)$. Suppose first that (B.7a) holds; from (17c,d), $A_j(y; K_j)$ must be well defined on $y \in [0, y_1]$. Suppose further that $A_j(y_1; K_j) < \pi/2$; it then follows that $w_j(y; K_j)$ and hence, H_{j+1} are well defined smooth functions on $y \in [0, y_1]$. But then from (B.3a) and (B.6) we obtain

$$H_{j+1}(y_1; K_j) > 0. \tag{B.8}$$

From Lemma 5 and (B.4a,b) we have the following result:

$$w_n(y; K) \to 0 \qquad \text{as } K \to -\infty \textit{ uniformly} \text{ in } n \in N \text{ and } y \in (0, y_1). \tag{B.9}$$

Hence from (B.9), we conclude that

$$H_{j+1}(y_1; K) \to -\infty \qquad \text{as } K \to -\infty. \tag{B.10}$$

From (B.8), (B.10), (B.3b) and the IVT it follows that there must exist a uniquely defined $K_{j+1} \in (-\infty, K_j)$ such that (22) is true for $n = j + 1$.

Next suppose that either (B.7b) holds or (B.7a) is true with $A_j(y_1; K_j) \geq \pi/2$. Now $A_j(y; K_j)$ is smooth for all $y \in [0, y_1)$ and either has a pole at $y = y_1$, in which case $A_j(y_1; K) \to +\infty$ as $K \to K_j^-$, or satisfies $A_j(y_1; K_j) \geq \pi/2$. From (B.9) we find that $A_j(y_1; K) \to 0^+$ as $K \to -\infty$. Hence by the IVT and (B.3d) we infer that for both of the above possibilities there exists a uniquely defined $\kappa_j \in (-\infty, K_j]$ such that $A_j(y_1; \kappa_j) = \pi/2$.

But then we may deduce that $w_j(y; \kappa_j)$ is smooth on $y \in [0, y_1)$ and has a pole at $y = y_1$. We conclude from the relevant equations that

$$H_{j+1}(y_1; K) \to +\infty \qquad \text{as } K \to \kappa_j^- . \tag{B.11}$$

From (B.11), (B.10), (B.3b) and the IVT it follows that there must exist a uniquely defined $K_{j+1} \in (-\infty, \kappa_j)$ such that (22) is true for $n = j + 1$.

It remains to consider the possibility that (B.7c) is true, and we assume this in what follows. From (B.9), we obtain

$$H_j(y_+(K); K) \to -\infty \qquad \text{as } K \to -\infty. \tag{B.12}$$

Hence from (B.7c), (B.12) and the IVT, there must exist a $k_j \in (-\infty, K_j)$ such that

$$H_j(y_+(k_j); k_j) = 0. \tag{B.13}$$

Now (B.6) and (B.3b) necessarily imply that $y_+(k_j) \in (0, y_1)$; that is, $y_+(k_j)$ is required to be an interior point. Hence by definition of the maximum, we must have

$$H_j'(y_+(k_j); k_j) = 0. \tag{B.14}$$

But (B.13) and (B.14) imply that $A_j(y_1; K) \to +\infty$ as $K \to k_j^-$. The rest of the proof is identical to that for the case (B.7b); there must exist a κ_j such that (B.11) is true and therefore there must exist a uniquely defined $K_{j+1} \in (-\infty, \kappa_j)$ such that (22) is true for $n = j + 1$.

To summarize, we have shown that (22) is true for $n = 1$, and if (22) holds for some $n = j \in N, j \geq 1$, it must hold for $n = j + 1$ as well; further, the ordering given in (23) must be preserved. It is clear that we have proved by induction that (22) and (23) hold for all positive $n \in N$, as claimed.

We now demonstrate that the limit K_∞ of the sequence $\{K_n\}$ defined by (22) must be finite. Assume that $K_n \to -\infty$ as $n \to \infty$. From (B.9) we would then find that $H_n(y_1; K_n) \to -\infty$ as $n \to \infty$. But this implies that for sufficiently large $n \in N$, (22) would be violated; we have arrived at a contradiction because we have just proved that (22) holds for any positive $n \in N$. It follows that the sequence $\{K_n\}$ is bounded; since it is also ordered as in (23), it must converge to the finite limit K_∞ as claimed.

Next we show that the limit function $w_\infty(y; K_\infty)$ is uniformly bounded on $y \in [0, y_1]$. It is clear from the preceding construction of K_∞ that the sequence $\{w_n(y; K_\infty)\}$ is well defined for all $n \in N$. Assume that $w_n(y; K_\infty) \to \infty$ as $n \to \infty$ for some $y_- \in (0, y_1]$. It follows from (21b) and (17e,f) that $H_{n+1}(y_-; K_\infty)$ becomes positive for some large $n = m \in N$. But this leads to a contradiction, for $w_{m+1}(y_-; K_\infty)$ would now be non-existent. We conclude that the sequence $\{w_n(y; K_\infty)\}$ is uniformly bounded on $y \in [0, y_1]$ as $n \to \infty$. Since it is also an ordered sequence (as in (B.3e), it follows that it must converge uniformly on $y \in [0, y_1]$ to the bounded smooth limit function $w_\infty(y; K_\infty)$ as claimed. Upon taking the limit $n \to \infty$ in (20) it is easy to demonstrate that $w_\infty(y; K_\infty)$ is an exact solution of (15a) on $y \in [0, y_1]$.

Define $H_\infty(y; K)$ by (14a) with w replaced by $w_\infty(y; K)$ (or equivalently, by the limit of the sequence $\{H_n(y; K)\}$). It is clear from the above iterative construction of K_∞ that $H_\infty(y; K_\infty) < 0$ on $y \in [0, y_1)$. We next prove that $H_\infty(y_1; K_\infty) = 0$. Note that the sequences $\{H_n(y; K)\}$ and $\{w_n(y; K)\}$ must converge uniformly on $y \in [0, y_1]$ for all $K \in (-\infty, K_\infty]$, since these are ordered sequences (as in (B.3a,e)) and bounded by $H_\infty(y; K_\infty)$ and $w_\infty(y; K_\infty)$ respectively (as can be inferred from (15a), (17) and (B.3b,f)). Therefore the limit functions $H_\infty(y; K)$ and $w_\infty(y; K)$ (which is the exact solution of (15a)) must also satisfy the following equivalent of (B.3b,f) uniformly on $y \in (0, y_1)$ and $K \in (-\infty, K_\infty)$:

$$\partial H_\infty/\partial K > 0 \quad \text{and} \quad \partial w_\infty/\partial K > 0. \tag{B.15a,b}$$

Now we suppose $H_\infty(y_1; K_\infty) < 0$ (which is the only possibility if this quantity does not vanish) and arrive at a contradiction. By Lemma 2 and (B.15a,b), the solution $w_\infty(y; K)$ would remain smooth (in fact infinitely differentiable, by Lemma 3) as K is increased to some K_* such that $0 > K_* > K_\infty$. But then $H_\infty(y; K_*)$ should exist and be negative on $y \in [0, y_1)$; further, from (15a), (17) and (B.3a,e), we conclude that the ordered sequences $\{w_n(y; K_*)\}$ and $\{H_n(y; K_*)\}$ converge to $w_\infty(y; K_*)$ and $H_\infty(y; K_*)$ respectively. Noting that the sequence $\{K_n\}$ converges to K_∞, let us take some $n = m$ sufficiently large that $K_\infty < K_m < K_*$. Now since $H_m(y_1; K_*) < H_\infty(y_1; K_*) \le 0$, it follows from (B.3b) that we must have $H_m(y_1; K_m) < 0$; but this violates (22), which was proved earlier, and we have the desired contradiction. We have proved that $H_\infty(y_1; K_\infty) = 0$.

By Lemmas 2 and 3, $w_\infty(y; K_\infty)$ must be infinitely differentiable on $y \in [0, y_1]$ and therefore we must necessarily have $H'_\infty(y_1; K_\infty) > 0$; otherwise $w'_\infty(y_1; K_\infty)$ would be non-existent, as can be seen from (13) and (17c,d) and this contradicts Lemma 2. This infinite differentiability automatically implies that the solution $w_\infty(y; K_\infty)$ can be extended to some $y > y_1$, as we have noted in the paragraph below Lemma 3; therefore $H_\infty(y; K_\infty)$ must change sign and become positive as y increases past y_1 (since it has a positive derivative at y_1).

It remains to show that the solution $w_\infty(y; K_\infty)$ is smooth for all $y \in [0, y_0]$, to complete the proof of Theorem 2. We suppose that $w_\infty(y; K_\infty)$ is smooth for all $y \in [0, y_5)$ but has a singularity at $y = y_5 \in (y_1, y_0]$, and arrive at a contradiction. Now by Lemma 2, we must necessarily have

$$H_\infty(y_5; K_\infty) = 0. \tag{B.16a}$$

If $y_5 = y_0$ and if, say, $h''(y_0)$ vanishes but $h'''(y_0)$ does not, then (B.16a) must be supplemented by the vanishing of $H'_\infty(y_5; K_\infty)$. Further, we must also have

$$H_\infty(y; K_\infty) > 0 \qquad \text{on } y \in (y_1, y_5), \tag{B.16b}$$

with H_∞ vanishing at the end points of the above interval. From (B.16a) and (17c) we find that $w'_\infty(y; K_\infty) \to +\infty$ as $y \to y_5^-$. The Remark below Lemma 2 establishes that this singularity in w'_∞ is integrable and therefore $w_\infty(y_5; K_\infty)$ is finite. These facts, along with (19) and (17f) imply that

$$H'_\infty(y; K_\infty) \to +\infty \qquad \text{as } y \to y_5^- . \tag{B.16c}$$

But (B.16c) is not compatible with (B.16a,b) and we have arrived at the desired contradiction. For example, if (B.16a) is true, then (B.16c) implies that $H_\infty(y; K_\infty)$ becomes negative as $y \to y_5^-$, which contradicts (B.16b). Thus $w_\infty(y; K_\infty)$ is infinitely differentiable on $y \in [0, y_0]$ and satisfies (18a,b); the proof of Theorem 2 is complete.

Appendix C. Proof of Lemma 5 of Appendix B

Let $k \equiv -K > 0$. Define $\eta > 0$ and $\xi > 0$ by

$$\eta \equiv -\int_0^{y_1} \frac{h''(r) - h(r)}{2\cosh(r)}\, dr, \tag{C.1a}$$

and

$$\xi \equiv \max_{y \in [0, y_1]} \left[\frac{\displaystyle\int_0^y \lambda(s, y)\, ds - 3h'(y)}{\cosh(y)} \right], \tag{C.1b}$$

where $\lambda(s, y)$ is given by (14b). Consider the sequence $\{\beta_n\}$, defined by

$$\beta_0 \equiv \tan(\eta/k) > 0, \tag{C.2a}$$

and

$$\beta_{n+1} \equiv \tan[\eta/(k - \beta_n \xi)] > 0, \qquad \forall n \in N. \tag{C.2b}$$

Here, it is assumed that k is sufficiently large that β_n is well-defined for any given $n \in N$ (in particular, $0 < [\eta/(k - \beta_n \xi)] < \pi/2$). From (20)–(21) and (17), we find that

$$\beta_n > w_n(y; K) \geq 0, \qquad \forall n \in N \text{ and } \forall y \in [0, y_1]. \tag{C.3}$$

Let $\beta_\infty > 0$ be defined to satisfy

$$\Delta(\beta_\infty) \equiv \beta_\infty - \tan[\eta/(k - \beta_\infty \xi)] = 0. \tag{C.4}$$

From (C.4), we find that for any given $\varepsilon > 0$, and for sufficiently large $k = k(\varepsilon)$, the following must hold:

$$\Delta(0) < 0 \quad \text{and} \quad \Delta(\varepsilon) > 0. \tag{C.5}$$

We conclude from (C.5) that there must exist a $\beta_\infty \in (0, \varepsilon)$ such that (C.4) holds. From (C.2a,b) and (C.4), it is easy to see that the following ordering must hold:

$$\beta_\infty > \beta_{n+1} > \beta_n, \qquad \forall n \in N. \tag{C.6}$$

The sequence $\{\beta_n\}$, which is ordered and bounded, must necessarily converge to the limit β_∞. But the sequence $\{w_n\}$ is also ordered as in (B.3e) and bounded by β_∞ (as can be seen from (C.3) and (C.6)); it must therefore converge. Further, upon taking the limit $\varepsilon \to 0^+$ in (C.5), it follows that $\beta_\infty \to 0^+$ as $k \to \infty$, and (B.4b) follows. Equation (B.4a) may now be easily deduced by induction from (B.3e), (15a), (17) and (20)–(21). For if (B.4a) holds for $n = j$, it must also hold for $n = j + 1$, and (B.4a) obviously holds for $n = 0$. Note that "sufficiently large $|K|$" in Lemma 5 may now be precisely defined to mean "sufficiently large $k(= |K|)$ such that (C.4) has a positive root for β_∞." Finally, upon taking the limit $n \to \infty$ in (20), it is easily shown that $w_\infty(y; K)$ is an exact solution of (15a). The proof of Lemma 5 is now complete.

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Abstract

The primary goal of this paper is to specify sufficient conditions for the inviscid instability of a general class of plane parallel shear flows. For given complex eigenvalue c and real wave number α, and for given $h(y)$, the real part of the adjoint eigenfunction, Rayleigh's equation is converted into a nonlinear integral equation for the basic velocity profile $U(y)$. Sufficient conditions are deduced for the existence and uniqueness of solutions to this integral equation, subject to appropriate homogeneous boundary conditions on the eigenfunction $\phi(y)$; the velocity profiles $U(y)$ so derived are guaranteed to be unstable. Also separately described in this paper is a method to obtain a general class of new, exact neutrally stable solutions of Rayleigh's equation; given any real c and α, and a function $\Phi(U)$, the velocity profile $U(y)$ and the eigenfunction $\Phi(U(y))$ may be determined theoretically. A specific example of a class of neutrally stable solutions for jet-like profiles on an unbounded domain is given.

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