Multidimensional Systems and Signal Processing, 5, 41–60 (1994)

Output Feedback Stabilizability and Stabilization Algorithms for 2D Systems

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Received November 25, 1992; Revised April 5, 1993

Abstract. Alternative methods are proposed for test of output feedback stabilizability and construction of a stable closed-loop polynomial for 2D systems. By the proposed methods, the problems can be generally reduced to the 1D case and solved by using 1D algorithms or Gröbner basis approaches. Another feature of the methods is that their extension to certain special nD ($n > 2$) cases can be easily obtained.

Moreover, the "Rabinowitsch trick," a technique ever used in showing the well-known Hilbert's Nullstellensatz, is generalized in some sense to the case of modules over polynomial ring. These results eventually lead to a new solution algorithm for the 2D polynomial matrix equation $D(z, w)X(z, w) + N(z, w)Y(z, w) = V(z, w)$ with $V(z, w)$ stable, which arises in the 2D feedback design problem. This algorithm shows that the equation can be effectively solved by transforming it to an equivalent Bezout equation so that the Gröbner basis approach for polynomial modules can be directly applied.

Key Words: 2D system, polynomial matrices, unilateral equation, Bezout equation, Gröbner basis, module, output feedback, stabilizability, stabilization

Notation

$\mathbf{R}$	the field of real numbers
$\mathbf{C}$	the field of complex numbers
$\mathbf{R}[z, w]$	commutative ring of 2D polynomials in z and w with coefficients in $\mathbf{R}$
$\mathbf{M}(\mathbf{R}[z, w])$	set of matrices with appropriate dimensions with entries in $\mathbf{R}[z, w]$
$\mathbf{R}[z, w]^n$	module of ordered n-tuples in $\mathbf{R}[z, w]$
$\mathbf{R}[z, w]^{n \times m}$	set of $n \times m$ matrices with entries in $\mathbf{R}[z, w]$
$\bar{U}$	closed unit disc in $\mathbf{C}$, i.e., $\{z \in \mathbf{C} \mid \lvert z \rvert \leq 1\}$
$\bar{U}^2$	closed unit bidisc, i.e., $\{(z, w) \in \mathbf{C}^2 \mid \lvert z \rvert \leq 1, \lvert w \rvert \leq 1\}$
A^T	transpose of matrix A

1. Introduction

The synthesis problem of stabilizing compensators for 2D linear feedback systems has been investigated by a number of researchers (see, e.g., [1]–[10] and the references therein). As in the 1D case, the synthesis procedure can be reduced to solving certain linear equations for polynomial or polynomial matrices in two variables. In particular, it has been shown that a causal 2D multivariable plant given by a (left) matrix fraction description

(MFD) $D^{-1}(z, w)N(z, w)$, with $D(z, w), N(z, w) \in \mathbf{M}(\mathbf{R}[z, w])$ and $\det D(0, 0) \neq 0$, is output feedback (structurally) stabilizable by a causal 2D compensator $Y(z, w)X^{-1}(z, w)$ if and only if the equation

$$D(z, w)X(z, w) + N(z, w)Y(z, w) = V(z, w) \tag{1}$$

holds, where $X(z, w), Y(z, w), V(z, w) \in \mathbf{M}(\mathbf{R}[z, w])$, $\det X(0, 0) \neq 0$, and $\det V(z, w) \neq 0$ for any $(z, w) \in \bar{U}^2$ [1, 3, 9]. Once Equation (1) is solved, the solution to the Bezout identity over the ring of 2D stable rational function can be directly obtained, and the class of all stabilizing compensators can be explicitly parameterized according to the MFD approach [1, 11]. A well-known necessary and sufficient condition for Equation (1) to be solvable is that $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$ (see, e.g., [1, 3]), where $\mathcal{V}(\mathcal{I})$ is the variety of the ideal $\mathcal{I}$ generated by all the maximal order minors $a_i(z, w)$, $i = 1, \ldots, \beta$, of the matrix $[D(z, w) \; N(z, w)]$.

Several problems are naturally associated with the synthesis of 2D compensators. There is, first of all, the problem of testing the feedback stabilizability of a given system without explicit computation of $\mathcal{V}(\mathcal{I})$ or direct solution of Equation (1). Then arises the problem of constructing a suitable stable polynomial matrix $V(z, w)$ such that Equation (1) holds for some $X(z, w)$ and $Y(z, w)$, and the problem of computing such solution.

Fornasini [10] and Bisiacco et al. [5] have recently shown a criterion and a linear test algorithm for stabilizability of 2D multivariable systems. By this method, the test can be reduced to the solution of a finite family of Lyapunov equations and thus can be accomplished in a finite number of steps.

As for construction of a stable $V(z, w)$ and solution of Equation (1), roughly speaking, two kinds of procedures have been developed up to now. The first kind is due to the researches of Guiver and Bose [1], Bisiacco et al. [2]–[7], and Fornasini [10] which can be recalled as follows.

(i) Construct a stable 2D polynomial, say $s(z, w)$, which vanishes on the variety $\mathcal{V}(\mathcal{I})$.
(ii) By employing Hilbert's Nullstellensatz, then, it can be shown that there exist $x_i(z, w) \in \mathbf{R}[z, w]$, $i = 1, \ldots, \beta$, and some integer r such that

$$\sum_{i=1}^{\beta} a_i(z, w)x_i(z, w) = s^r(z, w). \tag{2}$$

(iii) Next, 2D polynomial matrices $X(z, w)$, $Y(z, w)$ such that

$$D(z, w)X(z, w) + N(z, w)Y(z, w) = s^r(z, w)I \tag{3}$$

can be computed from the results obtained in (ii) by the methods of [2, 12, 13].

For the purpose of (i), if $\mathcal{V}(\mathcal{I})$ is explicitly known and $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$, such $s(z, w)$ can be directly constructed [1]–[3]. In general, however, the explicit computation of $\mathcal{V}(\mathcal{I})$ is not an easy job. To avoid this difficulty, attempts have been made (see, e.g., [4, 5, 10]), and eventually a linear algorithm without such requirement has been achieved in [5].

To solve Equation (2), Gröbnor basis approach for polynomial ideal [14] or the method of [15] can be employed. However, since the bound of the degree r is not available in the above procedure, it would be a mere repetition to solve Equation (2) so that the computation would be very costly [9].

As essential steps of the procedure, all the maximal order minors $a_i(z, w)$, $i = 1, \ldots, \beta$, and the Gröbner basis of $\mathcal{G}$ have to be calculated. It has been indicated, however, that this may not be an easy task due to the fact that the number of the maximal order minors of $[D(z, w)\ N(z, w)]$ is rather large even when the dimensions of $D(z, w)$ and $N(z, w)$ are relatively small [9].

It has also been pointed out that the restricted form $V(z, w) = s^r(z, w)I$ in Equation (3) may result in a solution $X(z, w)$, $Y(z, w)$ with relatively high degree in z and w [9]. This is due to the fact that this form constrains, in fact, all the diagonal polynomial entries of $V(z, w)$ in $\mathcal{G}$. Lin [9], however, has shown an example for which $V(z, w)$ indeed has entries not in $\mathcal{G}$ but such that Equation (3) admits a solution.

Nevertheless, this procedure is attractive for the advantage that it is easy to be extended to nD ($n > 2$) cases. As a matter of fact, since Gröbner basis approach is generally applicable for multivariable polynomials, the procedure applies directly to nD ($n > 2$) cases as long as a suitable stable nD polynomial can be constructed.

In contrast with the kind of procedure mentioned above, another kind of procedure was initiated by Raman and Liu [8] for SISO (single-input/single-output) 2D systems and later developed by Lin [9] to MIMO (multi-input/multi-output) case. According to the procedure of [9], a stable $V(z, w)$ and a solution $X(z, w)$, $Y(z, w)$ can be constructed simultaneously without calculating the minors. Further, the obtained $V(z, w)$ in [9] has a more general form which may lead to solution $X(z, w)$, $Y(z, w)$ having less degree in z and w than the solution obtained by the first kind of procedure. In this procedure, however, one has to compute explicitly all the (unstable) roots of the entries of a diagonal 1D polynomials matrix $Q(z)$ which satisfies $D(z, w)X_0(z, w) + N(z, w)Y_0(z, w) = Q(z)$, and then, for every unstable root in every polynomial entry of $Q(z)$, successively perform an elimination procedure that involves in general Smith form transformation of certain 1D matrices. Therefore, the procedure may be computationally inefficient when the number of such roots and/or the dimension of $Q(z)$ are relatively large. Furthermore, this approach reduces the considered 2D problem to 1D case by substituting explicit value corresponding to a (unstable) zero into one of the two variables, so it is rather difficult, if not impossible, to be extended to nD ($n > 2$) cases.

In view of the above discussions, the main concerns of this paper are as follows. In Section 2, alternative methods are proposed for test of output feedback stabilizability and construction of a closed-loop stable polynomial of 2D systems. By these methods, the considered problems can be generally reduced to 1D case, and thus can be solved by using either 1D algorithms or Gröbner basis approaches. Although it is hard to say that the proposed methods provide computational advantage in general, they indeed exhibit some insights to the same problems of nD systems. In fact, the basic ideas adopted in Section 2 can be applied to establish similar results, at least, for some simple nD ($n > 2$) cases. In Section 3, then, the "Rabinowitsch trick" (see, e.g., [16, 17]) is generalized in some senses to the case of modules over polynomial ring. Based on these results, a new solution algorithm for Equation (1) is proposed, which solves Equation (1) via the solution of an equivalent Bezout equation over a polynomial overring of $\mathbf{R}[z, w]$ by means of the Gröbner basis approach for polynomial modules [18]–[20]. According to this algorithm, some problems of the existing methods, such as the unavailability of the degree r and the computation of minors or zeros, can be avoided. Nevertheless, a $V(z, w)$ having the general form as shown in [9] can be obtained, and it is possible to extend the algorithm to nD ($n > 2$) cases without essential difficulty.

2. Stabilizability and closed-loop stable polynomials

Consider a MIMO 2D linear system given by a left MFD $D^{-1}(z, w)N(z, w)$. Without loss of generality, we suppose that $D(z, w)$ and $N(z, w)$ are left factor coprime. In fact, if $D(z, w)$ and $N(z, w)$ are not left factor coprime, a left greatest common factor, say $R(z, w)$, can be constructively computed and removed [21]–[23], and the stability of det $R(z, w)$ can be checked (see, e.g., [24]). By the results of [21], then, we always have $X_1(z, w)$, $Y_1(z, w)$, $X_2(z, w)$ and $Y_2(z, w) \in \mathbf{M}(\mathbf{R}[z, w])$ such that

$$D(z, w)X_1(z, w) + N(z, w)Y_1(z, w) = V_1(z), \tag{4}$$

$$D(z, w)X_2(z, w) + N(z, w)Y_2(z, w) = V_2(w), \tag{5}$$

where $V_1(z) \in \mathbf{M}(\mathbf{R}[z])$, $V_2(w) \in \mathbf{M}(\mathbf{R}[w])$ are diagonal 1D polynomial matrices with nonzero determinants.

By employing the long division method, a 1D polynomial can be decomposed, without explicit computation of its roots, into a product of a stable polynomial and a completely unstable polynomial (having only unstable zeros) [25, 26]. We can, therefore, carry out the decompositions

$$V_1(z) = V_{1u}(z)V_{1s}(z), \tag{6a}$$

$$V_2(w) = V_{2u}(w)V_{2s}(w), \tag{6b}$$

or alternatively,

$$\det V_1(z) = \det V_{1u}(z) \det V_{1s}(z), \tag{7a}$$

$$\det V_2(w) = \det V_{2u}(w) \det V_{2s}(w), \tag{7b}$$

such that det $V_{1u}(\xi)$ and det $V_{2u}(\xi)$ are completely unstable, while det $V_{1s}(\xi)$ and det $V_{1s}(\xi)$ are stable.

For 1D polynomials $g_1(z) \in \mathbf{R}[z]$ and $g_2(w) \in \mathbf{R}[w]$, we define the notation

$$\Gamma\{g_1(z), g_2(w)\} = \{(z, w) \in \mathbf{C}^2 \mid g_1(z) = 0, g_2(w) = 0\}. \tag{8}$$

Then the following results can be given.

THEOREM 1. For given left factor coprime MFD $D^{-1}(z, w)N(z, w)$ with $D(z, w) \in \mathbf{R}[z, w]^{n\times n}$, and $N(z, w) \in \mathbf{R}[z, w]^{n\times m}$, define

$$
\begin{aligned}
F(z, w) &= [D(z, w) \quad N(z, w)] \\
&= \begin{bmatrix} f_{1,1} & \cdots & f_{1,m+n} \\ \vdots & & \vdots \\ f_{n,1} & \cdots & f_{n,m+n} \end{bmatrix} \\
&= [\vec{f}_1 \quad \cdots \quad \vec{f}_{m+n}]
\end{aligned} \tag{9}
$$

where

$$
\vec{f}_j = [f_{1,j} \;\; f_{2,j} \;\; \cdots \;\; f_{n,j}]^T \in \mathbf{R}[z, w]^n, \qquad j = 1, \ldots, m + n, \tag{10}
$$

and denote by $\mathcal{V}(\mathcal{I})$ the variety of the ideal $\mathcal{I}$ generated by all the nth-order minors of $F(z, w)$, i.e., $a_i(z, w)$, $i = 1, \ldots, \beta$, $\beta = (m + n)!/(m!n!)$.

Then the following statements are equivalent:

(i) The 2D system given by $D^{-1}(z, w)N(z, w)$ is output feedback stabilizable;
(ii) For any $(z_0, w_0) \in \Gamma\{\det V_{1u}(z), \det V_{2u}(w)\}$, the matrix $[D(z_0, w_0) \quad N(z_0, w_0)]$ is full rank;
(iii) A non-zero constant is an element (the only element) included in the Gröbner basis (the reduced Gröbner basis) of the ideal generated by $\det V_{1u}(z)$, $\det V_{2u}(w)$, and $a_i(z, w)$, $i = 1, \ldots, \beta$;
(iv) For $i = 1, \ldots, n$, $\vec{e}_i$ is an element of the Gröbner basis of the module generated by

$$
\left\{ \vec{f}_1, \ldots, \vec{f}_{m+n}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det V_{1u}(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} *, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det V_{2u}(w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * \right\} \tag{11}
$$

where $\vec{e}_i$ denotes the n-tuple having 1 as the element at the ith position and zeros at the other positions, and $*$ denotes the ith position of the related tuples.

Proof. (i) $\Leftrightarrow$ (ii) As mentioned in the introduction, the output feedback stabilizability of 2D system $D^{-1}(z, w)N(z, w)$ is equivalent to the solvability of Equation (1). By the Cauchy-Binet theorem and Equations (4) and (5), it is clear that if (z_0, w_0) is a common zero of $a_i(z, w)$, $i = 1, \ldots, \beta$, namely, $(z_0, w_0) \in \mathcal{V}(\mathcal{I})$, then $\det V_1(z_0) = 0$ and $\det V_2(w_0) = 0$ simultaneously. This fact obviously implies that

$$
\mathcal{V}(\mathcal{I}) \subset \Gamma\{\det V_1(z), \det V_2(w)\}. \tag{12}
$$

By Equation (12) and the definitions of $\det V_{1u}(z)$ and $\det V_{2u}(w)$, it is easy to see

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 \subset \Gamma\{\det V_{1u}(z),\ \det V_{2u}(w)\} \subset \bar{U}^2. \tag{13}$$

On the other hand, it has been mentioned that

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset \tag{14}$$

is a necessary and sufficient condition for (i) to be true. In view of this fact and Equation (13), then, one can readily conclude that (i), or equivalently, Equation (1) holds if and only if

$$\mathcal{V}(\mathcal{I}) \cap \Gamma\{\det V_{1u}(z),\ \det V_{2u}(w)\} = \emptyset, \tag{15}$$

or equivalently, (ii) is true.

(ii) ⇔ (iii) As a matter of fact, Equation (15) implies, and is also implied by, the zero coprimeness of the polynomials $\det V_{1u}(z)$, $\det V_{2u}(w)$ and $a_i(z, w)$, $i = 1, \ldots, \beta$. By the result of [12], this is equivalent to the solvability of the equation

$$\sum_{i=1}^{\beta} a_i(z, w)\tilde{x}_i(z, w) + \bar{x}_1(z, w) \det V_{1u}(z) + \bar{x}_2(z, w) \det V_{2u}(w) = 1, \tag{16}$$

where $\tilde{x}_i, \bar{x}_j \in \mathbf{R}[z, w]$, $i = 1, \ldots, \beta$, $j = 1, 2$. According to the properties of Gröbner basis Equation (16) holds if and only if (iii) is true [14].

(iii) ⇔ (iv) Suppose that (iii) holds true, then we have the result of Equation (16), and we can rewrite it as

$$\sum_{i=1}^{\beta} a_i(z, w)\tilde{x}_i(z, w) = 1 - \bar{x}_1(z, w) \det V_{1u}(z) - \bar{x}_2(z, w) \det V_{2u}(w). \tag{17}$$

By the methods of [2], [12], and [13], we can obtain $Z(z, w) = [X^T(z, w)\quad Y^T(z, w)]^T \in \mathbf{R}[z, w]^{(n+m)\times n}$ such that

$$F(z, w)Z(z, w) = (1 - \bar{x}_1(z, w) \det V_{1u}(z) - \bar{x}_2(z, w) \det V_{2u}(w))\, I_n, \tag{18}$$

where I_n is the $n \times n$ identity matrix. Obviously, this equation can also be written in the form

$$z_{1,i}(z, w)\vec{f}_1(z, w) + \cdots + z_{m+n,i}(z, w)\vec{f}_{m+n}(z, w)$$
$$+ \bar{x}_{1,i}(z, w) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det V_{1u}(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * + \bar{x}_{2,i}(z, w) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \det V_{2u}(w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} *,$$
$$i = 1, \ldots, n, \tag{19}$$

where $z_{1,i}(z, w), \ldots, z_{m+n,i}(z, w)$ correspond to the entries of the ith column of $Z(z, w)$, and $\bar{x}_{1,i}(z, w) = \bar{x}_1(z, w)$ and $\bar{x}_{2,i}(z, w) = \bar{x}_2(z, w)$. In view of the properties of Gröbner basis for polynomial modules (see, e.g., [18]), the claim of (iv) is concluded.

Conversely, if (iv) is true, then by using Gröbner basis approach for polynomial modules we can find $z_{1,i}(z, w), \ldots, z_{m+n,i}(z, w)$, and $\bar{x}_{1,i}, \bar{x}_{2,i}$, $i = 1, \ldots, n$, such that Equation (19) is satisfied, which can be written in the matrix form as

$$F(z, w)Z(z, w) = \begin{bmatrix} v_{1,1}(z, w) & 0 & \cdots & 0 \\ 0 & v_{2,2}(z, w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_{n,n}(z, w) \end{bmatrix} \tag{20}$$

where

$$v_{i,i}(z, w) = 1 - \bar{x}_{1,i}(z, w) \det V_{1u}(z) - \bar{x}_{2,i}(z, w) \det V_{2u}(w), \quad i = 1, \ldots, n. \tag{21}$$

Applying Cauchy-Binet theorem to Equation (20), we get

$$\begin{aligned} \sum_{k=1}^{\beta} a_k(z, w)\tilde{x}_k(z, w) &= \prod_{i=1}^{n} v_{i,i}(z, w) \\ &= \prod_{i=1}^{n} (1 - \bar{x}_{1,i}(z, w) \det V_{1u}(z) - \bar{x}_{2,i}(z, w) \det V_{2u}(w)) \end{aligned} \tag{22}$$

where $a_k(z, w)$, $\tilde{x}_k(z, w)$ correspond to the $n \times n$ minors of $F(z, w)$ and $Z(z, w)$. Further, expanding the product on the right-hand side of Equation (22), we can have

$$\sum_{k=1}^{\beta} a_k(z, w)\tilde{x}_k(z, w) = 1 - \bar{x}_1(z, w) \det V_{1u}(z) - \bar{x}_2(z, w) \det V_{2u}(w) \tag{23}$$

for some $\bar{x}_1(z, w), \bar{x}_2(z, w) \in \mathbf{R}[z, w]$. This is obviously identical to Equation (16) that implies (iii). □

THEOREM 2. Suppose that $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$. Then the polynomial $s(z, w)$ defined as

$$s(z, w) = \det V_{1s}(z) \det V_{2s}(w) \tag{24}$$

vanishes on $\mathcal{V}(\mathcal{I})$ and is stable, namely, devoid of zeros in $\bar{U}^2$.

Proof. Suppose that $(z_0, w_0) \in \mathcal{V}(\mathcal{I})$. Since $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset$ by assumption, it follows that

$$|z_0| > 1 \quad \text{and/or} \quad |w_0| > 1. \tag{25}$$

By using Equation (12) and taking into account the constraint of Equation (25), it can be concluded that $\det V_{1s}(z_0) = 0$ and/or $\det V_{2s}(w_0) = 0$. In either case, however,

$$s(z_0, w_0) = \det V_{1s}(z_0) \det V_{2s}(w_0) = 0, \tag{26}$$

which shows that $s(z, w)$ vanishes at every $(z_0, w_0) \in \mathcal{V}(\mathcal{J})$. By the definitions of $\det V_{1s}(z)$ and $\det V_{2s}(w)$, it is clear that $s(z, w)$ possesses no zeros in $\bar{U}^2$. □

Remark 1. It is noted that Theorem 1 provides three different approaches to test the stabilizability. In the statement (ii), it is possible to compute the zeros of $\det V_{1u}(\xi)$ and $\det V_{2u}(\xi)$ by using well-developed 1D algorithms. However, the computation may be inefficient when the degrees of $\det V_{1u}(\xi)$ and/or $\det V_{2u}(\xi)$ are high. The statement (iii) shows that by using Gröbner basis approach for polynomial ideal, one need not compute the above zeros. As mentioned earlier, however, it would be computationally demanding to calculate all the minors $a_i(z, w)$, $i = 1, \ldots, \beta$, and the Gröbner basis corresponding to these minors. In contrast with the above two, the statement (iv) reveals that the Gröbner basis approach for polynomial modules can be directly employed to solve the problem, without requiring computation of the zeros or minors.

Remark 2. Since the decompositions of Equations (6) and (7) are in fact performed by an approximate method [25, 26], the above proposed approaches do not appear to provide particular advantage from a viewpoint of practical computation. But these results give a significant insight to some structural properties of the nD stabilizability problem. In other words, these results show a possible way to reduce the stabilizability test of nD systems to 1D case. In fact, based on some results obtained by Gröbner basis approach in [14], the basic ideas adopted in Theorem 1 and Theorem 2 can be directly employed to establish procedures for the stabilizability test and the construction of closed-loop stable polynomial of nD ($n > 2$) systems, provided that the ideal $\mathcal{J}$ is of zero dimension.

3. Construction of 2D feedback compensator

The purpose of this section is to develop a solution procedure of Equation (1) by applying the Gröbner basis approach for modules over polynomial ring (see, e.g., [18]–[20]). In particular, we expect that the developed procedure should share the advantages, and at the same time, should not suffer from the disadvantages, which we discussed in the introduction for the two kinds of known procedures.

It is noted that, first of all, some results on estimation of the degree r in Equation (2) have been recently obtained (see, e.g., [16, [27–29]). More directly, the "Rabinowitsch trick" (see, e.g., [16, 17]) can be applied to solve Equation (2) without requiring any previous knowledge about r. Let t be a new indeterminate. Then it is evident that the polynomials $(1 - ts(z, w))$ and $a_i(z, w)$, $i = 1, \ldots, \beta$, share no common zeros. According to Hilbert's

Nullstellensatz, the ideal generated by these polynomials must be the unit ideal. This means that there exist $\bar{x}(z, w, t)$, $\tilde{x}_i(z, w, t) \in \mathbf{R}[z, w, t]$, $i = 1, \ldots, \beta$, such that

$$\sum_{i=1}^{\beta} a_i(z, w)\tilde{x}_i(z, w, t) + (1 - ts(z, w))\bar{x}(z, w, t) = 1 \tag{27}$$

Then, substituting $1/s(z, w)$ for t and clearing out the denominators yield a relation of the form of Equation (2). By this method, however, we cannot yet remove the necessity of computation of the minors $a_i(z, w)$, $i = 1, \ldots, \beta$, and the restriction of $V(z, w) = s^r(z, w)I$.

In the following, therefore, we first extend the "Rabinowitsch trick" to the case of modules over polynomial ring in the sense of Lemmas 1 and 2. Then, a general consequence from these results is summarized in Theorem 3. Based on these results, we will propose a solution algorithm for Equation (1).

Moreover, since Lin [9] has shown that a strictly causal solution $X(z, w)$, $Y(z, w)$, namely which satisfy $\det X(0, 0) \neq 0$ and $N(0, 0) = \mathbf{0}$, can always be obtained from any particular solution of Equation (1) whenever the plant $D(z, w)^{-1} N(z, w)$ is causal, in what follows, we will assume the causality of $D(z, w)^{-1} N(z, w)$ and only pursue a particular solution to Equation (1) without considering its causality.

LEMMA 1. *Define $D(v, w)$, $N(v, w)$, $F(v, w)$ and $\mathcal{V}(\mathcal{I})$ as in Theorem 1. Then there exist $\bar{x}_i(z, w, t_i)$ and $\tilde{x}_{i,j}(z, w, t_i) \in \mathbf{R}[z, w, t_i]$, $j = 1, \ldots, m + n$, such that*

$$\tilde{x}_{i,1}(z, w, t_i)\vec{f}_1(z, w) + \cdots + \tilde{x}_{i,m+n}(z, w, t_i)\vec{f}_{m+n}(z, w)$$

$$+ \bar{x}_i(z, w, t_i) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 - t_i s(z, w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} *, \quad i = 1, \ldots, n, \tag{28}$$

holds for some stable 2D *polynomial $s(z, w)$ if and only if*

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset, \tag{29}$$

where t_i are new indeterminates and $*$ denotes the ith position of the related n-tuples.

Proof. Sufficiency: In view of the result of Theorem 2 or [5], a stable polynomial $s(z, w)$ vanishing on $\mathcal{V}(\mathcal{I})$ can be constructed whenever the condition (29) is satisfied. Let t be a new interdeterminate. Then it is obvious that $1 - ts(z, w)$ and the maximal minors $a_j(z, w)$, $j = 1, \ldots, \beta$ of $[D(z, w)\ N(z, w)]$ are zero coprime and thus the equation

$$\sum_{j=1}^{\beta} a_j(z, w)\tilde{x}_j(z, w, t) = 1 - \bar{x}(z, w, t)(1 - ts(z, w)) \tag{30}$$

holds true. As we did in the proof of Theorem 1, by applying the methods of [2], [12], and [13], we can obtain[1] the solution for Equation (28) from the results of Equation (30). In particular, we have $t_i = t$ and $\bar{x}_i(z, w, t_i) = \bar{x}(z, w, t)$.

Necessity: Suppose that there is a stable $s(z, w)$ such that Equation (28) holds. Then, the solutions $\tilde{x}_{i,1}(z, w, t_i), \ldots, \tilde{x}_{i,m+n}(z, w, t_i), \bar{x}_i(z, w, t_i)$ can be obtained by using Gröbner basis approach for polynomial modules. Substituting $t_i = 1/s$ into Equation (28) and clearing out the denominators, we get

$$x_{i,1}(z, w)\vec{f}_1(z, w) + \cdots + x_{i,m+n}(z, w)\vec{f}_{m+n}(z, w) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ s^{r_i}(z, w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * $$

$$i = 1, \ldots, n, \tag{31}$$

where $x_{i,j}(z, w) \in \mathbf{R}[z, w], j = 1, \ldots, m + n$, and r_i are some positive integers. Equation (31) can be rewritten in the matrix form of Equation (1), i.e.,

$$FZ = [D\ N] \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} s^{r_1}(z, w) & 0 & \cdots & 0 \\ 0 & s^{r_2}(z, w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{r_n}(z, w) \end{bmatrix} \tag{32}$$

with $Z = [X\ Y]^T = [x_{i,j}(z, w)], i = 1, \ldots, n, j = 1, \ldots, m + n$. By Cauchy-Binet theorem, the relation

$$\sum_{j=1}^{\beta} a_j(z, w)z_j(z, w) = s^r(z, w), \qquad r = r_1 + \cdots + r_n, \tag{33}$$

is obtained, where $z_j(z, w), j = 1, \ldots, \beta$, are the maximal minors of Z. Therefore, $s(z, w)$ have to vanish on $\mathcal{V}(\mathcal{I})$ and can never be stable if $\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 \neq \emptyset$ □

More generally, the following result can be given.

LEMMA 2. *Define* $D(z, w), N(z, w), F(z, w)$ and $\mathcal{V}(\mathcal{I})$ *as in Theorem 1. Then, there exists* $\tilde{x}_{i,j}(z, w, \mathbf{t}_i) \in \mathbf{R}[z, w, \mathbf{t}_i]$, *with* $\mathbf{t}_i = (t_{i1}, \ldots, t_{il}), j = 1, \ldots, m + n$, *and* $\bar{x}_{i,k}(z, w, \mathbf{t}_i) \in \mathbf{R}[z, w, \mathbf{t}_i], k = 1, \ldots, l$, *such that*

$$\tilde{x}_{i,1}(z, w, \mathbf{t}_i)\vec{f}_1(z, w) + \cdots + \tilde{x}_{i,m+n}(z, w, \mathbf{t}_i)\vec{f}_{m+n}(z, w)$$

$$+ \bar{x}_{i,1}(z, w, \mathbf{t}_i)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 - t_{i1}\, s_1(z, w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * + \cdots + \bar{x}_{i,l}(z, w, \mathbf{t}_i)\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 - t_{il}\, s_l(z, w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} *,$$

$$i = 1, \ldots, n, \tag{34}$$

for some stable 2D *polynomials* $s_1(z, w), \ldots, s_l(z, w)$ *if and only if*

$$\mathcal{V}(\mathcal{I}) \cap \bar{U}^2 = \emptyset, \tag{35}$$

where t_{ik}, $k = 1, \ldots, l$, *are new indeterminates. (Here, again,* $*$ *denotes the ith position of the related n-tuples.)*

Proof. Without loss of generality, suppose that the stable polynomial $s(z, w)$ obtained by the method of Theorem 2 or [5] under the condition (35) is represented in the decomposed form $s(z, w) = s_1(z, w) \cdots s_l(z, w)$. Since $s(z, w)$ vanishes on $\mathcal{V}(\mathcal{I})$, for any $(z_0, w_0) \in \mathcal{V}(\mathcal{I})$ we have that

$$s_1(z_0, w_0) = 0 \;\text{ and/or }\; s_2(z_0, w_0) = 0 \;\cdots\; \text{ and/or }\; s_l(z_0, w_0) = 0. \tag{36}$$

It is now obvious that for z, w and the newly introduced indeterminates $t_1, \ldots, t_l$, the polynomials $a_1(z, w), \ldots, a_\beta(z, w)$ and $1 - t_1\, s_1(z, w), \ldots, 1 - t_l\, s_l(z, w)$ have no common zeros, namely, they are zero coprime on $\mathbf{R}[z, w, \mathbf{t}]$ where $\mathbf{t} = (t_1, \ldots, t_l)$. By Hilbert's Nullstellensatz, then, there exist $\tilde{y}_i(z, w, \mathbf{t}), \bar{y}_j(z, w, \mathbf{t}) \in \mathbf{R}[z, w, \mathbf{t}]$, for $i = 1, \ldots, \beta$ and $j = 1, \ldots, l$, such that

$$\sum_{i=1}^{\beta} \tilde{y}_i(z, w, \mathbf{t}) a_i(z, w) + \sum_{j=1}^{l} \bar{y}_j(z, w, \mathbf{t})(1 - t_j\, s_j(z, w)) = 1. \tag{37}$$

In view of this fact, we can show the proof in the same way as for Lemma 1. □

Based on the above lemmas, the following theorem can be readily established.

THEOREM 3. For given left factor coprime MFD $D^{-1}(z, w)N(z, w)$ where $D(z, w) \in \mathbf{R}[z, w]^{n\times n}$, $N(z, w) \in \mathbf{R}[z, w]^{n\times m}$, there exist $X(z, w) \in \mathbf{R}[z, w]^{n\times n}$ and $Y(z, w) \in \mathbf{R}[z, w]^{m\times n}$ such that

$$D(z, w)X(z, w) + N(z, w)Y(z, w) = \begin{bmatrix} s^{r_1}(z, w) & 0 & \cdots & 0 \\ 0 & s^{r_2}(z, w) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{r_n}(z, w) \end{bmatrix}, \quad (38)$$

or more generally,

$$D(z, w)X(z, w) + N(z, w)Y(z, w) =$$

$$\begin{bmatrix} s_1^{r_{11}} s_2^{r_{12}} \cdots s_l^{r_{1l}} & 0 & \cdots & 0 \\ 0 & s_1^{r_{21}} s_2^{r_{22}} \cdots s_l^{r_{2l}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_1^{r_{n1}} s_2^{r_{n2}} \cdots s_l^{r_{nl}} \end{bmatrix}, \quad (39)$$

if and only if

$$\mathcal{V}(\mathcal{G}) \cap \bar{U}^2 = \emptyset \quad (40)$$

where

$$s(z, w) = s_1(z, w)s_2(z, w) \cdots s_l(z, w) \quad (41)$$

is a stable polynomial vanishing on $\mathcal{V}(\mathcal{G})$, and r_i, r_{ik} are some positive integers for $k = 1, \ldots, l$, $i = 1, \ldots, n$.

Proof. Sufficiency: According to Lemmas 1 and 2, if the condition (40) is satisfied, Equations (28) and (34) are solvable. Further, Gröbner basis approach for polynomial modules can be used to find the solutions to Equations (28) and (34). For $i = 1, \ldots, n$, then, substituting $t_i = 1/s$ into Equation (28), or $t_{ij} = 1/s_j(z, w)$, $j = 1, \ldots, l$, into Equation (34), clearing out all the denominators and writing the equation in matrix form, we obtain the results for Equations (38) and (39).

Necessity: Simply follows from Cauchy-Binet theorem (see, e.g., [1]). □

The results of Lemmas 1, 2, and Theorem 3 strongly suggest that Gröbner basis approach for polynomial modules can be employed to solve Equation (1) effectively. A solution algorithm can be stated as follows, provided that a stable polynomial $s(z, w)$ vanishing on $\mathcal{V}(\mathcal{G})$ has been obtained by the methods proposed in Section 2 or [5].

ALGORITHM 1

Input: $F(z, w) = [D(z, w) \ N(z, w)] \in \mathbf{R}[z, w]^{n\times(n+m)}$, a stable polynomial $s(z, w) \in \mathbf{R}[z, w]$ vanishing over $\mathcal{V}(\mathcal{G})$.

Output: $X(z, w)$, $Y(z, w)$ and $V(z, w)$ for Equation (1).

step 1. Calculate a Gröbner basis $G' = \{\vec{g}'_1, \ldots, \vec{g}'_{q_0}\}$ for the module generated by $\vec{f}_1, \ldots, \vec{f}_{m+n}$ that correspond to the columns of $F(z, w)$. For this purpose, the algorithms proposed in [18, 19] can be applied.

For $i = 1, \ldots, n$ do step 2 – step 5. (Here we only consider the general situation when $s(z, w)$ is given as $s(z, w) = s_1(z, w) \cdots s_l(z, w)$.)

step 2. Add the n-tuples

$$\vec{h}_{ik} = [0 \cdots 0 \;\; \underset{\substack{\uparrow \\ \text{the } i\text{th position}}}{1 - t_{ik}\, s_k(z, w)} \;\; 0 \cdots 0]^T, \qquad k = 1, \ldots, l, \tag{42}$$

to G', then calculate a Gröbner basis $G_i = \{\vec{g}_{i1}, \ldots, \vec{g}_{iq_i}\}$ for the module generated by $\{\vec{g}'_1, \ldots, \vec{g}'_{q_0}, \vec{h}_{i1}, \ldots, \vec{h}_{il}\}$.

step 3. By tracing the construction procedure performed in steps 1 and 2, construct $\tilde{u}_{i,j,k}(z, w, \mathbf{t}_i)$ and $\bar{u}_{i,j,r}(z, w, \mathbf{t}_i) \in \mathbf{R}[z, w, \mathbf{t}_i]$, with $\mathbf{t}_i = (t_{i1}, \ldots, t_{il})$, $k = 1, \ldots, m + n$, $r = 1, \ldots, l$, such that

$$\vec{g}_{ij} = \sum_{k=1}^{m+n} \tilde{u}_{i,j,k} \vec{f}_k + \sum_{r=1}^{l} \bar{u}_{i,j,r}\, \vec{h}_{il}, \qquad j = 1, \ldots, q_i. \tag{43}$$

Notice: Denote by $\vec{e}_i$ the n-tuple having 1 as the element at the ith position and zeros at the other positions. Then, by the properties of Gröbner basis for modules, Equation (34) is solvable if and only if $\vec{e}_i$ can be reduced to zero with respect to G_i. According to this fact and Theorem 1, therefore, when Equation (1) is solvable, there must be a $\vec{g}_{ij}$ for certain j such that $\vec{g}_{ij} = \gamma \vec{e}_i$ with $\gamma \in \mathbf{R}$. Without loss of generality, we assume that $\gamma = 1$. If this is not true, then Equation (1) has no solution. This can also serve as an alternative test for the stabilizability. Further, the explicit calculation of $\bar{u}_{i,j,r}(z, w, \mathbf{t}_i)$ is in fact not necessary for obtaining the solution to Equation (1).

step 4. Pick out from $G_i = \{\vec{g}_{i1}, \ldots, \vec{g}_{iq_i}\}$ the element, say $\vec{g}_{ib}$, $b \in \{1, \ldots, q_i\}$, that satisfies

$$\vec{e}_i = \vec{g}_{ib} = \sum_{k=1}^{m+n} \tilde{u}_{i,b,k} \vec{f}_k + \sum_{r=1}^{l} \bar{u}_{i,b,r}\, \vec{h}_{il} \tag{44}$$

then by comparing Equations (44) and (34), we have

$$\tilde{x}_{i,j}(z, w, \mathbf{t}_i) = \tilde{u}_{i,b,j}(z, w, \mathbf{t}_i), \qquad j = 1, \ldots, m + n, \tag{45}$$

$$\bar{x}_{i,k}(z, w, \mathbf{t}_i) = \bar{u}_{i,b,k}(z, w, \mathbf{t}_i), \qquad k = 1, \ldots, l. \tag{46}$$

step 5. Substituting $t_{ij} = 1/s_j(z, w)$, $j = 1, \ldots, l$, into Equation (34) and clearing out the denominators, we obtain the solution $x_{ij}(z, w)$, $j = 1, \ldots, m + n$, such that

$$x_{i,1}(z,w)\vec{f}_1(z,w) + \cdots + x_{i,m+n}(z,w)\vec{f}_{m+n}(z,w) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ s_1^{r_{i1}}(z,w) \cdots s_l^{r_{il}}(z,w) \\ 0 \\ \vdots \\ 0 \end{bmatrix} * \tag{47}$$

where r_{ik}, $k = 1, \ldots, l$, are some positive integers.

step 6. Finally, from the results $x_{i,j}(z, w)$, $i = 1, \ldots, n, j = 1, \ldots, m + n$, obtained in Equation (47), we have a solution $X(z, w)$, $Y(z, w)$ to Equation (1) as

$$[X^T(z,w) \quad Y^T(z,w)] = [x_{i,j}(z,w)], \qquad i = 1, \ldots, n, \quad j = 1, \ldots, m+n, \tag{48}$$

and a stable $V(z, w)$ in the form of the right-hand side of Equation (39).

Remark 3. According to the above results, we see that Equation (1) can be equivalently transformed to a Bezout equation, and consequently its solution can be obtained by nothing else than finding the Gröbner bases of certain polynomial modules. In this way, we can obviously avoid estimation of any degree (e.g., r) and computation of any minors or zeros which are required in the two existing methods discussed previously.

Remark 4. By comparing Equations (3), (38), and (39), it is easy to see that Equations (38) and (39) give less restrictive forms for the stable matrix $V(z, w)$ of Equation (1), for the constraint $V(z, w) = s(z, w)^r I$ is no longer necessary. In [13], it has been shown that when Equation (3) holds, a general $V(z, w)$ with entries in $\mathcal{G}$ can also be constructed by using the result of Equation (2). It is worth noting, however, that Theorem 3 does not demand the entries of $V(z, w)$ in Equations (38) or (39) to be necessarily in $\mathcal{G}$. This feature is of significance because there indeed exists $V(z, w)$ which has entries not belonging to $\mathcal{G}$ but such that Equation (1) admits a solution [9]. In fact, it is possible to obtain such a solution by using Algorithm 1 but impossible by the algorithm of [13]. Further, in view of the fact that exponents of some factors may be zero, the form for $V(z, w)$ given in Equation (39) is essentially identical to the one obtained in [9]. As mentioned earlier, this form may give solution $X(z, w)$, $Y(z, w)$ to Equation (1) with relatively lower degree in z and w. Since an $s(z, w)$ constructed by the method of Theorem 2 is in the separated form, i.e., a product of two 1D polynomials in z and w respectively, further decompositions are always possible by 1D approaches if it is required.

4. Example

Consider the 2D system

$$P(z, w) = \begin{bmatrix} -\dfrac{w - 3z}{2z - 5} & \dfrac{2z - 5}{3(2z - 1)} \\ \\ \dfrac{2z - 1}{8w + 6z - 15} & \dfrac{w^2}{2z - 1} \end{bmatrix}. \tag{49}$$

We can represent $P(z, w)$ by the left factor coprime MFD

$$P(z, w) = D^{-1}(z, w)N(z, w) \tag{50}$$

where

$$D(z, w) = \begin{bmatrix} \dfrac{3(2z - 1)(2z - 5)}{16} & 0 \\ \\ -\dfrac{3w^2(2w - 3)(2z - 5)}{2} & 2(8w + 6z - 15) \end{bmatrix},$$

$$N(z, w) =$$

$$\begin{bmatrix} -\dfrac{3(2z - 1)(w - 3z)}{16} & \dfrac{(2z - 5)^2}{16} \\ \\ \dfrac{6w^4 - 18w^3z - 9w^3 + 27w^2z + 8z - 4}{2} & -\dfrac{w^2((2w - 3)(2z - 5) - 8w)}{2} \end{bmatrix}.$$

First, by using the results given in Section 2, we investigate the stabilizability of $P(z, w)$ and construct a stable polynomial $s(z, w)$ vanishing on $\mathcal{V}(\mathcal{I})$. Applying the method of [21] to the above $D(z, w)$ and $N(z, w)$, the solutions to Equations (4) and (5) can be obtained with the following $V_1(z)$ and $V_2(w)$. (For brevity, the results for $X_1(z, w)$, $Y_1(z, w)$, $X_2(z, w)$, and $Y_2(z, w)$ are omitted here.)

$$V_1(z) = \begin{bmatrix} 3(2z - 1)^2(2z - 5) & 0 \\ \\ 0 & 6(2z - 1)^2(2z - 5) \end{bmatrix},$$

$$V_2(w) = \begin{bmatrix} 96(2w - 3)(2w - 15)w & 0 \\ \\ 0 & 64(2w - 3)(2w - 15)w \end{bmatrix}.$$

Then, $V_1(z)$ and $V_2(w)$ can be decomposed as in Equations (6) with

$$V_{1u}(z) = \begin{bmatrix} 3(2z-1)^2 & 0 \\ 0 & 6(2z-1)^2 \end{bmatrix},$$

$$V_{1s}(z) = \begin{bmatrix} (2z-5) & 0 \\ 0 & (2z-5) \end{bmatrix},$$

$$V_{2u}(w) = \begin{bmatrix} 96w & 0 \\ 0 & 64w \end{bmatrix},$$

$$V_{2s}(w) = \begin{bmatrix} (2w-3)(2w-15) & 0 \\ 0 & (2w-3)(2w-15) \end{bmatrix}.$$

It is easy to see

$$\Gamma\{\det V_{1u}(z), \det V_{2u}(w)\} = \{(1/2, 0)\} \tag{51}$$

By checking

$$\operatorname{rank}\,[D(1/2, 0) \quad N(1/2, 0)] = 2, \tag{52}$$

we conclude that $(1/2, 0)$ is not a common zero of all the 2×2 minors of $[D(z, w)$ $N(z, w)]$, and according to Theorem 1, $P(z, w)$ is stabilizable.

On the other hand, the stable polynomial $s(z, w)$ can be constructed by using all the mutually coprime factors in $\det V_{1s}(z)$ and $\det V_{2s}(w)$ as

$$s(z, w) = (2z-5)(2w-3)(2w-15). \tag{53}$$

Now, following Algorithm 1, we can solve Equation (1) by using the Gröbner basis approach for polynomial modules. Let

$$F(z, w) = [D(z, w) \quad N(z, w)], \tag{54}$$

so we have

$$\vec{f_1} = \begin{bmatrix} \dfrac{3(2z-1)(2z-5)}{16} & -\dfrac{3w^2(2w-3)(2z-5)}{2} \end{bmatrix}^T,$$

$$\vec{f_2} = [0 \quad 2(8w+6z-15)]^T,$$

$$\vec{f}_3 = \left[-\frac{3(2z - 1)(w - 3z)}{16} \quad \frac{6w^4 - 18w^3z - 9w^3 + 27w^2z + 8z - 4}{2} \right]^T,$$

$$\vec{f}_4 = \left[\frac{(2z - 5)^2}{16} \quad -\frac{w^2((2w - 3)(2z - 5) - 8w)}{2} \right]^T.$$

In addition, let

$$\vec{h}_1 = \begin{bmatrix} 1 - t_1 s(z, w) \\ 0 \end{bmatrix}, \quad \vec{h}_2 = \begin{bmatrix} 0 \\ 1 - t_2 s(z, w) \end{bmatrix}.$$

By the method of, for example, [25], we can find the Gröbner bases G_1 and G_2 for the modules generated by $\{\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4, \vec{h}_1\}$ and $\{\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4, \vec{h}_2\}$ respectively as

$$G_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \tag{55}$$

$$G_2 = \left\{ \begin{bmatrix} w - 15/2 \\ 0 \end{bmatrix}, \begin{bmatrix} z - 5/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \tag{56}$$

and obtain the results

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \sum_{k=1}^{4} \tilde{x}_{1,k}(z, w, t_1) \vec{f}_k(z, w) + \tilde{x}_1(z, w, t_1) \vec{h}_1(z, w, t_1) \tag{57}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \sum_{k=1}^{4} \tilde{x}_{2,k}(z, w, t_2) \vec{f}_k(z, w) + \tilde{x}_2(z, w, t_2) \vec{h}_2(z, w, t_2), \tag{58}$$

where

$$\tilde{x}_{1,1} = (1 - 12(2w - 15)(2z - 5)t_1)(27w^3 - 81w^2z + 32w + 24z - 108)$$
$$(2w - 3)t_1/12,$$

$$\tilde{x}_{1,2} = (1 - 12(2w - 15)(2z - 5)t_1)(8w - 6z + 15)(2w - 3)w^2t_1/4,$$

$$\tilde{x}_{1,3} = 9(1 - 12(2w - 15)(2z - 5)t_1)(2w - 3)(2z - 5)w^2t_1/4,$$

$$\tilde{x}_{1,4} = -(1 - 12(2w - 15)(2z - 5)t_1)(8w + 6z - 15)(2w - 3)t_1,$$

$$\bar{x}_1 = 1 - 12(2w - 3)(2z - 5)t_1,$$

$$\tilde{x}_{2,1} = 9(1 - 12(2w - 15)(2z - 5)t_2)(w - 3z)(2z - 1)t_2/32,$$

$$\tilde{x}_{2,2} = (1 - 12(2w - 15)(2z - 5)t_2)(8w - 6z - 9)(2z - 5)t_2/32,$$

$$\tilde{x}_{2,3} = 9(1 - 12(2w - 15)(2z - 5)t_2)(2z - 1)(2z - 5)t_2/32,$$

$$\tilde{x}_{2,4} = 0,$$

$$\bar{x}_2 = 1 - 12(2w - 3)(2z - 5)t_2.$$

By substituting $t_i = 1/s$, $i = 1, 2$ into Equations (57) and (58) and clearing out the denominators, we can get the solution to Equation (1), i.e.,

$$D(z, w)X(z, w) + N(z, w)Y(z, w) = V(z, w)$$

with

$$X(z, w) = \begin{bmatrix} 27w^3 - 81w^2z + 32w + 24z - 108 & 9(w - 3z)(2z - 1) \\ 3(8w - 6z + 15)w^2 & (8w - 6z - 9)(2z - 5) \end{bmatrix},$$

$$Y(z, w) = \begin{bmatrix} 27w^2(2z - 5) & 9(2z - 1)(2z - 15) \\ -12(8w + 6z - 15) & 0 \end{bmatrix},$$

$$V(z, w) = \begin{bmatrix} 12(2w - 3)(2z - 5) & 0 \\ 0 & 32(2w - 3)^2(2z - 5) \end{bmatrix}.$$

5. Conclusions

Some alternative methods have been proposed for testing the 2D output feedback stabilizability and constructing a 2D stable closed-loop polynomial. By these methods, the problems can be reduced to the 1D case and solved by using 1D algorithms or Gröbner basis approaches. The idea adopted here can be applied to establish similar procedures for nD ($n > 2$) cases under certain conditions.

Moreover, some generalizations for the "Rabinowitsch trick" have been obtained in the senses of Lemmas 1 and 2. These results lead to a new solution algorithm for Equation (1). According to the proposed procedure, it is possible to solve Equation (1) by transforming it to an equivalent Bezout equation so that Gröbner basis approach for polynomial modules can be directly applied. In consequence, some questions of the existing methods can be avoided and yet a general form for $V(z, w)$ as shown in [9] can be achieved.

Recently, Shankar and Sule [30], have developed a general theory of feedback stabilization for systems described by transfer functions over a general integral domain, which extends the well-known coprime factorization approach. By using this theory, then, they clarified necessary and sufficient condition for the solvability of nD stabilization problem in terms of affine varieties. It seems, however, that the task remains to find some constructive and effective algorithms to test the conditions and to construct the stabilizing compensators. Since the approaches proposed in this paper have good potentiality for extension to nD ($n > 2$) cases, we believe that it would be easy to generalize these approaches without essential difficulties.

Note

1. One of the anonymous reviewers has also indicated a different simple method in terms of the "Rabinowitsch" identification, by which the result of Equation (28) can easily be derived from Equation (3).

Acknowledgment

The authors wish to thank Professor S. Sakata for helpful comments, and B. Wall and T. Matsunaga for their help in computing the example. The authors would also like to acknowledge the anonymous reviewers for valuable comments and suggestions.

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