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An Analysis of Self-Diagnosis Model by Conditional Fault Set

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Diagnosability and syndrome decoding of a self-diagnosis model is studied with a conditional fault set. The conditional fault set is a fault set which is induced under such a condition that some subset of units are faulty (or fault-free).

The diagnosability defined on the model is generalized to include such information as (1) the maximum number of units to be identified as faulty; (2) the maximum number of units to be identified as fault-free; and (3) the maximum number of units whose states are definitely identified, when the upper bound on the number of faulty units is assumed.

Furthermore, we discuss the problem of finding minimal fault set. This problem is formulated in mathematical programming with the conditional fault set. A syndrome decoding algorithm is also presented which uses the conditional fault set in a similar manner to Hamming distance used in syndrome decoding of error-correcting codes.

KEY WORDS: Fault diagnosis; graph theory; self-diagnosis; diagnosability; syndrome decoding.

1. INTRODUCTION

The concept of system diagnosis is becoming important with the development of highly integrated digital systems and complicated computer networks.(1–4) Especially, self-diagnosis models (SDM) have been studied extensively.(5–8) SDM consists of n units, each of which can test and be tested by other units. This SDM can be expressed by a directed graph $G(V, E)$ where V is a set of vertices $\{v_i\}$ corresponding to a set of units of SDM, and E is a set of arcs $\{a_{ij}\}$ such that:

$$\begin{cases} a_{ij} \in E: \text{if a unit } x_i \text{ tests a unit } x_j \\ a_{ij} \notin E: \text{otherwise} \end{cases}$$

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Each test (arc a_{ij}) of SDM of PMC-type[5] (BGM-type[6]) produces binary test outcomes t_{ij}:

$$t_{ij} = \begin{cases} 0: & \text{if both unit } x_i \text{ and } x_j \text{ are fault-free.} \\ 1: & \text{if a unit } x_i \text{ is fault-free and a unit } x_j \text{ is faulty. (if both a unit } x_i \text{ and } x_j \text{ is faulty.)} \\ d: & \text{otherwise, where } d \text{ indicates that the test outcome can be either 1 or 0.} \end{cases}$$

For this SDM, many diagnosabilities such as t-fault diagnosability (t-fd), t-fault diagnosability with repair (t-fdwr), and t out of s diagnosability ($t/s-d$) have been proposed.[8–12] These diagnosabilities are defined under the common assumption that the number of faulty units does not exceed t. The algorithm for identifying the fault present is also studied under this assumption; a system is called t-fd iff all the faulty units are identified exactly, t-fdwr iff at least one faulty unit is identified, and $t/s-d$ if all the faulty units are specified within a subset of units whose cardinality is not greater than s.[13] A generalized diagnosability introduced here totally expresses parameters which appear one by one in the previously mentioned diagnosabilities.

Definition. A system is called $t/s/r/w-d$ iff the following conditions are satisfied provided the number of faulty units present does not exceed t.

(1) All the faulty units are specified within a set of at most s units;

(2) All the fault-free units are specified within a set of at most r units whenever the number of fault-free units is not greater than r;

(3) The states of at least w units can be identified.

When some of these parameters can not be specified, they are denoted by a dot $\cdot$. With this generalized diagnosability, t-fd, t-fdwr, and $t/s-d$ are termed $t/./.n-d$, $t/./(n-1)/1-d$, and $t/s/./(n-s)-d$ respectively where n is the number of units.

In Section 2, we define a conditional fault set. Conditions for the generalized diagnosability are expressed with the conditional fault set. In Section 3, a graph-theoretical expression of the conditional fault set of SDM is discussed. The expression of only PMC-type is obtained. In Section 4, we consider the diagnostic problem of finding the minimal fault sets consistent with a given syndrome. This problem is formulated as a mathematical programming problem with constraints and objective functions expressed by the conditional fault set defined in Section 2.

Another syndrome decoding algorithm is also proposed which uses the conditional fault set in a similar manner to that of Hamming distance used in syndrome decoding of error-correcting codes.

2. CONDITIONS FOR THE GENERALIZED DIAGNOSABILITY

A key concept of this paper is to define a conditional fault set. We discuss conditions for the generalized diagnosability in terms of the conditional fault set. Then, the generalized diagnosability of SDM is investigated as an example. We use the following notations.

1) $F(\subseteq X)$ denotes a fault set: the subset of all units which are faulty and σ denotes a syndrome: the subset of all the test arc which produces 1 as its test outcomes. We use $\sigma(F)$ to denote a syndrome which is consistent with the fault set F. If a fault pattern and syndrome is expressed in vector notation, we mention them fault pattern vector and syndrome vector respectively.
2) We use $\emptyset$ to denote a null set and $P(X)$ to denote the power set of a set X. $|S|$ denotes the cardinality of a set S.

Definition. The Conditional Fault Set of $H(S, R,)$ is defined as follows:

$$H^{\sigma}(S, R) = \min\{F \subseteq X: \sigma(F) = \sigma \text{ such that } F \supseteq S \text{ and } \bar{F} \supseteq R\}$$

where $\bar{F} = X - F$ and the minimum is taken with respect to the cardinality of a fault set F. $|H^{\sigma}(S, R)|$ denotes the cardinality of the minimal fault sets, whereas it is interpreted as if there is no consistent fault set. The conditional fault set $H^{\sigma}(S, R)$ means the minimal fault sets consistent with a syndrome under condition that units in S are all faulty and those in R all fault-free. The conditional fault set with the restricted condition that a unit x_i is faulty has already been proposed to express the condition for t-fdwr.[10] Properties of the conditional fault set $H^{\sigma}(S, R)$ are summarized Lemma 2 in the Appendix. With the conditional fault set, the generalized diagnosability $t/s/r/w-d$ is expressed as follows:

Proposition 1. A system is $t/s/r/w-d$ iff

(1) $\min_{\sigma \in \Sigma_A} |S^1(t, \sigma)| \geqslant n - s$

(2) $\min_{\sigma \in \Sigma_{n-r}} |S^0(t, \sigma)| \geqslant n - r$

(3) $\min_{\sigma \in \Sigma_A} (|S^0(t, \sigma)| + |S^1(t, \sigma)|) \geqslant w$

where

$$S^{1(0)}(t, \sigma) = \{x_i \in X: |H^\sigma(x_i, \phi)| \ (|H^\sigma(\phi, x_i)|) \geq t+1\}$$

and $\sum_A$ is a set of all possible syndromes: Σ_A is a set of all possible syndromes: $\Sigma_A = \{\sigma \subseteq E: \sigma(F) = \sigma$ for some $F \subseteq X\}$, Σ_{n-r} is a set syndrome which is consistent with only fault sets whose cardinality is less than $n-r$ where $n = |X|$. That is, $\Sigma_{n-r} = \{\sigma \subseteq E: \sigma(F) = \sigma$ for some $F \subseteq X$ such that $|F| \leq n-r\}$.

The conditions (1), (2), and (3) corresponds to the condition of $t/s/./.-d$, $t/./r/.-d$, and $t/./. /w-d$ respectively. Among these conditions, conditions (1) and (2) are expressed in a different form. The next conditions (1)′ and (2)′ is equivalent to (1) and (2) respectively by Lemma 3 in the Appendix.

A system is $t/s/r/w-d$ iff

(1)′ $\min_{\sigma \in \Sigma_A} \left(\max_{|S_i| = n-s} \left(\min_{x_i \in S_i} |H^\sigma(x_i, \phi)| \right) \right) \geq t+1$

(2)′ $\min_{\sigma \in \Sigma_{n-r}} \left(\max_{|S_i| = n-r} \left(\min_{x_i \in S_i} |H^\sigma(\phi, x_i)| \right) \right) \geq t+1$

Conditions (1) and (2) show a convenient form to obtain parameters s and r of the generalized diagnosability for a certain t. On the other hand, conditions (1)′ and (2)′ are used to investigate the permissible upper bound of t for given parameters s and r of the generalized diagnosability. The relation among these parameters of the generalized diagnosability, always hold that, $w \geq 2n-s-r$; since $n-s$ units are known to be fault-free and at least $n-r$ units are known to be faulty by the definition of these parameters where, $n = |X|$.

$t/s/./.-d$ is the same concept as $t/s-d$, which was first proposed by Friedman about SDM.[11] Necessary and sufficient condition for $t/s-d$ has not yet been obtained except for some special cases.[12] $t/./r/.-d$ is a new concept. However, the special case of this, when $r = n-1$ is known as t-fdwr about SDM. The condition for t-fdwr will be obtained by substituting $r = n-1$ in the condition (2), and this condition is the same as that proposed in Ref. 10.

The concept of $t/./r/.-d$ also plays an important role in the situation of sequential diagnosis, which proceeds by replacing faulty units with fault-free units and diagnosing for the renewed syndromes iteratively. The diagnosis will terminate in this way as long as the graph of SDM is strongly connected.[6] The number of exchanged units in a step will increase as the number r of the generalized diagnosability decrease, and the number of steps of the sequential diagnosis will decrease as well.

These are discussions about diagnosability. As for detectability, a similar discussion to that of diagnosability can be done with $H^\sigma(x_i, \varnothing)$. A system is said to be t-fault detectable (t-fdt); iff all the syndromes consistent with the null fault set $\phi \in P(X)$ (all units are fault-free) are distinguishable from those consistent with a fault set, $F \in P(X)$ where, $F \neq \emptyset$. The condition of t-fdt can also be characterized by $H^{\sigma}(x_i, \emptyset)$.

Proposition 2. A condition for t-fdt system is iff

$$\min_{x_i \in X} |H^{\sigma_0}(x_i, \emptyset)| \geqslant t+1$$

where

$$\sigma_0 = \sigma(\emptyset),$$

i.e. syndrome produced when all the units are fault-free.

Proof. $|H^{\sigma_0}(x_i, \emptyset)| \geqslant t+1$ implies that $\sigma(F_i) \neq \sigma_0$ for all $F_i \neq \emptyset$, and thus the formal diagnosis model is t-fdt. While, $|H^{\sigma_0}(x_i, \emptyset)| < t$ implies that the syndrome σ_0 is consistent with some fault set F_i such that $F_i \ni x_i$, hence $F_i \neq \emptyset$. ▮

Furthermore, the condition for the state of at least one unit can be identified as characterized with respect to the syndrome σ_1, which is consistent with the fault set $X \in P(X)$.

Theorem. (See Theorem 7 in Ref. 6.)
The state of at least one unit can be identified under the assumption that the number of faulty units present does not exceed t iff

(1) $\max_{x_i \in X} (|H^{\sigma_1}(\emptyset, x_i)|, |H^{\sigma_1}(x_i, \emptyset)|) \geqslant t+1$ where $\sigma_1 = \sigma(X)$.

(2) σ_1 is the only syndrome with which all the units can be faulty or fault-free under the assumption.

Proof. The condition (2) of the Theorem is automatically satisfied for SDM of BGM-type[6] and FDM.[14] For these models, only condition (1) is necessary and sufficient for the state where at least one unit can be identified.

3. CHARACTERIZATION OF THE CONDITIONAL FAULT SET

In the last section, the generalized diagnosability is expressed by the conditional fault set. Thus, if we can characterize the conditional fault set, these diagnosabilities are also characterized. In this section, we characterize the conditional fault set in graph-theoretical terms by using the recursive property of the conditional fault set.

By the definition of $H^\sigma(S, R)$ and that of SDM, the next formulas follow:

Lemma 1.

(1) For SDM of PMC-type (symmetric invalidation),

$$H^\sigma(S, R) = H^\sigma(S \cup \Gamma_0^{-1}(S) \cup \Gamma_1(R) \cup \Gamma_1^{-1}(R), \quad R \cup \Gamma_0(R))$$

(2) For SDM of BGM-type (asymmetric invalidation),

$$H^\sigma(S, R) = H^\sigma(S \cup \Gamma_1(R) \cup \Gamma_1^{-1}(R),$$
$$R \cup \Gamma_0(R) \cup \Gamma_0(S) \cup \Gamma_0^{-1}(S))$$

where

$$\Gamma_{1(0)}(S) = \{x_i \in X: (x_j, x_i) \in E \text{ and } t_{ji} = 1(0) \text{ for } x_j \in S\} - S$$

and

$$\Gamma_{1(0)}^{-1}(S) = \{x_j \in X: (x_j, x_i) \in E \text{ and } t_{ji} = 1(0) \text{ for } x_i \in S\} - S$$

Proof. (1) By the definition of $H^\sigma(S, R)$, the set of units S and R are assumed to be all faulty and fault-free, respectively. Since R is a set of fault-free units, a set of units $\Gamma_1(R) \cup \Gamma_1^{-1}(R)$ are regarded as all faulty, and $\Gamma_0(R)$ as all fault-free by the definition of symmetric invalidation. Further, since S is a set of faulty units, it follows that $\Gamma_0^{-1}(S)$ are also faulty again by the definition of symmetric invalidation. Thus, altogether

$$S \cup \Gamma_0^{-1}(S) \cup \Gamma_1(R) \cup \Gamma_1^{-1}(R)$$

and $R \cup \Gamma_0(R)$ are induced to be all faulty and fault-free, respectively from the assumption that S are all faulty and that R are all fault-free.

The proof of condition (2) is done in a similar manner to that of condition (1) using the definition of the conditional fault set $H^\sigma(S, R)$ and that of asymmetric invalidation. ∎

As for SDM of PMC-type, both $H^\sigma(x_i, \varnothing)$ and $H^\sigma(\varnothing, x_j)$ which are used to express the generalized diagnosability (the conditions (1), (2), and (3)) are characterized in graph-theoretical terms.

Theorem 1. For SDM of PMC-type, let S_k and R_k be the set of units induced to be all faulty and fault-free respectively by the k-th iteration of formula of Lemma 1 – condition (1).

(1) $S_n = \bigcup_{k=0}^{n-1} (\Gamma_0^{-k}(x_i))$ for $S_1 = x_i$ and $R_1 = \varnothing$.

(2) $S_n = \bigcup_{k=0}^{n-1} (\Gamma_0^{-k}(Q))$ for $S_1 = \varnothing$ and $R_1 = x_j$.

where $Q = \Gamma_1(P) \cup \Gamma_1^{-1}(P)$, $P = \bigcup_{k=0}^{n-1} (\Gamma_0^k(x_j))$, $n = |X|$ and $\Gamma_{0(1)}^0(S) = S$.

Proof. The following recurrence formulas follow by Lemma 1 – (1):

$$S_{k+1} = S_k \cup \Gamma_0^{-1}(S_k) \cup \Gamma_1(R_k) \cup \Gamma_1^{-1}(R_k) \ldots (3.1)$$
$$R_{k+1} = R_k \cup \Gamma_0(R_k) \ldots (3.2)$$

Further, since the test graph is finite, $S_{k+1} = S_k$ and $R_{k+1} = R_k$ for $k = n(= |X|)$, and the next formulas follow:

$$S_n = S_n \cup \Gamma_0^{-1}(S_n) \cup \Gamma_1(R_n) \cup \Gamma_1^{-1}(R_n) \ldots (3.3)$$
$$R_n = R_n \cup \Gamma_0(R_n) \ldots (3.4)$$

For $S_1 = x_i$, $R_1 = \varnothing$, it is easily obtained that $R_n = \varnothing$ by (3.2) and hence $S_n = \bigcup_{k=0}^{n-1} (\Gamma_0^{-k}(x_i))$ by (3.1). In fact, these S_n and R_n satisfy (3.3) and (3.4).

In case of $S_1 = \varnothing$ and $R_1 = x_j$, first, R_n is obtained by (3.2):

$$R_n = \bigcup_{k=0}^{n-1} (\Gamma_0^k(x_j))$$

Since this R_n is the set of all units induced to be fault-free, faulty units directly induced from these fault-free units are $\Gamma_1(R_n) \cup \Gamma_1^{-1}(R_n)(= Q)$, and $\bigcup_{k=1}^{n-1} (\Gamma_0^{-k}(Q))$ are also faulty units induced from these faulty units Q in turn. Thus, altogether, $\bigcup_{k=0}^{n-1}(\Gamma_0^{-k}(Q))$ are all units induced to be faulty. This S_n also satisfy (3.3) with $R_n = \bigcup_{k=0}^{n-1} (\Gamma_0^k(x_j))$. Thus, the expression (2) of this theorem is concluded. ∎

As for SDM of BGM-type, S_n for $S_1 = x_i$, $R_1 = \varnothing$ and $S_1 = \varnothing$ $R_1 = x_j$ cannot be expressed in a simple form due to the fact that S_k and R_k are dependent on each other as known by Lemma 1-condition (2).

Example 1. Consider a system whose test graph is shown in Fig. 1. Let $S_\infty(S_1, R_1)$, $R_\infty(S_1, R_1)$ denote the converged set of S_k, R_k respectively, of the recurrence formulas for the initial set S_1, R_1. Suppose a given syndrome vector is $(t_{12} t_{13} t_{23} t_{24} t_{34} t_{35} t_{45} t_{41} t_{51} t_{52}) =$ (1 1 0 0 1 0 0 0 0 1). Then, for SDM of PMC-type some converged set are obtained as follows by Theorem 1:

$$S_\infty(x_4, \varnothing) = \{x_2, x_4\} \; R_\infty(x_4, \varnothing) = \varnothing$$
$$S_\infty(\varnothing, x_4) = \{x_2, x_3\} \; R_\infty(\varnothing, x_4) = \{x_1, x_4, x_5\}$$

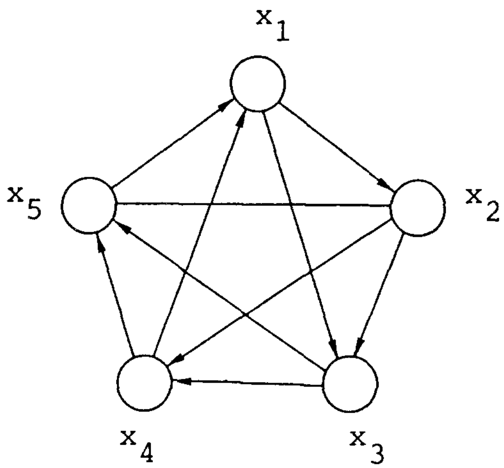

Fig. 1. An example of the graph that expresses the test relation among units of Self-Diagnosis Model (SDM) known as 2 – fd.

whereas

$$H^{\sigma}(x_4, \varnothing) = \{x_2, x_3, x_4\}$$
$$H^{\sigma}(\varnothing, x_4) = \{x_2, x_3\}$$

As SDM of BGM-type with the same syndrome, these converged sets are obtained by Lemma 1-condition (2):

$$S_{\infty}(x_4, \varnothing) = \{x_4\} \; R_{\infty}(x_4, \varnothing) = \{x_1, x_2, x_3, x_4, x_5\}$$

Since $S_{\infty}(x_4, \varnothing) \cap R_{\infty}(x_4, \varnothing) \neq \varnothing$, contradiction arises. On the other hand,

$$S_{\infty}(\varnothing, x_4) = \{x_2, x_3, x_4\}, \; R_{\infty}(\varnothing, x_4) = \{x_1, x_2, x_3, x_4, x_5\}$$

Thus, again $S_{\infty}(\varnothing, x_4) \cap R_{\infty}(\varnothing, x_4) \neq \varnothing$ and hence it is known that the syndrome is not possible syndrome as SDM of BGM-type.

Theorem 2. For SDM of PMC-type, let $\mathrm{VC}(\sigma)$ be a set of vertex covers of subgraph composed of all the arcs associated with test outcome 1. Then,

$$H^{\sigma}(S_{\infty}(S_1, \varnothing), \varnothing) = S_{\infty}(S_1, \varnothing) \text{ for all } S_1 \in \mathrm{VC}(\sigma)$$

Proof. A fault set F is consistent with a syndrome σ, if the next two requirements are satisfied:

(i) Arcs from $\bar{F}$ to F produces test outcomes 1.

(ii) Arcs of both initial vertex and terminal vertex is in $\bar{F}$ produces test outcomes 0. These requirements directly follow from the definition of SDM of PMC-type.

For all the arcs a_{ij}, such that, both $x_i, x_j \in S_\infty(S_1, \varnothing)$ the test outcomes $t_{ij}=0$, for suppose otherwise $x_i \notin S_1$ or $x_j \notin S_1$ by the hypothesis. Thus, the condition (i) is met for $S_\infty(S_1, \varnothing)$.

As to the condition (ii), since $S_\infty(S_1, \varnothing)$ is a converged solution of the recurrence formulas (3.1) and (3.2), $S_\infty(S_1, \varnothing)$ includes all the units which test S_1 with test outcome 0. Hence, $t_{ij}=0$, for all, $x_i \in \overline{S_\infty(S_1, \varnothing)}$, $x_j \in \overline{S_\infty(S_1, \varnothing)}$. Thus, the condition (ii) is also met. ∎

Theorem 2 together with the conditions (1), (2), and (3) (or (1)′, (2)′, and (3)′) of the generalized diagnosability, the diagnosability of PMC-type is characterized in graph-theoretical terms. However, it is difficult to check them. This is because the domain of syndromes over which the minimum is taken is sizable for large-scale systems. The domain can be reduced to a great extent when the graph of SDM has symmericity. Permutation

$$P=\begin{pmatrix} 1, & 2,..., & j \\ i_1, & i_2,..., & i_j \end{pmatrix}$$

is called possible permutation if some rotation (around axis or point) result in the same system as that whose suffix is permutated by P. Syndrome σ_i is regarded as equivalent to σ_j if σ_i is converted to σ_j by a possible permutation P. And the minimum of conditions of the generalized diagnosability are taken over the domain which consists of all the syndromes that are not equivalent with each other. The cardinality of this set of syndromes is obtained by Burnside's Theorem.[15]

Example 2. Figure 2 shows the graph of an example of SDM of PMC-type. The number of the cardinality of a set of syndromes Σ_A over which conditions of Proposition 1 are checked is reduced to twenty four. This reduced set of syndromes is denoted by Σ'. $\min_{\sigma \in \Sigma'} S^k(t, \sigma)$, $k=0, 1$ are shown for several t in Table 1. It is known by the conditions (1), (2), and (3) of the Proposition 1 that this SDM is $1/3/6/3-d$, $2/4/6/3-d$, and $3/6/6/0-d$. It is also known that this SDM is not even 1-fdt by Proposition 2. If we suppose the syndrome σ_0 never occurs, then the parameter r of the generalized diagnosability becomes 5 when $t=1, 2$.

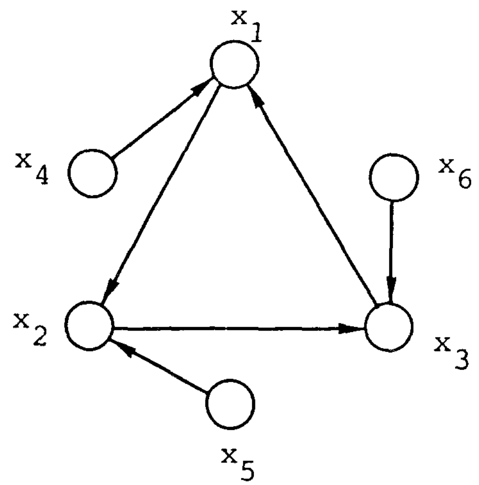

Fig. 2. An example of the graph of Self-Diagnosis Model (SDM).

4. SYNDROME DECODING BY THE CONDITIONAL FAULT SET

As discussed in the previous sections, the conditional fault set $H^{\sigma}(S, R)$ is used to characterize the generalized diagnosability. However, this conditional fault set is more suitably used for syndrome decoding. Finding fault set consistent with a given syndrome is called syndrome decoding. Most of syndrome decodings are carried out under t-fault assumption (i.e. the cardinality of a fault set present does not exceed t).[13] However, these algorithm fail to work if the system is not t-fd or if the cardinality of the existing fault set exceeds t.

In this section, we focus on finding minimal fault set so that systems are not required to be t-fd. A syndrome decoding algorithm is also presented which utilizes the conditional fault set $H^{\sigma}(S, R)$. The algorithm uses the

Table I. Parameters of Generalized Diagnosability for several *t* for the example of Self-Diagnosis Model (SDM)

t	$\min_{\sigma \in \Sigma'} (\lvert S^1(t, \sigma)\rvert)$	$\min_{\sigma \in \Sigma'} (\lvert S^0(t, \sigma)\rvert)$	$\min_{\sigma \in \Sigma'} (\lvert S^1(t, \sigma)\rvert + \lvert S^0(t, \sigma)\rvert)$
1	3	0	3
2	2	0	3
3	6	6	0

cardinality of conditional fault set in a similar manner to Hamming distance used in the syndrome decoding algorithm of error-correcting codes. We first formulate the problem as a mathematical programming.

4.1. Syndrome Decoding as a Mathematical Programming

Syndrome decoding is formulated as a mathematical programming using the conditional fault set in both an objective function and a constraint. Dual problem is also formulated using the dual feature of the conditional fault set. Obviously, the problem of finding the minimal fault set consistent with a syndrome is induced into the next two mathematical programming problems dual to each other. And the minimal fault set is given by solving either of these mathematical programming problems.

a. Problem

$$\min_{S_j \in P(X)} |S_\infty(S_j, \varnothing)|$$

under constraint that

$$H^\sigma(S_\infty(S_j, \varnothing), \varnothing) = S_\infty(S_j, \varnothing)$$

b. Dual Problem

$$\max_{S_k \in P(X)} |R_\infty(\varnothing, S_k)|$$

under constraint that

$$H^\sigma(\overline{R_\infty(\varnothing, S_k)}, \varnothing) = \overline{R_\infty(\varnothing, S_k)}$$

Example 3. For the same SDM of PMC-type as that of Example 1 with the same syndrome, we solve the problem $\min_{S_j \in P(X)} S_\infty(S_j, \varnothing)$ under constraint $H^\sigma(S_\infty, \varnothing) = S_\infty$. Fault sets which satisfy $H^\sigma(S_\infty, \varnothing) = S_\infty$ can be obtained by Theorem 2. First, the vertex cover of the subgraph composed of the arcs associated with the test outcome 1 is obtained by the Boolean function $(X_1 + X_2)(X_1 + X_3)(X_2 + X_5)(X_3 + X_4)$. Since it is developed to the form

$$X_2X_3 + X_1X_2X_4 + X_1X_3X_5 + X_1X_4X_5,$$

$$\mathrm{VC}(\sigma) = \{\{x_2, x_3\}\{x_1, x_2, x_4\}\{x_1, x_3, x_5\}\{x_1, x_4, x_5\}\}$$

Using these vertex cover as the initial fault set S_1 and $R_1 = \varnothing$, the converged sets are

$$S_\infty(\{x_2, x_3\}, \varnothing) = \{x_2, x_3\}$$
$$S_\infty(\{x_1, x_2, x_4\}, \varnothing) = \{x_1, x_2, x_3, x_4, x_5\}$$
$$S_\infty(\{x_1, x_3, x_5\}, \varnothing) = \{x_1, x_2, x_3, x_4, x_5\}$$
$$S_\infty(\{x_1, x_4, x_5\}, \varnothing) = \{x_1, x_2, x_3, x_4, x_5\}$$

These converged sets satisfy $H^\sigma(S_\infty, \varnothing) = S_\infty$, however $\{x_2, x_3\}$ is the minimal set among them. Thus the minimal fault set consistent with the syndrome is $\{x_2, x_3\}$.

4.2. Syndrome Decoding Algorithms with a Measure

Syndrome decoding is carried out with such a measure that one can indicate how close some fault pattern is to the fault pattern present whose syndrome is only known. In syndrome decoding of error-correcting codes, Hamming distance is used as such a measure. Here, we present a measure for syndrome decoding of SDM and also a syndrome decoding algorithm using the measure. It is shown that the conditional fault set $H^\sigma(S, R)$ discussed so far can be used as such a measure. We present another measure of syndrome difference which is a similar concept to Hamming distance.

If a system has the measure $m(x_i, \sigma)$ as stated below, consistent fault set will be specified for a given syndrome σ.

Definition. If a system has such a measure $m(x_i, \sigma)$, the measure is called syndrome decoding measure of the system.

$$m(x_i, \sigma) < m(x_j, \sigma)$$

for all the units $x_i, x_j \in X$ such that $x_i \in F_c$ and $x_j \notin F_c$ where F_c is the fault set consistent with σ.

The next algorithm which is similar to syndrome decoding algorithm in coding theory can find consistent fault sets for all syndromes $\sigma \in P(E_t) = \{\sigma \subseteq E\colon \sigma(F) = \sigma,\ 0 < |F| \leqslant t\}$. If syndrome decoding is carried out for t-fd systems over the domain $P(E_t)$ then syndrome decoding is done correctly, since F_c is equal to the present fault set. Measure $m(x_i, \sigma)$ plays the same role in the algorithm as Hamming distance in coding theory.

Algorithm 1. Let σ_0 be a given syndrome vector

$$\sigma := \sigma_0$$

Step 1. Find x_i such that $m(x_i, \sigma) < m(x_j, \sigma)$ for all other single fault x_j. And add the unit x_i to the set variable X; $X := X + x_i$. If $m(x_i, \sigma) = 0$ then STOP else proceed to the next step 2.

Step 2. Update the syndrome vector σ to σ' supposing the unit x_i is fault-free. $\sigma := \sigma'$. In this updating, the notation $d(x_i)$ is used which means that $d(x_i) = 1$ if x_i is faulty and 0 otherwise. If the updated syndrome vector does not include 1 (i.e., all 0 and $d(x_i)$) then STOP else go back to Step 1.

This algorithm actually terminates, since at least one unit is assumed to be fault-free in every execution of Step 2. The next theorem states what measures are available as a syndrome decoding measure.

Theorem 3.

(1) $|H^{\sigma}(x_i, \varnothing)|$ is the syndrome decoding measure if the system is t-fd.

(2) $m_s(\sigma(x_i), \sigma) = \min_{x_i \in X} w(\sigma(x_i) \oplus \sigma)$ is the syndrome decoding measure if

$$\max_{x_i \in F_i} \{|Q(R, \{x_i, x_j\})| - |Q(F_i, \{x_i, x_j\})| - |Q(x_j, F_i - x_i)|\} \geqslant 0$$

for all F_i such that $0 < |F_i| \leqslant t$ where $Q(X, Y) = \{(x_i, y_j) \in E: x_i \in X, y_j \in Y\}$ and the system is t-fd. Here, $w(x)$, $\oplus$ denotes the weight of the Boolean vector x, element-wise exclusive or operation, respectively.

Proof. (See the Appendix)

The measure stated in condition (2); $\min_{x_i \in X} w(\sigma(x_i) \oplus \alpha)$ is called syndrome difference. It is needed to catalog all syndromes corresponding to all single faults for the syndrome decoding with the measure. Using $|H^{\sigma}(x_i, \varnothing)|$ as a syndrome decoding measure, at least one of the minimal fault sets consistent with a given syndrome is obtained even if the system is not t-fd. The next examples show how this algorithm works for diagnosable systems.

Example 4. It is known that the SDM shown in Fig. 1 is 2-fd. Syndrome vectors catalogued are:

$\sigma(x_1) = (d\ d\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0)$

$\sigma(x_2) = (1\ 0\ d\ d\ 0\ 0\ 0\ 0\ 0\ 1)$

$\sigma(x_3) = (0\ 1\ 1\ 0\ d\ d\ 0\ 0\ 0\ 0)$

$\sigma(x_4) = (0\ 0\ 0\ 1\ 1\ 0\ d\ d\ 0\ 0)$

$\sigma(x_5) = (0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ d\ d)$

Suppose the same syndrome vector as that of Example 1, $\sigma_0 = (1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1)$ is obtained, then the fault unit x_2 is chosen in Step 1, for $m_s(\sigma(x_2), \sigma_0)$ realizes the minimum. In Step 2, syndrome vector σ_0 is updated to $\sigma' = (0\ 1\ d(x_3)\, d(x_4)\ 0\ 0\ 0\ 0\ 0\ 0)$ where $d(x_i) = 1$ if x_i is faulty, and 0 otherwise. For this updated syndrome, Step 1 is executed again. Then, x_3 realizes the minimum of syndrome difference. And further, since $m_s(\sigma(x_3), \sigma') = d(x_4)$ which is equal to 0 when the unit x_4 is fault-free, the fault pattern vector $(0\ 1\ 1\ 0\ 0)$ is concluded.

Using $|H^\sigma(x_i, \varnothing)|$ as a syndrome decoding measure, the minimal fault set consistent with the syndrome is obtained by one execution of Steps 1 and 2, for

$$|H^\sigma(x_1, \varnothing)| = |\{x_1, x_2, x_3, x_4, x_5\}| = 5$$
$$|H^\sigma(x_2, \varnothing)| = |\{x_2, x_3\}| = 2$$
$$|H^\sigma(x_3, \varnothing)| = |\{x_2, x_3\}| = 2$$
$$|H^\sigma(x_4, \varnothing)| = |\{x_2, x_3, x_4\}| = 3$$
$$|H^\sigma(x_5, \varnothing)| = |\{x_1, x_2, x_3, x_4\}| = 4$$

CONCLUSION

Properties of the conditional fault set are mainly discussed. It is used for both characterizing the generalized diagnosability and syndrome decoding. The generalized diagnosability is an extension of the diagnosabilities incorporating most of the diagnosabilities so far proposed. A detailed diagnosis is possible by the generalized diagnosability, since this diagnosability incorporates information expressed in t-fault diagnosability, t-fault diagnosability with repair, and t out of s diagnosability.

Conditions for the generalized diagnosability are discussed with a concept of conditional fault set. Although these conditions are difficult to check, for they must be checked all over the possible syndromes, this domain will be reduced in some cases. By the conditional fault set, an algorithm for finding the minimal fault set consistent with a given syndrome is obtained as well. This problem is formulated as a mathematical programming problem with a conditional fault set.

As a result, once these conditional fault sets are obtained, the generalized diagnosability as well as the minimal fault set consistent with a syndrome is given.

Problem left for future study is to develop a heuristic algorithm for the calculation of these conditional fault sets efficiently. Expressing the conditional fault set of other diagnosis models such as probabilistic self-diagnosis model and self-diagnosis with incomplete test are also interesting problems.

APPENDIX

Lemma 2. (Properties of the conditional fault set)

(1) (i) If $H^\sigma(S_i, \varnothing) \neq \varnothing$, then $H^\sigma(S_i, \varnothing) \supseteq S_i$ where equality holds when $\sigma(S_i) = \sigma$.

(ii) $H^\sigma(\varnothing, \bar{S}_i) \subseteq S_i$ where equality holds when S_i is the minimal fault set consistent with a syndrome σ.

(iii) A fault set $F_i \in P(X)$ is the minimal fault set consistent with a syndrome σ iff $H^\sigma(\varnothing, \bar{F}_i) = H^\sigma(F_i, \varnothing)$ where $\bar{F}_i = X - F_i$.

(2) If $\{S_i\}$ $i = 1 \ldots q$ is a set of units such that, (a) $|S_i| \leqslant t$ $i = 1 \ldots q$; (b) $\sigma = \sigma(S_1) = \sigma(S_2) = \cdots = \sigma(S_q)$ and $\sigma(S_j) \neq \sigma$ for all $S_j \notin \{S_i\}$, then

(i) $|H^\sigma(x_k, \varnothing)| \geqslant t + 1$ for all x_k in $(\overline{\bigcup_{i=1}^q S_i})$

(ii) $|H^\sigma(\varnothing, x_k)| \geqslant t + 1$ for all x_k in $(\bigcap_{i=1}^q S_i)$

Proof. In Lemma 2 condition (1) (i) and (ii) is straightforward from the definition of $H^\sigma(S, R)$. Hence, we prove (iii).

Sufficiency: By the property of the conditional fault set in Lemma 2 condition (1) it follows that $H^\sigma(S_i, \varnothing) \supseteq S_i \supseteq H^\sigma(\varnothing, \bar{S}_i)$. Thus, $H^\sigma(\varnothing, \bar{F}_i) = H^\sigma(F_i, \varnothing)$ implies that $H^\sigma(\varnothing, \bar{F}_i) = H^\sigma(F_i, \varnothing) = F_i$.

Necessity: If a fault set F_i is consistent with a syndrome σ, then $H^\sigma(F_i, \varnothing) = F_i$ and $H^\sigma(\varnothing, \bar{F}_i) \subseteq F_i$. Furthermore, since F_i is the minimum fault set, there is no other fault set F_j such that $H^\sigma(\varnothing, \bar{F}_i) = F_j \subset F_i$.

(2) For the proof in Lemma 2 of condition (2)-(i), suppose there exists the unit x_k such that $|H^\sigma(x_k, \varnothing)| \leqslant t$ and $x_k \in (\overline{\bigcup_{i=1}^q S_i})$. This implies that there exist a fault set S which is consistent with the syndrome σ and that $S_j \notin \{S_i\}$. This fact contradicts (b). In the same manner, the existence of x_k such that $|H^\sigma(\varnothing, x_k)| < t + 1$ and $x_k \in (\bigcap_{i=1}^q S_i)$ also violates the hypothesis (b). ∎

Lemma 3. Relationship between $S^k(t, \sigma)$ and $H^\sigma(S, R)$, $k = 1, 0$.

(1) $\max_{|S_i| = n-s} (\min_{x_i \in S_i} |H^\sigma(x_i, \varnothing)|) \geqslant t + 1$ iff $|S^1(t, \sigma)| \geqslant n - s$

(2) $\max_{|S_i| = n-r} (\min_{x_i \in S_i} |H^\sigma(\varnothing, x_i)|) \geqslant t + 1$ iff $|S^0(t, \sigma)| \geqslant n - r$

Proof. $\min_{x_i \in S_i} H^\sigma(S_i, \emptyset)$ indicates all the fault sets consistent with the syndrome σ and satisfy the constraint that at least one unit is faulty in S_i. Therefore, the condition

$$\max_{|S_i| = n-s} \left(\min_{x_i \in S_i} |H^\sigma(x_i, \emptyset)|\right) \geq t+1$$

is equivalent to that all the fault sets consistent with the syndrome σ and whose cardinality is less than $t+1$ must satisfy the constraint that all the subsets of units whose cardinality is greater than $n-s$ consist of only fault-free units. And this fact is equivalent to the condition $|S^1(t, \sigma)| \geq n-s$. The proof of condition (2) is done in the same manner as this. ▮

Proof of Theorem 3.

(1) Suppose the system under consideration is t-fd and that there exist $x_i \in F_i$, $x_j \notin F_j$ such that

$$|H^\sigma(x_i, \emptyset)| \geq |H^\sigma(x_j, \emptyset)|$$

for some $F_i (0 < |F_i| \leq t)$ which is consistent with the syndrome σ. Since $x_i \in F_i$ and $\sigma(F_i) = \sigma$, it follows that $H^\sigma(x_i, \emptyset) \subseteq F_i$ by the definition of the conditional fault set. However, $H^\sigma(x_j, \emptyset)$ is also a fault set which is consistent with σ and that

$$|H^\sigma(x_j, \emptyset)| \leq |H^\sigma(x_i, \emptyset)| \leq |F_i| \leq t$$

Thus, it contradicts the fact that the system is t-fd, since $H^\sigma(x_j, \emptyset)$ is another fault set consistent with σ.

(2) Graph-theoretically, the syndrome difference of the SDM of PMC-type can be written as:

$$m_s(\sigma(F_i), \sigma(F_j)) = |Q(\overline{F_i \cup F_j}, F_i - (F_i \cap F_j))| + |Q(\overline{F_i \cup F_j}, F_j - (F_i \cap F_j))|$$

where

$$Q(X, X') = \{(x_i, x_j) \in E: x_i \in X \text{ and } x_j \in X'\}$$

Condition (2) describes the condition $m_s(\sigma(x), \sigma(x')) > 0$ in vector notation. This condition is obtained by substituting the syndrome difference of SDM expressed by graph-theoretical notations the definition of syndrome decoding measure: For all fault sets $F_i \subseteq S$, there exists a unit $x_i \in F_i$ that $m_s(\sigma(x_i), \sigma(F_i)) < m_s(\sigma(x_j), \sigma(F_i))$ for all other unit x_j such that $x_j \notin F_i$. ▮

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