Multidimensional Systems and Signal Processing, 6, 137–158 (1995)

Design of Zero-Phase Recursive 2-D Variable Filters with Quadrantal Symmetries

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Received July 19, 1993 ; Revised November 17, 1993

Abstract. The digital filters with adjustable frequency-domain characteristics are called variable filters. Variable filters are useful in the applications where the filter characteristics are needed to be changeable during the course of signal processing. In such cases, if the existing traditional constant filter design techniques are applied to the design of new filters to satisfy the new desired characteristics when necessary, it will take a huge amount of design time. So it is desirable to have an efficient method which can fast obtain the new desired frequency-domain characteristics. Generally speaking, the frequency-domain characteristics of variable filters are determined by a set of spectral parameters such as cutoff frequency, transition bandwidth and passband width. Therefore, the characteristics of variable filters are the multi-dimensional (M-D) functions of such spectral parameters. This paper proposes an efficient technique which simplifies the difficult problem of designing a 2-D variable filter with quadrantally symmetric magnitude characteristics as the simple one that only needs the normal one-dimensional (1-D) constant digital filter designs and 1-D polynomial approximations. In applying such 2-D variable filters, only varying the part of 1-D polynomials can easily obtain new desired frequency-domain characteristics.

Key Words: variable digital filter, constant digital filter, 1-D polynomials, outer product expansion, quadrantal symmetries

1. Introduction

Variable digital filters have various potential applications in acoustic signal processing, image processing and communication systems [1], [2], [3], [4]. In such applications, variable filters are required to change their coefficients frequently to satisfy the new desired variable frequency-domain characteristics. If the conventional constant filter design techniques such as nonlinear optimization ones are utilized to update the variable filter coefficients whenever such needs arise, it will take long design time. Thus, the technique that can easily obtain new desired frequency-domain characteristics are necessary.

Generally speaking, the frequency-domain characteristics of variable filters are determined by a set of spectral parameters such as cutoff frequency, transition bandwidth and passband width. Different spectral parameter values specify different frequency-domain characteristics, and thus need different filter coefficients. Evidently, the coefficients of variable filters are the multi-dimensional (M-D) functions of the spectral parameters. From this viewpoint, Fahmy et al. proposed some techniques for designing recursive one-dimensional

(1-D) and two-dimensional (2-D) variable filters. The main objective of the techniques is to find the variable filter coefficients as the M-D polynomials of the spectral parameters. At first, the spectral parameters within specified ranges are uniformly sampled. Then the normal 1-D or 2-D constant filters corresponding to the sampled spectral parameter values are designed. By this step, a set of coefficients of the normal constant filters are obtained. Next, the coefficients of a variable filter are assumed to be the form of M-D polynomials of the spectral parameters, and then they are determined one by one by using an M-D curve fitting technique to best fit the resulting constant filter coefficients. The techniques are flexible for designing variable filters with arbitrary desired variable characteristics, and the coefficients of the resulting variable filters can be easily obtained only by computing the M-D polynomial values. However,

(1) Since many constant filters have to be designed first by using a nonlinear optimization method, and then a lot of M-D polynomials representing variable filter coefficients have to be determined, the technique is not computationally efficient.

(2) Since the denominator coefficients of the designed variable filters are also M-D polynomials of a set of spectral parameters, and are varied in the signal processing applications, the stability of the variable filters cannot be guaranteed.

This paper proposes a new technique for designing zero-phase recursive 2-D variable filters with quadrantally symmetric magnitude characteristics. The technique is based on the decomposition of the given 2-D variable magnitude specifications. At first, we uniformly sample the given 2-D variable magnitude specification. Using the samples, we construct an M-D array, which is the extended version of the normal matrix (2-D case). Then, an outer product expansion method is proposed for decomposing the M-D array into the sum of the outer products of vectors. The vectors are then regarded as the magnitude specifications of the 1-D normal constant filters and the specifications of 1-D polynomials. Finally, by performing the normal 1-D constant filter designs and 1-D polynomial approximations, we can easily obtain a 2-D variable filter. Since the normal 1-D constant filters are easy to design by using the existing conventional filter design techniques, and the optimal 1-D polynomials can be determined by solving linear equations, the proposed design technique simplifies the original 2-D variable filter design problem significantly. In applying the designed variable filters, since the part of the 1-D constant filters is fixed, and only the part of 1-D polynomials is varied, the stability of the resulting 2-D variable filters is always guaranteed so long as the 1-D constant filters are designed to be stable. Also, only adjusting the 1-D polynomials can easily obtain the new desired frequency-domain characteristics. Three design examples are given to illustrate the design technique.

2. Outer product expansion

Assume that $H_d[\omega_1, \omega_2, \Psi_1, \Psi_2, \cdots, \Psi_K]$ is the given quadrantally symmetric 2-D variable magnitude specification, where ω_1 and ω_2 are normalized frequencies. Since the specification $H_d[\omega_1, \omega_2, \Psi_1, \Psi_2, \cdots, \Psi_K]$ is quadrantally symmetric, we only need to consider it in the first quadrant. That is,

$$\omega_i \in [0, \pi], \qquad i = 1, 2. \tag{1}$$

In addition, $\{\Psi_1, \Psi_2, \cdots, \Psi_K\}$ are the parameters that define the desired variable frequency-domain characteristics, we call them the spectral parameters. They are specified as

$$\Psi_i \in [\Psi_{imin}, \Psi_{imax}], \qquad i = 1, 2, \cdots, K \tag{2}$$

where Ψ_{imin} and Ψ_{imax} are respectively the lower bound and upper bound of the spectral parameter Ψ_i.

Uniformly sampling the variable specification $H_d[\omega_1, \omega_2, \Psi_1, \Psi_2, \cdots, \Psi_K]$, we can obtain the samples

$$a(m, n, l_1, l_2, \cdots, l_K) = H_d[\omega_{1m}, \omega_{2n}, \Psi_1(l_1), \Psi_2(l_2), \cdots, \Psi_K(l_K)] \tag{3}$$

where

$$\begin{aligned} \omega_{1m} &= \pi(m-1)/(M-1), \qquad 1 \le m \le M \\ \omega_{2n} &= \pi(n-1)/(N-1), \qquad 1 \le n \le N \\ \Psi_i(l_i) &= \Psi_{imin} + (\Psi_{imax} - \Psi_{imin})(l_i - 1)/(L_i - 1), \qquad 1 \le l_i \le L_i. \end{aligned} \tag{4}$$

Using the samples $a(m, n, l_1, l_2, \cdots, l_K)$, we can construct a $(K+2)$-D array $\boldsymbol{A} \in \boldsymbol{R}^{M \times N \times L_1 \times L_2 \times \cdots \times L_K}$, where $a(m, n, l_1, l_2, \cdots, l_K)$ are its elements, i.e.,

$$\boldsymbol{A} = [a(m, n, l_1, l_2, \cdots, l_K)]. \tag{5}$$

2.1. *Decomposition-based design*

In this section, we propose a method for decomposing the $(K+2)$-D array $\boldsymbol{A}$ into the form

$$\boldsymbol{A} \approx \sum_{i=1}^{r} \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{P}_{i2} \otimes \cdots \otimes \boldsymbol{P}_{iK} \tag{6}$$

where the notation $\otimes$ denotes the outer product of vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i, \boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$, and $\boldsymbol{F}_i \in \boldsymbol{R}^{M \times 1}$, $\boldsymbol{G}_i \in \boldsymbol{R}^{N \times 1}$, $\boldsymbol{P}_{i1} \in \boldsymbol{R}^{L_1 \times 1}$, $\boldsymbol{P}_{i2} \in \boldsymbol{R}^{L_2 \times 1}$, $\cdots$, $\boldsymbol{P}_{iK} \in \boldsymbol{R}^{L_K \times 1}$ [5], [6]. The outer product expansion (6) can also be represented by using an element expression as

$$a(m, n, l_1, l_2, \cdots, l_K) \approx \sum_{i=1}^{r} F_i(m) G_i(n) P_{i1}(l_1) P_{i2}(l_2) \cdots P_{iK}(l_K) \tag{7}$$

where $\{F_i(m), G_i(n), P_{i1}(l_1), P_{i2}(l_2), \cdots, P_{iK}(l_K)\}$ are the elements of vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i, \boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$.

In 2-D case, using the existing conventional matrix decomposition methods such as the singular value decomposition (SVD) and the lower-upper (LU) triangular methods can obtain the decomposition (6), and it has been successfully applied to the designs and realizations of 2-D constant digital filters [7], [8], [9], [10]. From the viewpoint of 2-D variable filter designs, we perform the outer product expansion (6) subject to the following two constraints.

(a) Overall squared decomposition error

$$\begin{aligned} E_r &= \|\boldsymbol{A} - \sum_{i=1}^{r} \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{P}_{i2} \otimes \cdots \otimes \boldsymbol{P}_{iK}\|^2 \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{l_1=1}^{L_1}\sum_{l_2=1}^{L_2}\cdots\sum_{l_K=1}^{L_K}[a(m,n,l_1,l_2,\cdots,l_K) \\ &\quad - \sum_{i=1}^{r} F_i(m)G_i(n)P_{i1}(l_1)P_{i2}(l_2)\cdots P_{iK}(l_K)]^2 \end{aligned}$$

is minimum.

(b) Vectors $\boldsymbol{F}_i$ and $\boldsymbol{G}_i$ are non-negative because they will be regarded as the magnitude specifications of 1-D constant digital filters later.

Once the decomposition (6) is obtained, the next task is to approximate the vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i, \boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$. From (7) we know that the non-negative vectors $\boldsymbol{F}_i$ and $\boldsymbol{G}_i$ are the functions of frequencies ω_1 and ω_2, respectively, and the vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$ are the functions of the spectral parameters $\{\Psi_1, \Psi_2, \cdots, \Psi_K\}$. To obtain a 2-D variable filter, we use 1-D constant filter $f_i(z_1)$ with an arbitrary phase response to approximate the vector $\boldsymbol{F}_i^{1/2}$, and thus the magnitude specification vector $\boldsymbol{F}_i$ can be approximated by the zero-phase 1-D constant filter $f_i(z_1)f_i(z_1^{-1})$. Similarly, we use 1-D constant filter $g_i(z_2)$ with an arbitrary phase response to approximate the vector $\boldsymbol{G}_i^{1/2}$, and thus the magnitude specification vector $\boldsymbol{G}_i$ can be approximated by the zero-phase 1-D constant filter $g_i(z_2)g_i(z_2^{-1})$. Moreover, 1-D functions $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$ are used to approximate the vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$. By cascading zero-phase 1-D constant filters $\{f_i(z_1)f_i(z_1^{-1}), g_i(z_2)g_i(z_2^{-1})\}$ with 1-D functions $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$ and then putting them together in parallel, we can obtain a zero-phase 2-D variable filter

$$\begin{aligned} &H(z_1, z_2, \Psi_1, \Psi_2, \cdots, \Psi_K) \\ &= \sum_{i=1}^{r} f_i(z_1)f_i(z_1^{-1})g_i(z_2)g_i(z_2^{-1})p_{i1}(\Psi_1)p_{i2}(\Psi_2)\cdots p_{iK}(\Psi_K) \end{aligned} \quad (8)$$

which is shown in Figure 1. Since 1-D constant filters $f_i(z_1)$ and $g_i(z_2)$ are relatively easy to design, and the 1-D functions $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$ are easy to determine, this is shown in the next section, the original 2-D variable filter design problem can be easily solved. In applying such a 2-D variable filter, by just varying the 1-D functions $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$ we can obtain the new desired variable frequency-domain characteristics, but the zero-phase 1-D constant filters $\{f_i(z_1)f_i(z_1^{-1}), g_i(z_2)g_i(z_2^{-1})\}$ are always fixed. So $f_i(z_1)$, $g_i(z_2)$ may be designed to be recursive or nonrecursive. In any case, the resulting 2-D variable filter $H(z_1, z_2, \Psi_1, \Psi_2, \cdots, \Psi_K)$ is always stable so long as the designed 1-D constant filters $f_i(z_1)$ and $g_i(z_2)$ are stable. The above design approach is diagrammatized in Figure 2.

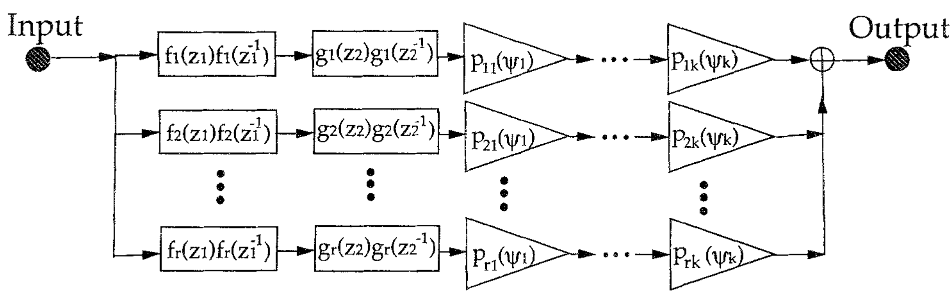

Figure 1. Zero-phase 2-D variable digital filter structure.

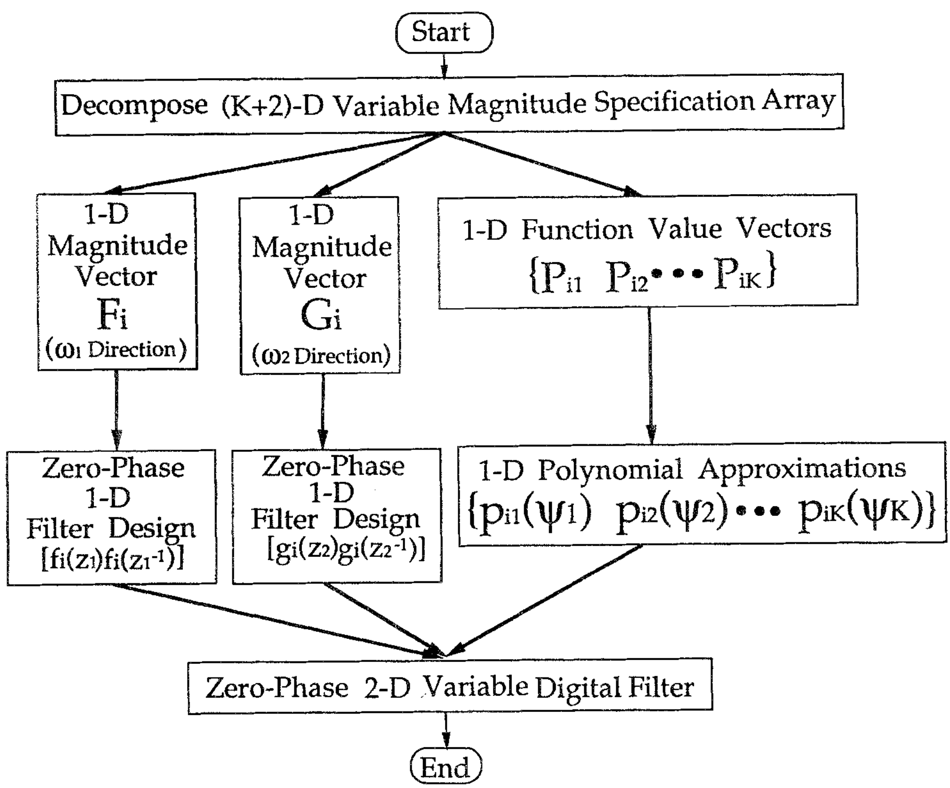

Figure 2. Efficient approach to 2-D variable filter design.

2.2. *Novel decomposition algorithm*

From above we can understand why the two constraints (a) and (b) are imposed on the outer product expansion (6). The constraint (a) is for reducing the number of parallel channels in Figure 1. This will save hardware cost in implementation. That is to say, for a given decomposition error E_r, the number r of the parallel channels is as small as possible. Conversely, for a given number r, the decomposition error E_r is as small as possible. On the other hand, the constraint (b) is necessary because the magnitude responses of digital filters are non-negative, and the vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i\}$ will be regarded as the magnitude specifications of the zero-phase 1-D constant filters $\{f_i(z_1)f_i(z_1^{-1}), g_i(z_2)g_i(z_2^{-1})\}$ respectively. Below, we propose a method for obtaining the outer product expansion (6). The method finds the vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i, \boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$, $i = 1, 2, \cdots, r$, successively following the next 6 steps, where i is the counter of the current decomposition stage, and r is a preset number of parallel channels. At first, we set $i = 1$.

Step 1. Compute the error array $\boldsymbol{E}$ at the i-th decomposition stage as

$$\boldsymbol{E} = \boldsymbol{A} - \sum_{j=1}^{i-1} \boldsymbol{F}_j \otimes \boldsymbol{G}_j \otimes \boldsymbol{P}_{j1} \otimes \boldsymbol{P}_{j2} \otimes \cdots \otimes \boldsymbol{P}_{jK}. \tag{9}$$

But if $i = 1$, we let $\boldsymbol{E} = \boldsymbol{A}$, where $\boldsymbol{A}$ is the constructed $(K+2)$-D magnitude specification array in (5). The elements of $\boldsymbol{E}$ are $e(m, n, l_1, l_2, \cdots, l_K)$, i.e.,

$$\boldsymbol{E} = [e(m, n, l_1, l_2, \cdots, l_K)]. \tag{10}$$

Convert the $(K + 2)$-D error array $\boldsymbol{E}$ to a 3-D array $\boldsymbol{B} = [b(m, n, q)]$ such that

$$b(m, n, q) = e(m, n, l_1, l_2, \cdots, l_K) \tag{11}$$

where

$$q = (l_1 - 1)L_2L_3 \cdots L_K + (l_2 - 1)L_3L_4 \cdots L_K + \cdots + (l_{K-1} - 1)L_K + l_K. \tag{12}$$

Then, convert the 3-D array $\boldsymbol{B}$ to a 2-D array (matrix) $\boldsymbol{C} = [c(p, q)]$ such that

$$c(p, q) = b(m, n, q) \tag{13}$$

where

$$p = (m - 1)N + n. \tag{14}$$

Next, separate the matrix $\boldsymbol{C}$ into the sum of the non-negative matrix $\boldsymbol{C}^+ = [c^+(p, q)]$ and the non-positive matrix $\boldsymbol{C}^- = [c^-(p, q)]$ as

$$\boldsymbol{C} = \boldsymbol{C}^+ + \boldsymbol{C}^- \tag{15}$$

where

$$\begin{aligned} c^+(p, q) &= max[c(p, q), 0] \\ c^-(p, q) &= min[c(p, q), 0]. \end{aligned} \tag{16}$$

Step 2. Perform the SVD on matrices $\boldsymbol{C}^{+}$ and $\boldsymbol{C}^{-}$. From Perron's non-negative matrix theory, we know that the matrices $\boldsymbol{C}^{+}$ and $\boldsymbol{C}^{-}$ can be best approximated by the outer products of the non-negative vector pairs $\{\boldsymbol{X}^{+}, \boldsymbol{Y}^{+}\}$ and $\{\boldsymbol{X}^{-}, \boldsymbol{Y}^{-}\}$ as

$$\begin{aligned} \boldsymbol{C}^{+} &\approx \boldsymbol{X}^{+} \otimes \boldsymbol{Y}^{+} \\ \boldsymbol{C}^{-} &\approx -\boldsymbol{X}^{-} \otimes \boldsymbol{Y}^{-}. \end{aligned} \tag{17}$$

Then restore the non-negative vector $\boldsymbol{X}^{+} = [x^{+}(p)]$ and $\boldsymbol{X}^{-} = [x^{-}(p)]$ to the non-negative matrices $\boldsymbol{D}^{+} = [d^{+}(m,n)]$ and $\boldsymbol{D}^{-} = [d^{-}(m,n)]$, respectively, such that

$$\begin{aligned} d^{+}(m,n) &= x^{+}(p) \\ d^{-}(m,n) &= x^{-}(p) \end{aligned} \tag{18}$$

where the relation between p and m, n is given in (14). Next, perform the SVD on the matrices $\boldsymbol{D}^{+}$ and $\boldsymbol{D}^{-}$, and best approximate them as

$$\begin{aligned} \boldsymbol{D}^{+} &\approx \boldsymbol{F}_i^{+} \otimes \boldsymbol{G}_i^{+} \\ \boldsymbol{D}^{-} &\approx \boldsymbol{F}_i^{-} \otimes \boldsymbol{G}_i^{-} \end{aligned} \tag{19}$$

where vectors $\{\boldsymbol{F}_i^{+}, \boldsymbol{G}_i^{+}\}$ and $\{\boldsymbol{F}_i^{-}, \boldsymbol{G}_i^{-}\}$ are non-negative. Thus the 3-D array $\boldsymbol{B}$ can be approximated by the outer product

$$\boldsymbol{B} \approx \boldsymbol{F}_i^{+} \otimes \boldsymbol{G}_i^{+} \otimes \boldsymbol{Y}^{+} \tag{20}$$

or

$$\boldsymbol{B} \approx -\boldsymbol{F}_i^{-} \otimes \boldsymbol{G}_i^{-} \otimes \boldsymbol{Y}^{-}. \tag{21}$$

Indeed, only one of the decompositions (20) and (21) , which results in a smaller decomposition error, will be used, and the other one will be neglected. The way to determine which one should be remained is given in the next step. This implies that only part of the error array $\boldsymbol{E}$ will be approximated in this i-th decomposition stage. The approximation will successively improved by the succeeding decomposition stages.

Step 3. Fix $\{\boldsymbol{F}_i^{+}, \boldsymbol{G}_i^{+}\}$, and then find a new vector $\boldsymbol{Y}_{new}^{+}$ such that

$$\begin{aligned} Error^{+} &= \|\boldsymbol{B} - \boldsymbol{F}_i^{+} \otimes \boldsymbol{G}_i^{+} \otimes \boldsymbol{Y}_{new}^{+}\|^2 \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{q=1}^{Q}[b(m,n,q) - F_i^{+}(m)G_i^{+}(n)Y_{new}^{+}(q)]^2 \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{q=1}^{Q}[F_i^{+}(m)G_i^{+}(n)Y_{new}^{+}(q) - b(m,n,q)]^2 \end{aligned}$$

is minimum, where $Q = L_1 L_2 \cdots L_K$. The optimal vector $\boldsymbol{Y}_{new}^{+}$ is determined as follows.

Differentiating $Error^+$ with respect to the l-th element of the vector $\boldsymbol{Y}^+_{new}$, we obtain

$$\frac{\partial Error^+}{\partial Y^+_{new}(l)} = \sum_{m=1}^{M}\sum_{n=1}^{N} 2[F_i^+(m)G_i^+(n)Y^+_{new}(l) - b(m,n,l)] \cdot F_i^+(m)G_i^+(n). \tag{22}$$

Equating $\frac{\partial Error^+}{\partial Y^+_{new}(l)}$ to zero, we get

$$Y^+_{new}(l) = \frac{\sum_{m=1}^{M}\sum_{n=1}^{N} F_i^+(m)G_i^+(n)b(m,n,l)}{\sum_{m=1}^{M}\sum_{n=1}^{N}[F_i^+(m)G_i^+(n)]^2} \tag{23}$$

where $l = 1, 2, \cdots, Q$, but

$$\sum_{m=1}^{M}\sum_{n=1}^{N}[F_i^+(m)G_i^+(n)]^2 \neq 0. \tag{24}$$

In the same way, we can also find a new vector $\boldsymbol{Y}^-_{new}$ such that

$$\begin{aligned} Error^- &= \|\boldsymbol{B} - \boldsymbol{F}_i^- \otimes \boldsymbol{G}_i^- \otimes \boldsymbol{Y}^-_{new}\|^2 \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{q=1}^{Q}[b(m,n,q) - F_i^-(m)G_i^-(n)Y^-_{new}(q)]^2 \\ &= \sum_{m=1}^{M}\sum_{n=1}^{N}\sum_{q=1}^{Q}[F_i^-(m)G_i^-(n)Y^-_{new}(q) - b(m,n,q)]^2 \end{aligned}$$

is minimum.

Next, compare $Error^+$ with $Error^-$.

If $Error^+ \leq Error^-$, we let

$$\begin{cases} \boldsymbol{F}_i = \boldsymbol{F}_i^+ \\ \boldsymbol{G}_i = \boldsymbol{G}_i^+ \\ \boldsymbol{Y}_1 = \boldsymbol{Y}^+_{new}. \end{cases} \tag{25}$$

If $Error^+ > Error^-$, we let

$$\begin{cases} \boldsymbol{F}_i = \boldsymbol{F}_i^- \\ \boldsymbol{G}_i = \boldsymbol{G}_i^- \\ \boldsymbol{Y}_1 = \boldsymbol{Y}^-_{new}. \end{cases} \tag{26}$$

As a result, we obtain

$$\boldsymbol{B} \approx \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{Y}_1. \tag{27}$$

Then, restore the vector $\boldsymbol{Y}_1 = [Y_1(q)]$ to a K-D array $\boldsymbol{A}_1 = [a_1(l_1, l_2, \cdots, l_K)]$ such that

$$a_1(l_1, l_2, \cdots, l_K) = Y_1(q). \tag{28}$$

The relation between q and $\{l_1, l_2, \cdots, l_K\}$ is given in (12). Thus the error array $\boldsymbol{E}$ can be expressed by a generalized outer product of vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i\}$ and the K-D array $\boldsymbol{A}_1$ as

$$\boldsymbol{E} \approx \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{A}_1 \tag{29}$$

which can also be represented using their elements as

$$e(m, n, l_1, l_2, \cdots, l_K) \approx F_i(m)G_i(n)a_1(l_1, l_2, \cdots, l_K). \tag{30}$$

Step 4. Convert the K-D array $\boldsymbol{A}_1 = [a_1(l_1, l_2, \cdots, l_K)]$ to a matrix $\boldsymbol{B}_1 = [b_1(l_1, q)]$ such that

$$b_1(l_1, q) = a_1(l_1, l_2, \cdots, l_K) \tag{31}$$

where

$$q = (l_2 - 1)L_3L_4 \cdots L_K + (l_3 - 1)L_4L_5 \cdots L_K + \cdots + (l_{K-1} - 1)L_K + l_K. \tag{32}$$

Perform the SVD on the matrix $\boldsymbol{B}_1$ and best approximate it by the outer product of the vector pair $\{\boldsymbol{P}_{i1}, \boldsymbol{Y}_2\}$ as

$$\boldsymbol{B}_1 \approx \boldsymbol{P}_{i1} \otimes \boldsymbol{Y}_2. \tag{33}$$

Then restore the vector $\boldsymbol{Y}_2 = [Y_2(q)]$ to a $(K-1)$-D array $\boldsymbol{A}_2 = [a_2(l_2, l_3, \cdots, l_K)]$ as

$$a_2(l_2, l_3, \cdots, l_K) = Y_2(q) \tag{34}$$

where the relation between q and $\{l_2, l_3, \cdots, l_K\}$ is given in (32). As (29), we obtain

$$\boldsymbol{E} \approx \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{A}_2. \tag{35}$$

Step 5. Convert the $(K-1)$-D array $\boldsymbol{A}_2 = [a_2(l_2, l_3, \cdots, l_K)]$ to a matrix $\boldsymbol{B}_2 = [b_2(l_2, q)]$ such that

$$b_2(l_2, q) = a_2(l_2, l_3, \cdots, l_K) \tag{36}$$

where

$$q = (l_3 - 1)L_4L_5 \cdots L_K + (l_4 - 1)L_5L_6 \cdots L_K + \cdots + (l_{K-1} - 1)L_K + l_K. \tag{37}$$

Perform the SVD on the matrix $\boldsymbol{B}_2$ and best approximate it by the outer product of the vector pair $\{\boldsymbol{P}_{i2}, \boldsymbol{Y}_3\}$ as

$$\boldsymbol{B}_2 \approx \boldsymbol{P}_{i2} \otimes \boldsymbol{Y}_3. \tag{38}$$

Then restore the vector $\boldsymbol{Y}_3 = [Y_3(q)]$ to a $(K-2)$-D array $\boldsymbol{A}_3 = [a_3(l_3, l_4, \cdots, l_K)]$ as

$$a_3(l_3, l_4, \cdots, l_K) = Y_3(q) \tag{39}$$

where the relation between q and $\{l_3, l_4, \cdots, l_K\}$ is given in (37). Thus we obtain

$$\boldsymbol{E} \approx \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{P}_{i2} \otimes \boldsymbol{A}_3. \tag{40}$$

Repeating the same operations on the gradually reduced dimensional arrays $\boldsymbol{A}_3$, $\boldsymbol{A}_4$, $\cdots$, $\boldsymbol{A}_{K-1}$ as above, we can obtain the final outer product expansion of the $(K+2)$-D error array $\boldsymbol{E}$ as

$$\boldsymbol{E} \approx \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{P}_{i2} \otimes \cdots \otimes \boldsymbol{P}_{iK}. \tag{41}$$

Let $i = i + 1$. If $i < r$, return to Step 1. Otherwise, proceed to the next step.

Step 6. Combining the results from Step 1 $\sim$ Step 5, we yield

$$\boldsymbol{A} \approx \sum_{i=1}^{r} \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{P}_{i2} \otimes \cdots \otimes \boldsymbol{P}_{iK}. \tag{42}$$

At this point, the overall decomposition error

$$E_r = \|\boldsymbol{A} - \sum_{i=1}^{r} \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{P}_{i2} \otimes \cdots \otimes \boldsymbol{P}_{iK}\|^2 \tag{43}$$

is not minimum. Next, we choose the result (42) as a starting point, and utilize a nonlinear optimization method to minimize the overall decomposition error E_r. At last, we can obtain the optimal outer product expansion (42).

Here we should notice that although a nonlinear optimization method is used for minimizing the overall decomposition error E_r, the computation time is not long because the result (42) is chosen as a starting point, and itself is usually a good approximation to the $(K+2)$-D array $\boldsymbol{A}$. In addition, to evaluate the proposed decomposition method, we use the normalized root mean square (rms) error

$$\frac{\|\boldsymbol{A} - \sum_{i=1}^{r} \boldsymbol{F}_i \otimes \boldsymbol{G}_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{P}_{i2} \otimes \cdots \otimes \boldsymbol{P}_{iK}\|}{\|\boldsymbol{A}\|} \times 100\% \tag{44}$$

as the evaluation criterion.

3. 2-D variable filter design

Once the optimal outer product expansion (42) is determined, the next job is to approximate the non-negative vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i\}$ by using zero-phase 1-D constant filters $\{f_i(z_1)f_i(z_1^{-1}), g_i(z_2)g_i(z_2^{-1})\}$, and the real-valued vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$ by using 1-D functions $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$. Since any 1-D functions can be approximated by using 1-D polynomials, and 1-D polynomials are mathematically tractable, in this section, we choose the 1-D functions $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$ to be 1-D polynomials. This section formulates the 1-D constant filter designs and 1-D polynomial approximations separately.

3.1. *Zero-phase 1-D constant filter designs*

Zero-phase 1-D constant filters $\{f_i(z_1)f_i(z_1^{-1}), g_i(z_2)g_i(z_2^{-1})\}$ are designed by best approximating the 1-D magnitude specification vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i\}$. To do this, we just need to approximate the vectors $\{\boldsymbol{F}_i^{1/2}, \boldsymbol{G}_i^{1/2}\}$ by 1-D constant filters $\{f_i(z_1), g_i(z_2)\}$ with arbitrary phase characteristics. Let 1-D constant filter $f_i(z_1)$ to be of the form

$$f_i(z_1) = \frac{A \cdot \prod_{k=1}^{M_1/2} (1 + a_{k,1}z_1^{-1} + a_{k,2}z_1^{-2})}{\prod_{k=1}^{M_1/2} (1 + \alpha_{k,1}z_1^{-1} + \alpha_{k,2}z_1^{-2})}. \tag{45}$$

The optimal filter coefficient vector

$$\boldsymbol{\Gamma}_1 = [\, A \;\; a_{k,1} \;\; a_{k,2} \;\; \alpha_{k,1} \;\; \alpha_{k,2} \,] \tag{46}$$

is determined by minimizing the squared error function

$$\begin{aligned} e_f(\boldsymbol{\Gamma}_1) &= \|\boldsymbol{F} - \boldsymbol{F}_i\|^2 \\ &= \sum_{m=1}^{M} [F(m) - F_i(m)]^2 \end{aligned} \tag{47}$$

using the Davidon-Fletcher-Powell (DFP) nonlinear minimization method. In (47), $\boldsymbol{F}$ is the magnitude response vector of the zero-phase 1-D constant filter $f_i(z_1)f_i(z_1^{-1})$, and $F(m)$ is its m-th element, and $F_i(m)$ is the m-th element of the non-negative vector $\boldsymbol{F}_i$, i.e.,

$$\begin{aligned} \boldsymbol{F} &= [F(1) \;\; F(2) \;\; \cdots \;\; F(M)]^t \\ \boldsymbol{F}_i &= [F_i(1) \;\; F_i(2) \;\; \cdots \;\; F_i(M)]^t. \end{aligned} \tag{48}$$

It should be mentioned that if no constraints are imposed on the denominator coefficients $\{\alpha_{k,1}, \alpha_{k,2}\}$ in the nonlinear minimization (47), the resulting 1-D filter $f_i(z_1)$ may be unstable.

It is known that the 1-D filter $f_i(z_1)$ is stable if and only if

$$\begin{cases} |\alpha_{k,1}| < 1 + \alpha_{k,2} \\ |\alpha_{k,2}| < 1. \end{cases} \tag{49}$$

So in our design, we first perform the denominator coefficient transformations

$$\begin{cases} \alpha_{k,1} = \sin\theta_{k,1}(1 + \sin\theta_{k,2}) \\ \alpha_{k,2} = \sin\theta_{k,2} \end{cases} \tag{50}$$

where

$$\begin{cases} \theta_{k,1} \neq \pi/2 + p\pi \\ \theta_{k,2} \neq \pi/2 + q\pi \end{cases} \tag{51}$$

and p, q are any integers. Then the optimal coefficient vector

$$\boldsymbol{\Gamma}_2 = [\, A \;\; a_{k,1} \;\; a_{k,2} \;\; \theta_{k,1} \;\; \theta_{k,2} \,] \tag{52}$$

is found by minimizing the error function (47). Once the vector $\boldsymbol{\Gamma}_2$ is obtained, the optimal coefficient vector $\boldsymbol{\Gamma}_1$ can be easily calculated from $\boldsymbol{\Gamma}_2$ by using the transformations (50). Here we should emphasize that the conditions (51) are always satisfied in the practical nonlinear minimization process, thus the designed 1-D filter $f_i(z_1)$ is always stable. Also, the zero-phase 1-D constant filter $g_i(z_2)g_i(z_2^{-1})$ is designed in the same way. After the zero-phase 1-D constant filters $\{f_i(z_1)f_i(z_1^{-1}), g_i(z_2)g_i(z_2^{-1})\}$ are obtained, the next job is to approximate the vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$ by using 1-D polynomials $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$. Assume that the magnitude response vectors of the designed zero-phase 1-D constant filters $\{f_i(z_1)f_i(z_1^{-1}), g_i(z_2)g_i(z_2^{-1})\}$ are $\{\boldsymbol{F}'_i, \boldsymbol{G}'_i\}$. Evidently,

$$\begin{aligned} \boldsymbol{F}'_i &\approx \boldsymbol{F}_i \\ \boldsymbol{G}'_i &\approx \boldsymbol{G}_i. \end{aligned} \tag{53}$$

Since the real-valued vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$ can be exactly approximated by using 1-D polynomials $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$, which will be shown below, we know that the final squared approximation error of the designed zero-phase 2-D variable filter is

$$E'_r = \|\boldsymbol{A} - \sum_{i=1}^{r} \boldsymbol{F}'_i \otimes \boldsymbol{G}'_i \otimes \boldsymbol{P}_{i1} \otimes \boldsymbol{P}_{i2} \otimes \cdots \otimes \boldsymbol{P}_{iK}\|^2. \tag{54}$$

From (54) it is known that if we hold the resulting vectors $\{\boldsymbol{F}'_i, \boldsymbol{G}'_i\}$ constant, and choose the vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$ as initial values, and then further minimize the error E'_r, the final design error of the zero-phase 2-D variable filter can be further reduced. So before approximating the vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$, we first reoptimize them by minimizing the error E'_r, and then approximate the new updated vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$.

3.2. 1-D polynomial approximations

As stated above, to approximate the real-valued vectors $\{\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \cdots, \boldsymbol{P}_{iK}\}$, the functions $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$ may be arbitrary 1-D functions such as exponential functions, trigonometric functions and polynomials. Among them, 1-D polynomials are most computationally efficient. In addition, from the Weierstrass approximation theorem it is known that 1-D polynomials can be used to approximate arbitrary 1-D functions with any desired approximation accuracy, so we choose $\{p_{i1}(\Psi_1), p_{i2}(\Psi_2), \cdots, p_{iK}(\Psi_K)\}$ to be 1-D polynomials in this paper. Below, we consider the problem of using the 1-D polynomial

$$p(\Phi) = \sum_{i=0}^{N_p} c_i \Phi^i \tag{55}$$

to approximate a real-valued specification vector $\boldsymbol{P} \in \boldsymbol{R}^{L_p \times 1}$, where N_p is the order of $p(\Phi)$. The squared approximation error is

$$e_p = \sum_{j=1}^{L_p} [\sum_{i=0}^{N_p} c_i \Phi_j^i - P(j)]^2 \tag{56}$$

where Φ_j is the j-th sample of the variable Φ.

Differentiating e_p with respect to c_q, $q = 0, 1, \cdots, N_p$, and setting it to zero, we obtain

$$\sum_{i=0}^{N_p} c_i \cdot [\sum_{j=1}^{L_p} \Phi_j^i \Phi_j^q] = \sum_{j=1}^{L_p} P(j) \Phi_j^q. \tag{57}$$

The Eq. (57) can be represented in the matrix form as

$$\boldsymbol{\Phi}\boldsymbol{\Phi}^t \boldsymbol{C} = \boldsymbol{\Phi}\boldsymbol{P} \tag{58}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \Phi_1^0 & \Phi_2^0 & \cdots & \Phi_{L_p}^0 \\ \Phi_1^1 & \Phi_2^1 & \cdots & \Phi_{L_p}^1 \\ \vdots & \vdots & & \vdots \\ \Phi_1^{N_p} & \Phi_2^{N_p} & \cdots & \Phi_{L_p}^{N_p} \end{bmatrix} \tag{59}$$

$$\boldsymbol{C} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{N_p} \end{bmatrix}^t \tag{60}$$

$$\boldsymbol{P} = \begin{bmatrix} P(1) & P(2) & \cdots & P(L_p) \end{bmatrix}^t. \tag{61}$$

Solving the simultaneous linear Eq. (58) can obtain the optimal coefficient vector $\boldsymbol{C}$.

4. Design examples

This section presents three design examples to show the usefulness of the proposed zero-phase 2-D variable filter design technique.

Lowpass Filter. A 2-D variable lowpass magnitude design specification is given by

$$H_d(\omega_1, \omega_2, \Psi_1) = \begin{cases} 1 & R \leq R_p \\ 0 & R \geq R_c \end{cases} \tag{62}$$

where

$$\begin{aligned} R &= \sqrt{\omega_1^2 + \omega_2^2}/\pi \\ R_p &= 0.22 + \Psi_1 \\ R_c &= 0.40 + \Psi_1 \\ \Psi_1 &\in [-0.08, 0.08]. \end{aligned} \tag{63}$$

The spectral parameter Ψ_1 controls the variable position of the transition band, but the transition bandwidth is constant [4]. To construct a 3-D magnitude specification array $\boldsymbol{A}$, we assume that the variable magnitude specification in the transition band varies linearly from the passband to stopband. In this example, we take $M = N = 21$, $L_1 = 9$, and thus a 3-D magnitude specification array $\boldsymbol{A} \in \boldsymbol{R}^{21\times21\times9}$ is constructed. Performing the outer product expansion on the 3-D array $\boldsymbol{A}$, we can obtain the decomposition errors shown in Table 1. Observing the Table 1, we know that the greater the number r of parallel channels, the smaller the normalized rms decomposition error. If $r = 4$, the normalized rms decomposition error is 4.35%. Thus in our designs, we only approximate the vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i, \boldsymbol{P}_{i1}\}$, $i = 1, 2, 3, 4$, and ignore the others. This is because taking more parallel channels will need extra hardware cost in implementation but hardly improve the design accuracy of the final resulting zero-phase 2-D variable filter.

Table 2 shows the normalized rms errors of the designed (4,4)-order zero-phase 2-D variable lowpass filter for some Ψ_1 samples. The order of 1-D polynomials $p_{i1}(\Psi_1)$ is 8. Figure 3 and Figure 4 illustrate the magnitude responses of the designed variable filter for $\Psi_1 = -0.08$ and $\Psi_1 = 0$, respectively. The design results are relatively satisfactory to some extent.

Compared with the Fahmy's technique, our proposed technique is more computationally efficient because it only needs 1-D constant filter designs and 1-D polynomial linear approximations. Especially, the stability of the resulting variable filters is always guaranteed, and their parallel structures are suitable for high speed signal processing. Also, the designed 2-D variable filters are zero-phase, so they are particularly important in image processing applications.

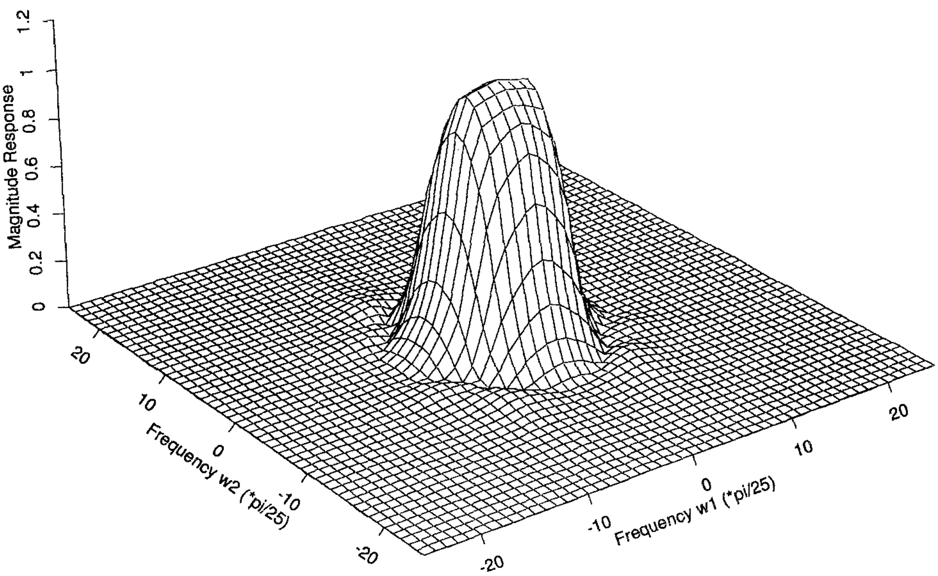

Figure 3. Variable magnitude response for $\Psi_1 = -0.08$.

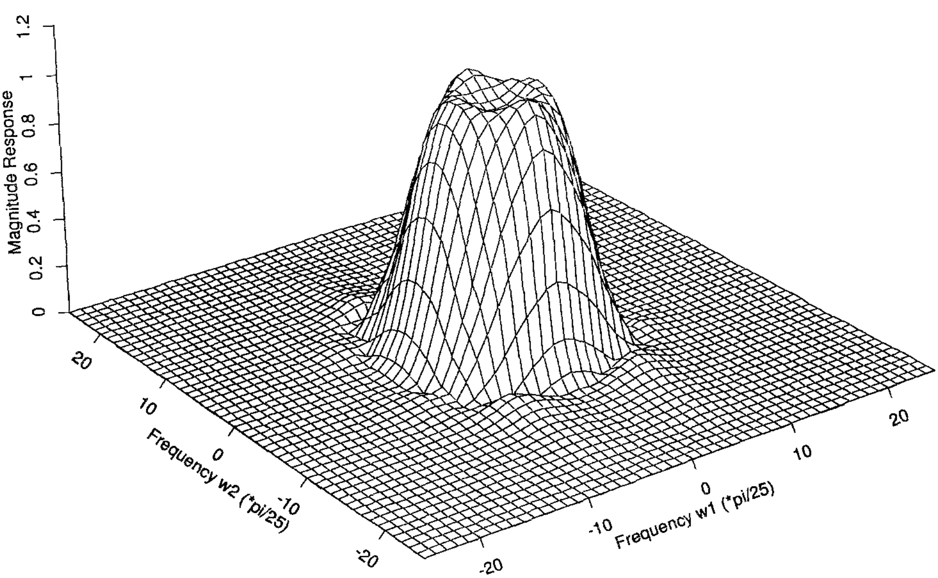

Figure 4. Variable magnitude response for $\Psi_1 = 0$.

Table 1. Decomposition errors of lowpass filter

Channel number [r]	Normalized rms error [%]
1	25.11
2	14.62
3	8.10
4	4.35
5	3.60
6	3.08
⋮	⋮

Table 2. Design errors of lowpass filter.

Sampled Ψ_1	Normalized rms error [%]
-0.08	9.99
-0.06	9.26
-0.04	9.29
-0.02	8.62
0	7.34
0.02	7.06
0.04	6.61
0.06	8.34
0.08	11.75

Fan Filter. The variable magnitude design specification of a 2-D variable fan filter is given by

$$H_d(\omega_1, \omega_2, \Psi_1) = \begin{cases} 1 & \omega_2 \geq \Psi_1\omega_1 \\ 0 & \omega_2 \leq \Psi_1\omega_1 - 0.5\pi \end{cases} \tag{64}$$

where $\Psi_1 \in [1, 2]$. The spectral parameter Ψ_1 controls the variable passband angle. The transition bandwidth is constant, and the specification in the transition band varies linearly. For constructing a 3-D array $\boldsymbol{A}$, we take $M = N = 21$, and $L_1 = 11$. Thus a 3-D magnitude specification array $\boldsymbol{A} \in \boldsymbol{R}^{21\times21\times11}$ is obtained. Performing the outer product expansion on the 3-D array $\boldsymbol{A}$, we obtain the decomposition errors given in Table 3. In variable filter design, we only approximate the vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i, \boldsymbol{P}_{i1}\}$, $i = 1, 2, \cdots, 6$, and ignore the others. Thus $r = 6$. In this case, the normalized rms error from the decomposition stage is 5.96%. Table 4 shows the final normalized rms errors of the designed (2,2)-order variable fan filter, the order of 1-D polynomials $p_{i1}(\Psi_1)$ is chosen to be 5.

Figure 5 and Figure 6 illustrate the magnitude responses of the designed (2,2)-order variable fan filter for $\Psi_1 = 1.5$ and $\Psi_1 = 2$, respectively. From the design results we know that although the filter order is just only (2,2), extremely good results have been obtained.

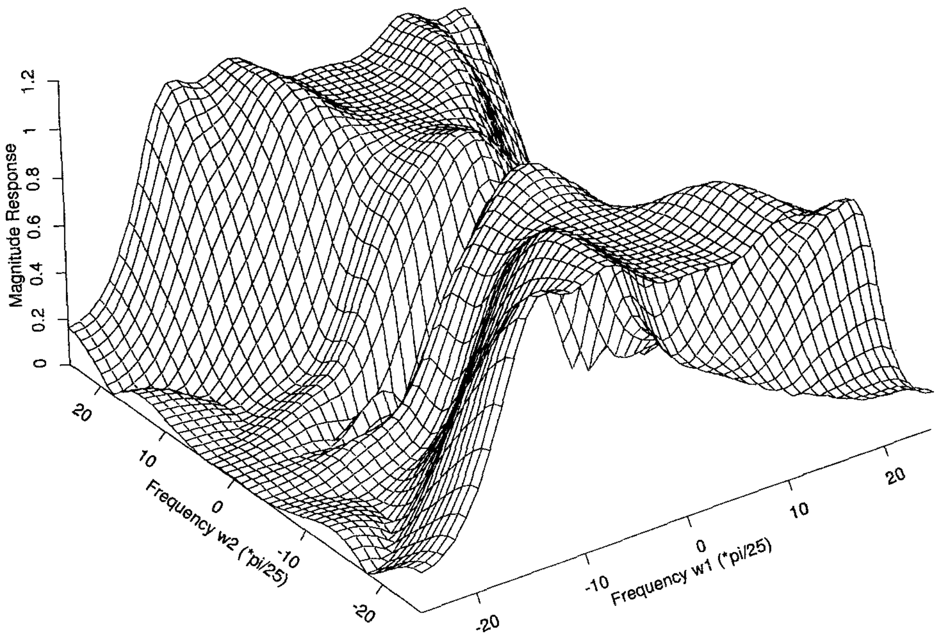

Figure 5. Variable magnitude response for $\Psi_1 = 1.5$.

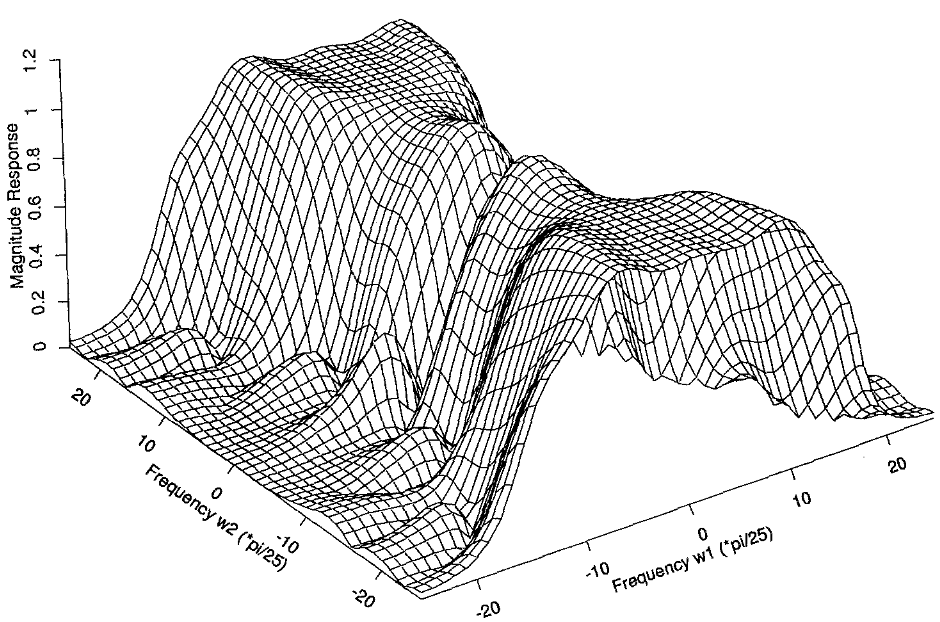

Figure 6. Variable magnitude response for $\Psi_1 = 2$.

Table 3. Decomposition errors of fan filter.

Channel number [r]	Normalized rms error. [%]
1	34.43
2	21.73
3	13.65
4	10.28
5	7.94
6	5.96
7	5.13
⋮	⋮

Table 4. Design errors of fan filter.

Sampled Ψ_1	Normalized rms error [%]
1.0	12.14
1.1	10.88
1.2	9.52
1.3	9.11
1.4	8.93
1.5	8.81
1.6	8.90
1.7	9.10
1.8	9.80
1.9	11.20
2.0	12.43

Highpass Filter. A 2-D variable highpass magnitude design specification is given by

$$H_d(\omega_1, \omega_2, \Psi_1, \Psi_2) = \begin{cases} 0 & R \leq \Psi_1 \\ 1 & R \geq \Psi_1 + \Psi_2 \end{cases} \tag{65}$$

$$\begin{aligned} R &= \sqrt{\omega_1^2 + \omega_2^2}/\pi \\ \Psi_1 &\in [0.3, 0.5] \\ \Psi_2 &\in [0.2, 0.3]. \end{aligned} \tag{66}$$

The spectral parameter Ψ_1 controls the variable stopband width, and Ψ_2 controls the transition bandwidth. Therefore, the stopband width and transition bandwidth can be independently adjusted. In addition, the specification in the transition band varies linearly.

As in the above two examples, we take $M = N = 21$, and $L_1 = 11$, $L_2 = 6$. Performing the outer product expansion on the 4-D specification array $\boldsymbol{A} \in \boldsymbol{R}^{21\times21\times11\times6}$, we obtain the

Table 5. Decomposition errors of highpass filter.

Channel number [r]	Normalized rms error [%]
1	30.36
2	11.28
3	6.66
4	5.92
⋮	⋮

Table 6. Design errors of highpass filter.

Sampled Ψ_1	Sampled Ψ_2	Normalized rms error [%]
0.30	0.20	6.61
	0.30	5.37
0.32	0.20	6.53
	0.30	5.24
0.34	0.20	6.35
	0.30	5.08
0.36	0.20	6.47
	0.30	5.18
0.38	0.20	6.22
	0.30	5.35
0.40	0.20	6.58
	0.30	6.09
0.42	0.20	6.59
	0.30	7.16
0.44	0.20	7.08
	0.30	8.66
0.46	0.20	8.50
	0.30	10.71
0.48	0.20	9.91
	0.30	12.77
0.50	0.20	11.10
	0.30	14.75

decomposition errors in Table 5. In variable filter design, the vectors $\{\boldsymbol{F}_i, \boldsymbol{G}_i, \boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}\}$, $i = 1, 2, 3$, are approximated, i.e., $r = 3$. In this case, the normalized rms error from the decomposition stage is 6.66%. Table 6 gives the final normalized rms errors of the designed (2,2)-order variable highpass filter. The orders of 1-D polynomials $p_{i1}(\Psi_1)$ and $p_{i2}(\Psi_2)$ are respectively 5 and 3.

Figure 7 illustrates the magnitude response of the designed (2,2)-order variable highpass filter for $\Psi_1 = 0.3$ and $\Psi_2 = 0.2$. Figure 8 illustrates that for $\Psi_1 = 0.5$ and $\Psi_2 = 0.3$. From

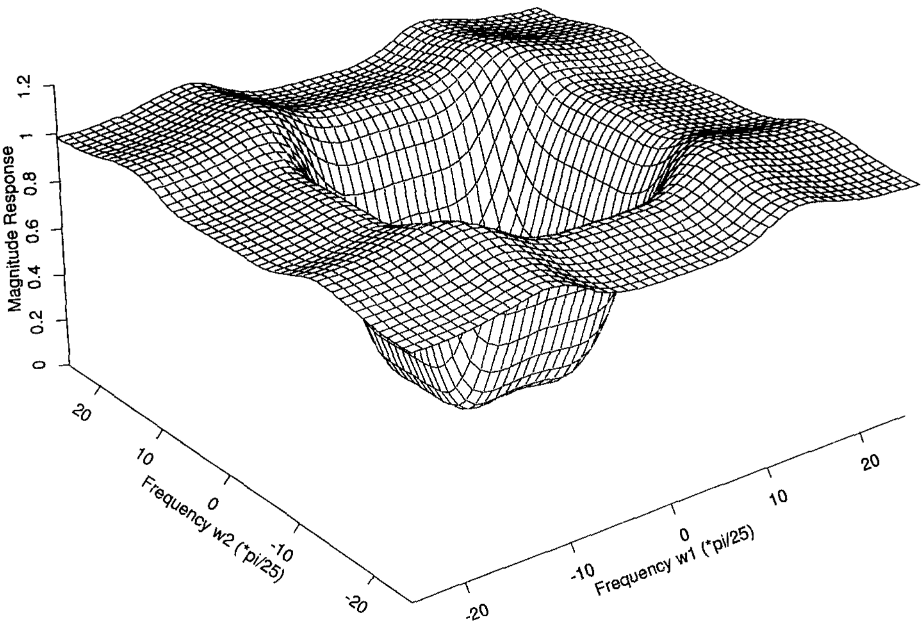

Figure 7. Variable magnitude response for $\Psi_1 = 0.3$, $\Psi_2 = 0.2$.

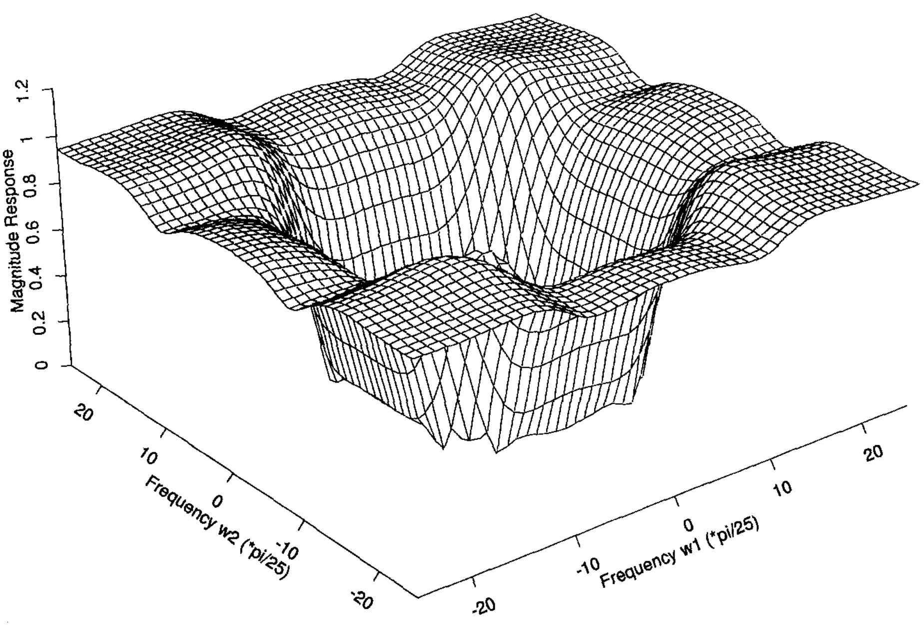

Figure 8. Variable magnitude response for $\Psi_1 = 0.5$, $\Psi_2 = 0.3$.

the design results we know that although the filter order is just only (2,2), very satisfactory variable characteristics have been obtained.

5. Conclusions

This paper has proposed an efficient technique for designing zero-phase 2-D variable digital filters with quadrantally symmetric magnitude characteristics. The technique is based on the decomposition of the given 2-D variable magnitude specifications. At first, we proposed a new outer product expansion method for decomposing the 2-D variable magnitude specifications into the magnitude specifications of the normal 1-D constant filters and the specifications of 1-D functions. Then the resulting 1-D magnitude specifications are approximated by using zero-phase 1-D constant filters, and the specifications of 1-D functions are approximated by using 1-D polynomials. At last, by interconnecting the obtained zero-phase 1-D constant filters and 1-D polynomials, we can easily obtain a zero-phase 2-D variable filter. The design technique is computationally efficient. In addition, since the part of the zero-phase 1-D constant filters is always fixed in signal processing applications, the resulting zero-phase 2-D variable filters are always stable so long as the zero-phase 1-D constant filters are designed to be stable. Moreover, the coefficients of the resulting 2-D variable filters can be easily obtained by computing the 1-D polynomials. However, the proposed technique can only design 2-D variable filters with quadrantally symmetric magnitude characteristics. The one for approximating arbitrary 2-D magnitude characteristics is under investigation.

Acknowledgements

The research was carried out under the Special Researchers' Basic Science Program, Science and Technology Agency, Japan.

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