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Complexity, Convexity, and Unimodality

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A class of polygons termed *unimodal* is introduced. Let $P = P_1, p_2, \ldots, p_n$ be a simple n-vertex polygon. Given a fixed vertex or edge, several definitions of the distance between the fixed vertex or edge and any other vertex or edge are considered. For a fixed vertex (edge), a distance measure defines a distance function as the remaining vertices (edges) are traversed in order. If for every vertex (edge) of P a specified distance function is unimodal then P is a unimodal polygon in the corresponding sense. Relationships between unimodal polygons, in several senses, and *convex* polygons are established. Several properties are derived for unimodal polygons when the distance measure is the euclidean distance between vertices of the polygons. These properties lead to very simple $0(n)$ algorithms for solving a variety of problems that occur in computational geometry and pattern recognition. Furthermore, these algorithms establish that convexity is not the key factor in obtaining linear-time-complexity for solving these problems. The paper closes with several open questions in this area.

KEY WORDS: Unimodality; convexity; polygons; algorithms; closer-pair problem; diameter; all-nearest-neighbor problem; all-furthest-neighbor problem; geometric complexity; computational geometry; pattern recognition; artificial intelligence.

1. INTRODUCTION

The notion of convexity is very important in pattern recognition, computational geometry, and mathematics in general. In pattern recognition, convexity appears in a variety of settings such as detecting whether a two-dimensional figure is convex,[(1)] decomposing simple polygons into convex sets,[(2,3)] and computing convex hulls of sets.[(4,19,27)] In computational geometry convexity plays an important role in the analysis and design of algorithms.[(5)] In mathematics it forms a discipline of its own with a rich and

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extensive history. Some very useful references on convexity in mathematical literature include the books by Yaglom and Boltyanskii,[6] Eggleston,[7] Boltyanskii and Gohberg,[8] and Benson,[9] as well as the collection of papers edited by Klee.[10]

In this paper we are concerned with a simple polygon $P = p_1, p_2, ..., p_n$, i.e., we are given a list of vertices, in clockwise order, along with their Cartesian coordinates. We assume the polygon is in *standard form*, i.e., the vertices are distinct and no three consecutive vertices are collinear. A pair of vertices $p_i p_{i+1}$ defines an edge of the polygon where $i = 1, 2, ..., n$ and $p_{n+1} = p_1$. If for every vertex p_k, the angle determined by the edges $p_{k-1} p_k$ and $p_k p_{k+1}$ is convex (i.e., the interior angle is less than 180°) then the polygon is *convex*. All indices are taken modulo n.

Recently, Snyder and Tang[11] proposed an algorithm (which they claim runs in $O(n)$ worst-case time) for finding the *diameter* of a convex polygon P. The *diameter* of P, denoted by $D(P)$, is defined as follows:

$$D(P) = \max_{i,j} \{d(p_i, p_j)\}$$

where $d(p_i, p_j)$ is the euclidean distance between vertices p_i and p_j. The algorithm in Ref. 11 is a "hill-climbing" method in which the following basic operation is repeated. Given a starting vertex p_s vertices $p_{s+1}, p_{s+2}, ..., p_{s-1}$ are visited in order, at each step computing the euclidean distance between p_s and the vertex being visited, and this scan stops at step i if $d(p_s, p_i) < d(p_s, p_{i-1})$. In Ref. 11 it is concluded that this step yields

$$d(p_s, p_{i-1}) = \max_{k} \{d(p_s, p_k)\}$$

This is a tacit assumption that the function $d(p_s, p_k)$, $k = s, s+1, ..., s-1, s$ is unimodal for convex polygons. It turns out that this assumption is false and the algorithm in Ref. 11 is not guaranteed to work. Other authors such as Dobkin and Snyder[12] provide a different diameter algorithm, but have also assumed that convex polygons exhibit this unimodality property. Thus it appears that the falsity of this property is counter-intuitive at first glance. For counter-examples to these algorithms the reader is referred to Refs. 13 and 14.

In mathematics literature almost no results are available concerning properties of distances in convex polygons, let alone unimodality properties of the resulting distance functions. This, together with the fact that such properties are useful in the design of efficient algorithms and in the study of geometric complexity, suggests that this is a fruitful area for future investigation.

Many distance measures can be defined on convex polygons between edges and vertices. Given a vertex as an "anchor" we can measure (1) the euclidean distance between the anchor vertex and the remaining vertices as the polygon is traversed or we can measure (2) the perpendicular distance between the anchor vertex and the lines collinear with the edges of the polygon. Alternately, given an edge as an anchor we can measure the perpendicular distance between the line collinear with the anchor edge and the vertices of the polygon.

In this paper we investigate the relationship between convex polygons and polygons which have the property that each of their vertices or edges has a unimodal distance function in each of the above senses. Polygons that exhibit such unimodal distance functions are termed unimodal polygons. Furthermore, if the sense is not specified then the first sense above is intended. These results and several useful properties of unimodal polygons are presented in Sections 2 and 3. Section 4 considers the implications that the results of Sections 2 and 3 have for computational geometry. It is well known that when n points form the vertices of a convex polygon many computational geometric problems such as (1) the closest pair problem; (2) the all-nearest-neighbor problem; and (3) computing the diameter of a set can all be solved in $0(n)$ worst-case running time.[15,16] For arbitrary sets of points the fastest existing algorithms require time $0(n \log n)$ and this is optimal for most of the above problems. For the all-furthest-neighbor problem, even for convex polygons the fastest known algorithm requires time $0(n \log n)$. One naturally asks whether the property of convexity is the key factor in obtaining the reduced linear time complexities. In Section 4 it is shown that this is not the case. First it is shown that the diameter algorithm in Ref. 11 does not work even for convex, unimodal polygons (note that unimodal polygons need not be convex). Next it is shown that the diameter algorithm in Ref. 12 does indeed work for unimodal convex polygons and that it can be used to solve the all-furthest-neighbor problem for unimodal convex polygons in $0(n)$ worst-case time. This represents the first instance of a linear time complexity for this problem. Finally, $0(n)$ algorithms are given for solving the remaining problems listed above for the case of unimodal polygons in general, thus showing that $0(n)$ time complexities can be realized for these problems even if the polygons are not convex. Furthermore, the algorithms presented here are extremely simple compared to the corresponding algorithms existing for convex polygons. Some concluding remarks and open problems are given in Section 5.

2. CONVEXITY AND UNIMODALITY

One of the most common definitions of convexity is the following[6]:

Definition. A simple polygon P is convex iff for every pair of points $p, q \in P$ the closed line segment $\overline{pq}$ lies entirely in P.

We now introduce some notation and definitions of distances that are useful in characterizing convex polygons.

Let $L(p_i, p_{i+1})$ (sometimes $L_{i,i+1}$ for short) denote the line collinear with points p_i and p_{i+1}. Let $H(p_i, p_{i+1})$ denote the interior closed half-plane determined by $L(p_i, p_{i+1})$, i.e., since the vertices are ordered in a clockwise direction, $H(p_i, p_{i\pm 1})$ lies to the right of $\overline{p_i p_{i+1}}$ and it includes the line $L(p_i, p_{i+1})$. Let $\bar{H}(p_i, p_{i+1})$ denote the complement of $H(p_i, p_{i+1})$. As before, $d(p_i, p_j)$ denotes the euclidean distance between vertices p_i and p_j. Let $d(p_k \perp L_{ij})$ denote the perpendicular distance between vertex p_k and $L(p_i, p_j)$. Furthermore, if p_k lies in $\bar{H}(p_i, p_j)$ then the distance is negative. With these two distance measures we can define several distance functions.

(1) Given a vertex p_i,

$$f(p_i, p_j) \triangleq d(p_i, p_j), \qquad j = i, i+1, \ldots, i-1, i$$

(2) Given a vertex p_k,

$$f(p_k \perp L_{j,j+1}) \triangleq d(p_k \perp L_{j,j+1}), \qquad j = k, k+1, \ldots, k-2, k-1$$

(3) Given and edge $\overline{p_i p_{i+1}}$,

$$f(L_{i,i+1} \perp p_k) \triangleq d(L_{i,i+1} \perp p_k), \qquad k = i+1, i+2, \ldots, i-1, i$$

Thus a polygon $P = p_1, p_2, \ldots, p_n$ specifies n distance functions for each of the above definitions.

In this paper we consider polygons which have the property that for every vertex or edge, as the case may be, the corresponding distance functions, as defined above, are unimodal, and we investigate the relationship between such polygons and convex polygons. A real function f defined on the integers $1, 2, \ldots, n$ is said to be unimodal if there exist two integers i, j, where $1 < i \leqslant j < n$ such that f is strictly increasing on the interval $[1, i]$, strictly decreasing on the interval $[j, n]$ and such that $f(k) = f(k+1)$ for $k = i, i+1, \ldots, j-1$. In other words, if a function contains one "peak" only it is unimodal. Similarly, functions with two "peaks" are bimodal and with many peaks, multimodal.

Definition. If a polygon $P = p_1, p_2, \ldots, p_n$ is such that for every vertex $p_i, i = 1, 2, \ldots, n$, $f(p_i \perp L_{j,j+1})$ is unimodal then P is said to be a unimodal polygon in this sense. Alternately, P is termed $f(p_i \perp L_{j,j+1})$—unimodal.

Similar definitions hold for the other distance measures and for k-modality where $k > 1$.

It is tempting at first glance to assert that convex polygons are unimodal in all three senses defined. Indeed such polygons exist as evidenced by the equilateral triangle. However, it was shown in Ref. 14 that for the case of $f(p_i, p_j)$ convex polygons may contain vertices with as many as $0(n)$ modes. We will extend these results to hold for the case of $f(p_i \perp L_{j,j+1})$ also.

The following properties of convex polygons established in Ref. 6 will be useful in proving some theorems.

Property 1. A polygon P is convex iff every line passing through an interior point of P intersects the boundary of P at two points.

Definition. A line L is called a supporting line of a polygon P if it passes through at least one boundary point of P and if P lies entirely to one side of L.

Property 2. In each direction there exist two parallel supporting lines to a convex polygon.

Theorem 1. A convex polygon is $f(L_{i,i+1} \perp p_j)$—unimodal.

Proof. Consider any edge $\overline{p_i p_{i+1}}$ of the convex polygon P. This edge defines a direction ϕ. By property 2 there exist two parallel supporting lines in direction ϕ and $L(p_i, p_{i+1})$ is one of them. Let $L(p_k, p_{k+1})$ be the other. Note that k may equal $k+1$. These supporting lines decompose P into two polygonal chains $C_1 = p_{i+1}, p_{i+2},..., p_k$ and $C_2 = p_{k+1}, p_{k+1},..., p_i$. By property 1 the line $L(x)$ passing through an interior point x of P in direction ϕ intersects the boundary of P at two points. Since the two chains C_1 and C_2 span $L_{i,i+1}$ and $L_{k,k+1}$, $L(x)$ intersects C_1 and C_2. Therefore $L(x)$ intersects $C_1$and C_2 each at precisely one point. Therefore the chains C_1 and $C_2$are monotonic in the direction orthogonal to ϕ. Therefore $f(L_{i,i+1} \perp p_j)$ is unimodal. Since this is true for every edge of P the theorem follows. ∎

Theorem 2. A polygon that is $f(L_{i,i+1} \perp p_j)$—unimodal is convex.

Proof. Assume a polygon P is not convex. Then P must contain at least one reflex vertex (i.e., a vertex, the interior angle of which is greater than 180°). Let p_k be one such vertex. Now consider $f(L_{k,k+1} \perp P_i)$. That there must exist vertices p_j such that $d(L_{k,k+1} \perp p_j) > 0$ follows from Jordan's curve theorem. However, $d(L_{k,k+1} \perp p_{k-1}) < 0$ since p_k is a reflex vertex. Furthermore $d(L_{k,k+1} \perp p_k) = 0$. Therefore $f(L_{k,k+1} \perp p_i)$ is at least bimodal and cannot be unimodal, proving Theorem 2. ∎

Thus we see that Theorems 1 and 2 provide us with yet another definition of convexity. A polygon P is convex iff for every edge $\overline{p_i p_{i+1}}$ of P the function $f(L_{i,i+1} \perp p_j)$ is unimodal. This unimodality property of convex polygons is fairly well known. It allows Chazelle[5] to design efficient algorithms for detecting whether two convex polygons intersect or not. It also implies that certain other functions of edges of convex polygons are unimodal. For example consider the area of a set of triangles, each determined by the base as an edge $\overline{p_i p_{i+1}}$ of a convex polygon P and the top of the triangle consisting of vertex p_k where $k = i+2, i+3, \ldots, i-1$. The area of a triangle equals half the base times the height. The base is constant and the height is precisely $d(L_{i,i+1} \perp p_k)$. Therefore the area is unimodal. This is one of the properties that allows Dobkin and Snyder[12] to obtain an $0(n)$ algorithm for finding the maximum-area triangle inscribed in a convex polygon.

Consider now what happens if we interchange the role of edge and vertex in Theorems 1 and 2; in other words consider the function $f(p_i \perp L_{j,j+1})$. This function appears not to have been previously investigated.

Theorem 3. A convex polygon need not be $f(p_i \perp L_{j,j+1})$—unimodal.

Proof. Consider the convex pentagon in Fig. 1. The coordinates of the vertices are as follows: $a = (0, 0), b = (1, 5), c = (2, 5), d = (2, -5),$

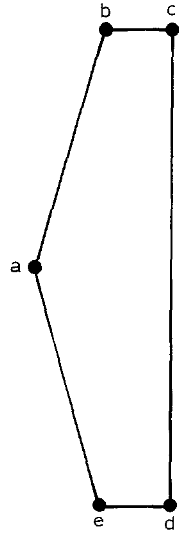

Fig. 1. Illustration of Theorem 3.

$e = (1, -5)$. Now consider $f(p_i \perp L_{j,j+1})$ where $p_i = a$. We have the following distance function: (0, 5, 2, 5, 0) which is clearly bimodal. ∎

This result can be strengthened by Theorem 4.

Theorem 4. A convex n-vertex polygon may contain vertices whose distance function $f(p_i \perp L_{j,j+1})$ exhibits as many as $0(n)$ modes or local maxima.

Proof. We shall construct a convex polygon P with the required property. Let p_i be any vertex of P and let $d(p_i, p_{i-1}) = d(p_i, p_{i+1}) = r$, and refer to Fig. 2. Construct an arc of radius r from p_{i+1} to p_{i-1} with center at p_i. Place vertices p_{i+k}, $k = 3, 5, 7, 9,\ldots$ on the circular arc an equal distance apart. Place vertices p_{i+j}, $j = 4, 8, 12, 16,\ldots$ on the circular arc such that p_{i+j} bisects the arc defined by p_{i+j-1} and p_{i+j+1}. Let x_{i+l}, $l = 2, 6, 10, 14, 18,\ldots$ denote the intersection of $\overline{p_{i+k}p_{i+k+2}}$ with the line through p_i that bisects $\overline{p_{i+k}p_{i+k+2}}$, $k = 1, 5, 9, 13,\ldots$. Similarly let y_{i+l} denote the intersection of this same line with arc (p_{i+k}, p_{i+k+2}). Place the remaining vertices p_{i+l}, $l = 2, 6, 10, 14, 18,\ldots$ anywhere on the open line segment $\overline{x_{i+l}y_{i+l}}$. First we need to prove that this polygon is convex. Consider any vertex p_{i+j} where $j = 3, 4, 5, 7, 8, 9, 11, 12, 13,\ldots$. Since the adjacent vertices of p_{i+j} lie either on or in the interior of the circle of radius r, such a vertex is convex. Now consider the vertices p_{i+j}, where $j = 2, 6, 10, 14, 18,\ldots$. Since p_{i+j} lies in $\bar{H}(p_{i+j-1}, p_{i+j+1})$, p_{i+j} is also convex. Finally, p_i, p_{i+1}, and p_{i-1} are all convex and thus all the vertices of P are convex establishing the convexity of P. Now we show that $f(p_i \perp L_{j,j+1})$ contains $0(n)$ modes or local maxima. Consider any pair of edges such as $\overline{p_{i+j}p_{i+j+1}}$ and $\overline{p_{i+j+1}p_{i+j+2}}$ where $j = 1, 5, 9,\ldots$. Let $d(p_i \perp L_{i+j,i+j+1}) = d_s$. Also let d_c

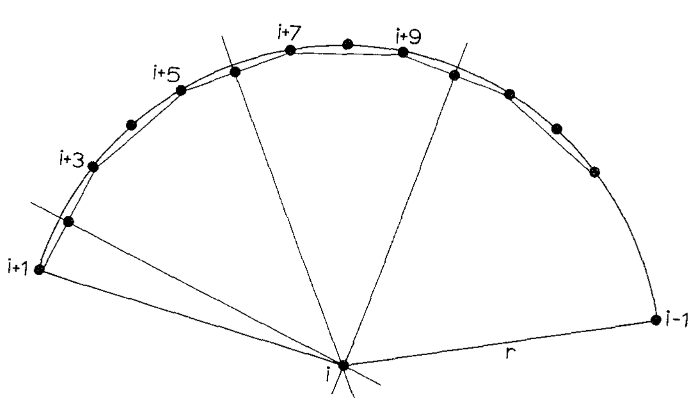

Fig. 2. Illustration of Theorem 4.

denote the length of the chord defined by extending the line segment $\overline{p_{i+j}p_{i+j+1}}$ to intersect the circular arc. Clearly $d_c > d(p_{i+3}, p_{i+4})$ for example. Note that the perpendicular bisector of the chord defining d_c precisely specifies d_s. Now consider edges such as $\overline{p_{i+3}p_{i+4}}$. The perpendicular bisector of $\overline{p_{i+3}p_{i+4}}$ specifies $d(p_i \perp L_{i+3,i+4})$. From the chordal property of circles it follows that since $d_c > d(p_{i+3}, p_{i+4})$, distances such as

$$d(p_i \perp L_{i+1,i+2}) < d(p_i \perp L_{i+3,i+4})$$

Since this relationship alternates we see that we obtain a local maximum every time four edges are traversed starting from $\overline{p_{i+2}, p_{i+3}}$. ∎

Although a convex polygon need not be $f(p_i \perp L_{j,j+1})$—unimodal, the converse is nevertheless true as the next theorem demonstrates.

Theorem 5. A polygon that is $f(p_i \perp L_{j,j+1})$—unimodal must be convex.

Proof. By contradiction, let P be a nonconvex simple polygon and let p_k be a reflex vertex of P. Consider vertex p_{k-1}. From Jordan's curve theorem it follows that there exist edges $\overline{p_i p_{i+1}}$ of P such that $p_{k-1} \in H(p_i, p_{i+1})$ and therefore for which $d(p_{k-1} \perp L_{i,i+1})$ is positive. But $d(p_{k-1} \perp L_{k,k+1}) < 0$. Therefore $f(p_i \perp L_{j,j+1})$ is at least bimodal. ∎

We turn now to consider the function $f(p_i, p_j)$ investigated in Ref. 14. As mentioned earlier, it was shown in Ref. 14 that a convex polygon need not be $f(p_i, p_j)$—unimodal and that vertices of convex $n$gons could have as many as $O(n)$ local maxima in their distance functions. It is tempting nevertheless to assert that convex polygons must surely have at least one unimodal vertex. That this is also false was discovered independently by Jones[17] and Lantuejoul,[18] who exhibited convex hexagons for which $f(p_i, p_j)$ is bimodal for $i = 1, 2, 3, 4, 5, 6$. An alternate proof of their result is given in Theorem 6.

Theorem 6. A convex polygon need not contain a single vertex whose distance function $f(p_i, p_j)$ is unimodal.

Proof. Construct an equilateral triangle T the vertices of which coincide with those of a Rouleaux polygon[6] (see Fig. 3). Three vertices, (p_1, p_3, and p_5) will form the required hexagon. Let the coordinates of p_1 be $(0, 0)$ and $p_5 = (1, 0)$. Construct an equilateral triangle inscribed in T determined by one of its vertices (a in Fig. 3) being placed at $(1/3, 0)$ and one of its edges being collinear with the line $x = 1/3$. Let $L(b, a)$ intersect arc (p_5, p_1) at a', let $L(a, c)$ intersect arc (p_3, p_5) at c', and let $L(c, b)$ intersect

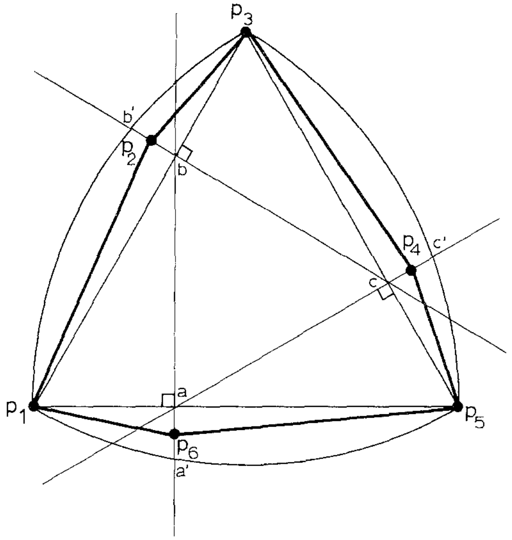

Fig. 3. Illustration of Theorem 6.

arc (p_1, p_3) at b'. Clearly if vertices p_2, p_4, and p_6 were placed at b, c, and a, respectively, bimodality would be assured as is easy to verify with elementary geometry. However, for P to be a convex hexagon we cannot allow collinear triplets of adjacent vertices. Thus we place p_2 in the open interval (b, b'), p_4 in (c, c') and p_6 in (a, a'). We now have a convex hexagon. For $i = 1, 3, 5$ $f(p_i, p_j)$ is clearly still bimodal. For $i = 2, 4, 6$ we must be more careful because if we move the vertices too close to the boundary of the Rouleaux triangle $f(p_i, p_j)$ will become unimodal. Referring to Fig. 4 let ε be the distance that p_4 is translated in the direction of $\overline{ac}$. Let

$$d(p_6, p_4) = d^*, d(p_6, p_5) = d', d(a, a'') = \varepsilon$$

and let

$$\delta \equiv d(a, p_5) - d(a, c) = \frac{2}{3}(1 - \cos 30°)$$

Clearly, $d(p_6, p_4)' < d(p_6, p_3)$. In order to ensure bimodality of p_6 we require that $d(p_6, p_4)$ also be less than $d(p_6, p_5)$. In other words we want $d^* < d'$. Now $d^* < d(a'', p_4) = d(a, c) + 2\varepsilon$ and $d' > d(a, c) + \delta$. Therefore it is sufficient to choose ε so that $d(a, c) + 2\varepsilon < d(a, c) + \delta$, i.e., $\varepsilon < \delta/2$. By symmetry the result follows. ∎

With the previous distance functions we have seen in Theorems 2 and 5 that unimodality implies convexity. Surprisingly, polygons that are $f(p_i, p_j)$—unimodal need not be convex.

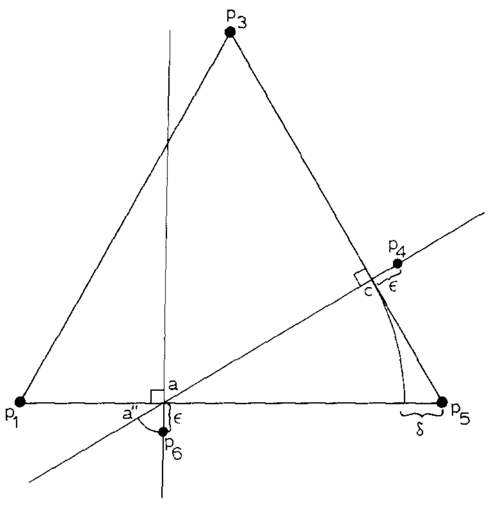

Fig. 4. Illustration of Theorem 6.

Theorem 7. Unimodal polygons need not be convex.

Proof. Consider two right angled triangles abc and edc where b and d are the right angles and $d(a, b) = d(e, d) = 2, d(b, c) = d(c, d) = 1$ and join them at c such that $\langle bcd = 90°$ to form a nonconvex pentagon as shown in Fig. 5. It is elementary to show that the polygon is unimodal. ∎

Furthermore, it is not difficult to construct unimodal n-vertex polygons that contain $0(n)$ reflex vertices and this is left as an exercise for the reader. Thus unimodality in the $f(p_i, p_j)$ sense and convexity appear to be quite distinct notions.

In summary we have the following relationships between convexity and unimodality in the three senses discussed.

convexity $\Leftrightarrow f(L_{i,i+1} \perp p_j)$—unimodality

convexity $\Leftarrow f(p_j \perp L_{i,i+1})$—unimodality

convexity $\not\Leftrightarrow f(p_i, p_j)$—unimodality

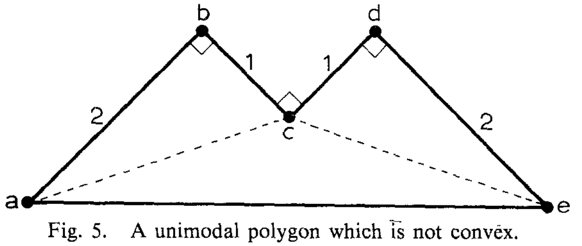

Fig. 5. A unimodal polygon which is not convex.

3. SOME PROPERTIES OF UNIMODAL POLYGONS

In this section we derive some properties of unimodal polygons that will allow us to obtain very simple algorithms to solve some geometric problems in Section 4.

Given a simple polygon $P = p_1, p_2, \ldots, p_n$, let the graph consisting of only the vertices and edges of P be denoted by PG(P). Let NNG(P) denote the nearest neighbor graph of P.

Theorem 8. If a simple polygon P is unimodal then the nearest neighbor of every vertex p_i is an adjacent vertex of p_i.

Proof. By contradiction, let $\overline{p_i p_j}$ be an edge in NNG(P) and not in PG(P). Thus p_i and p_j are not adjacent to each other. Thus there must exist at least one vertex p_i' in the polygonal chain $C_{ij} = p_i, \ldots, p_i', \ldots, p_j$ and similarly there must exist at least one vertex p_j' in the chain $C_{ji} = p_j, \ldots, p_j', \ldots, p_i$. Furthermore, since $\overline{p_i p_j} \in$ NNG(P) it follows that either p_i is a nearest neighbor of p_j or the reverse is true. Assume the former, without loss of generality. Therefore, $d(p_j, p_i') > d(p_i, p_j)$ and $d(p_j, p_j') > d(p_i, p_j)$. It follows that $f(p_i, p_j)$ is at least bimodal, a contradiction. ∎

A similar argument establishes Corollary 2.

Corollary 2. If a simple polygon P is unimodal then the closest pair of vertices forms an edge in $PG(P)$.

Definition. A pair of vertices of P are antipodal if they admit parallel lines of support.

Definition. A pair of vertices $p_i, p_j \in P$ are global symmetric furthest neighbors (*GSFN*) iff

$$d(p_i, p_j) = \max_k \{d(p_i, p_k)\}$$

and

$$d(p_i, p_j) = \max_k \{d(p_k, p_j)\}$$

for $k = 1, 2, \ldots, n$.

Definition. A pair of vertices $p_i, p_j \in P$ are local symmetric furthest neighbors (LSFN) iff

$$\mathrm{d}(p_i, p_j) > d(p_i, p_{j-1}) \qquad \text{and} \qquad d(p_i, p_{j+1})$$

and

$$d(p_i, p_j) > d(p_{i-1}, p_j) \quad \text{and} \quad d(p_{i+1}, p_j).$$

It is clear that a GSFN pair is also a LSFN pair. Furthermore, it is easy to show that a LSFN pair is antipodal.[20] In Ref. 20 it is shown that a convex *n*gon may have as many as $\lfloor n/2 \rfloor$ GSFN pairs of vertices. For some additional distance properties in convex polygons the reader is referred to Moser[21] and Altman.[22]

Theorem 9. If a simple polygon P is unimodal then every local SFN pair of vertices of P is also a global SFN pair.

Proof. Let p_i, p_j be a local SFN pair of vertices of a unimodal polygon P. It follows that

$$d(p_i, p_j) > d(p_i, p_k) \quad \text{for} \quad k = j-1, j+1$$

and

$$d(p_i, p_j) > d(p_k, p_j) \quad \text{for} \quad k = j-1, j+1$$

Since P is unimodal we have

$$d(p_i, p_k) \leqslant d(p_i, p_{j-1}), \quad k = i, i+1, \ldots, j-1$$

and

$$d(p_i, p_k) \leqslant d(p_i, p_{j+1}), \quad k = j+1, j+2, \ldots, i$$

Therefore

$$d(p_i, p_j) = \max_k \{d(p_i, p_k)\}$$

Similarly it follows that

$$d(p_i, p_j) = \max_k \{d(p_k, p_j)\}$$

Therefore p_i, p_j is a global SFN pair. ∎

Note that the converse of Theorem 9 is not true. Consider the octagon *a–h* formed by (1) defining a square and letting *a*, *c*, *e*, and *g*, the vertices of the square, be four vertices of the octagon, (2) placing vertex *b* at the midpoint of $\overline{ac}$, *d* at the midpoint of $\overline{ce}$, *f* at the midpoint of $\overline{eg}$ and *h* at the midpoint of $\overline{ga}$, and (3) perturbing vertices *b*, *d*, *f*, and *h*, very slightly so that no three consecutive vertices are collinear. Then all the LSFN pairs (namely *a*, *e* and *c*, *g*) are also GSFN pairs and yet the octagon is not unimodal since $f(p_i, p_j)$ is not unimodal for $p_i = b, d, f, h$. Furthermore, the polygon need not be convex either.

4. GEOMETRIC ALGORITHMS AND COMPLEXITY

In this section we investigate the two diameter algorithms in Refs. 11 and 12 in light of the results of the previous sections and then propose algorithms for solving the problems mentioned in the introduction.

4.1. The Diameter

4.1.1. The Algorithm of Snyder and Tang[11]

Snyder and Tang[11] have proposed the following algorithm for finding the diameter of a convex polygon. We denote their algorithm by Algorithm D-1. It turns out this algorithm does not necessarily yield the correct answer.[13]

Algorithm D-1.

Input: A convex polygon $P = p_1, p_2, \ldots, p_n$.
Output: The diameter of P.
Step 1. Select an arbitrary starting vertex; call it p_0.
Step 2. Perform a linear search of the vertices of P, testing adjacent vertices in turn, searching for the vertex with a maximum distance from p_0. Call the new vertex p_1.
Step 3. With p_1 as an "anchor" point, starting at p_0, search clockwise or counter-clockwise for the vertex furthest from p_1. This search is carried out only in the direction in which the distance is increasing.
If the distance decreases in both directions, EXIT with p_0 and p_1 as the extrema determining the diameter; ELSE continue.
Step 4. Find a new point p_2, such that $d(p_1, p_2) > d(p_1, p_0)$, and assign $p_0 \leftarrow p_1$, $p_1 \leftarrow p_2$; then GO TO Step 3.

Sufficient conditions on convex polygons which guarantee that Algorithm D-1 will fail are given in Ref. 13. Several questions arise concerning (a) how "bad" can the answer be, (b) does the algorithm solve another interesting problem for convex polygons and (c) is the algorithm in fact guaranteed to find the diameter for another class of polygons?

The answer to (b) is easy. From the stopping criterion used in the algorithm it follows that the algorithm is guaranteed to find a LSFN pair of vertices of a convex polygon P. In fact not only can the algorithm fail to find the diameter but also it can fail to exit with a GSFN pair. This partially answers (a). We now expand on (a); we show that given a convex $n$gon P with $O(n)$ GSFN pairs, Algorithm D-1 can exit with the closest such pair.

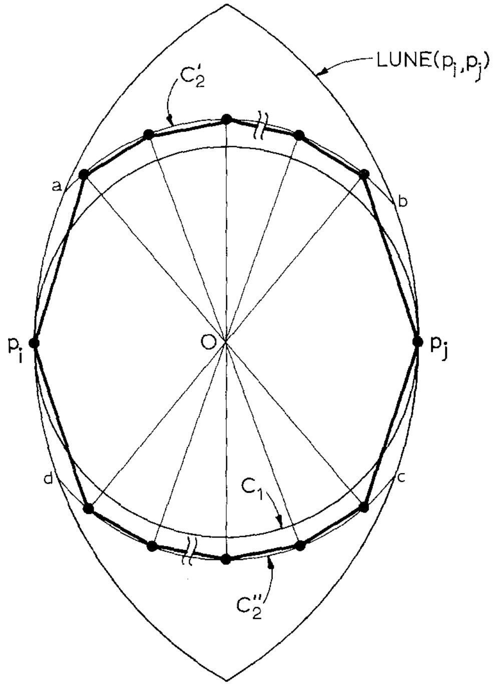

Fig. 6. Algorithm $D - 1$ can exit with the closest GSFN pair of vertices.

Construct a convex polygon, illustrated in Fig. 6, as follows. Place two vertices of P, (p_i, p_j) a unit distance apart, and construct LUNE (p_i, p_j). Let 0 denote the midpoint of $\overline{p_i p_j}$. Construct two circles C_1 and C_2 with center 0, C_1 with radius 1/2 and C_2 with radius r such that $1/2 < r < \sqrt{3}/2$. Let C_2 intersect LUNE (p_i, p_j) at points a, b creating arc C_2' and c, d creating arc C_2''. Now place the remaining vertices in diametrically opposite pairs on C_2' and C_2''. The polygon is clearly convex. Also each diametrically opposite pair on C_2' and C_2'' as well as pair p_i, p_j is a GSFN pair. Therefore there are $n/2$ GSFN pairs when n is even. Now Algorithm D-1 clearly exists with (p_i, p_j), which is the closest GSFN pair.

We turn now to the question of whether there exist interesting classes of polygons for which the algorithm is guaranteed to find the diameter. Since the algorithm tacitly assumes that the polygons are unimodal one hopes that the algorithm will work in this situation. We show that the algorithm fails even for unimodal polygons. In fact we show more; we show that there exist very restricted classes of polygons that are convex and unimodal, and can be circumscribed in a circle, for which the algorithm can fail. Consider the circle of radius one with center 0 and refer to Fig. 7. Let L be the horizontal

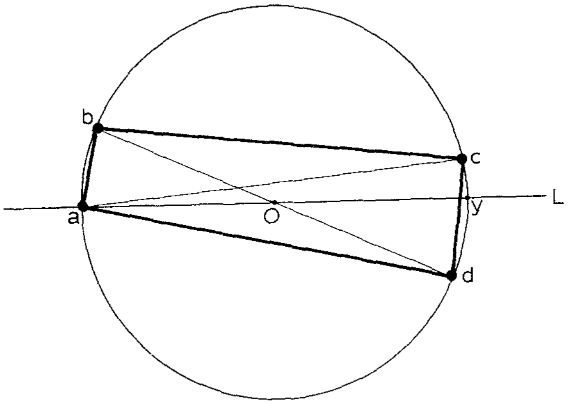

Fig. 7. Algorithm $D-1$ can fail on a convex-unimodal polygon.

line through 0 that intersects the circle at a and y. Let a be the first vertex of P. We choose b so that the arc length from a to b is some small value 2ε, c such that the arc length from c to y equals ε, and d such that the arc length from y to d is 2ε. Clearly the quadrilateral is circumscribed on the circle, it is convex and unimodal. The diameter of P is 2 as specified by $\overline{bd}$. Elementary geometrical observations lead to the following distance relations:

$$d(b, c) < d(b, y) = d(a, d) < d(a, c) < 2$$

from which it follows that Algorithm D-1 exist with the incorrect answer $\overline{ac}$ as the diameter. Thus it would appear that Algorithm D-1 is rather hopeless for finding the diameter of a polygon. Note however that for unimodal polygons, due to Theorem 9, the algorithm finds a "better" solution than for convex polygons in the sense that now the algorithm always exits with a GSFN pair.

4.1.2. The Algorithm of Dobkin and Snyder[12]

We now turn our attention to an elegant class of linear running time algorithms proposed in Ref. 12 to find an area maximizing $k$gon inscribed in a convex $n$gon. A proof for the case $k = 3$ is given in Ref. 12. However, as was shown in Ref. 14 this algorithm cannot be used for the case $k = 2$. For this case the algorithm is intended to find the diameter and is described as Algorithm D-2.

Algorithm D-2.

Input: Convex $n$gon $P = p_1, \ldots, p_n$.
Output: Vertices A, B of the diameter.

Legend: All additions are performed modulo n.

loop α while β: γ repeat is due to Knuth and is equivalent to the ALGOL-60 code

```
loop: α;
      if β then go to L1 else go to L2;
L1: γ;
      go to loop
L2:
```

```
begin A ← p_1; B ← p_2; a ← 1; b ← 2;
    loop
        loop
        while d(p_a, p_b) ⩽ d(p_a, p_{b+1}):
        b ← b + 1;
        repeat
        if d(A, B) < d(p_a, p_b) then begin A ← p_a; B ← p_b; end;
        a ← a + 1;
        if a = b then b ← b + 1;
    while a ≠ 1:
    repeat
end
```

We now prove that Algorithm D-2, unlike D-1, does work for unimodal convex polygons.

Theorem 10. Algorithm D-2 finds the diameter of a convex unimodal $n$gon in $0(n)$ time.

Proof. Consider a polygon $P = p_1, p_2, \ldots, p_n$ and let Algorithm D-2 start at vertex p_i. With p_i as an "anchor" vertex subsequent vertices are scanned until a vertex p_j is found such that $d(p_i, p_{j+1}) < d(p_i, p_j)$. Now $d(p_i, p_j)$ is marked as the ith candidate for the diameter. Since the polygon is unimodal it follows that p_j is the furthest neighbor of p_i. Therefore if we were to repeat this procedure for each vertex p_i, $i = 1, \ldots, n$ and select the largest candidate, the correct answer would clearly be obtained but at a cost of an $0(n^2)$ time complexity. Linearity is obtained in Algorithm D-2 by limiting the search for subsequent "anchor" vertices. In particular when p_{i+1} becomes the "anchor" vertex, vertices are scanned starting from p_j, and all vertices p_k for $i + 2 < k < j$ are ignored in searching for the furthest neighbor of p_{i+1}. Thus to prove that Algorithm D-2 works we still need to show that if p_j is the furthest neighbor of p_i, then all distances $d(p_{i+1}, p_k)$ for $i + 2 < k < j$ are less than or equal to $d(p_{i+1}, p_j)$. Since the polygon is unimodal it is sufficient to show that $d(p_{i+1}, p_{j-1}) \leqslant d(p_{i+1}, p_j)$. Let

$p_j = \max_k \{d(p_i, p_k)\}$ and refer to Fig. 8. We have that $d(p_i, p_j) \geqslant d(p_i, p_{j-1})$. Therefore

$$\angle p_i p_{j-1} p_j \geqslant \angle p_{j-1} p_j p_i$$

Since

$$\angle p_{i+1} p_{j-1} p_j = \angle p_{i+1} p_{j-1} p_i + \angle p_i p_{j-1} p_j$$

and

$$\angle p_{j-1} p_j p_{i+1} = \angle p_{j-1} p_j p_i - \angle p_{i+1} p_j p_i$$

it follows that $\angle p_{i+1} p_{j-1} p_j \geqslant \angle p_{j-1} p_j p_{i+1}$. Therefore $d(p_{i+1}, p_j) \geqslant d(p_{i+1}, p_{j-1})$. ∎

4.2. The All-Furthest-Neighbor Problem

Given a set of n points on the plane (vertices in the case of a polygon) the all-furthest-neighbor problem consists of finding the furthest point for each point, i.e., for each p_i, $i = 1, 2, \ldots, n$ we need to find p_j such that

$$d(p_i, p_j) = \max_k \{d(p_i, p_k)\}$$

Shamos[23] proposes on $0(n \log n)$ algorithm to solve this problem, based on searching edges of the dual of the furthest-point-Voronoi diagram of the set. However it was shown in Ref. 24 that this algorithm does not work and an alternate $0(n \log n)$ algorithm is proposed. No faster algorithms are available for the case of convex polygons. However, from Theorem 10 it follows that for each vertex of a convex unimodal polygon the candidate diameter is a furthest-neighbor pair. Therefore Algorithm D-2 solves the all-furthest-neighbor problem for convex unimodal polygons in $0(n)$ time.

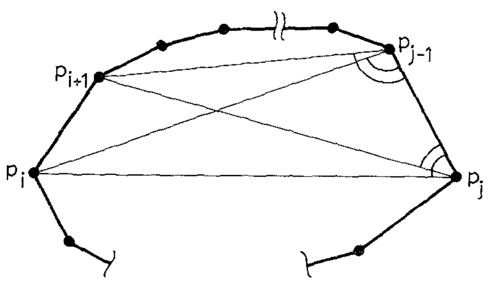

Fig. 8. Illustrating the proof of Theorem 10.

4.3. The All-Nearest-Neighbor Problem

Given a set of n points on the plane (vertices in the case of polygons) the all-nearest-neighbor problem consists of finding the nearest point to each point. Note that the solution to this problem also solves the closest-pair problem, Shamos and Hoey[25] have shown that this problem can be solved in $0(n \log n)$ time. Lee and Preparata[26] show that if the set is a convex polygon this problem can be solved in $0(n)$ time. Although the algorithm in Ref. 26 is linear it is nevertheless quite involved. We now give a trivially simple $0(n)$ algorithm for solving the all-nearest-neighbor problem for unimodal polygons.

Algorithm ANN

Input: A unimodal polygon $P = p_1, p_2, \ldots, p_n$.
Output: The closest vertex to every vertex.
Step 1. For each vertex p_i exit with p_{i+1} as its nearest neighbor if $d(p_i, p_{i+1}) < d(p_i, p_{i-1})$; otherwise exit with p_{i-1}.

Theorem 11. Algorithm ANN solves the all-nearest-neighbor problem for unimodal polygons in $0(n)$ time.

Proof. That the algorithm gives the correct solution follows from Theorem 8 in Section 3. For each vertex p_i its nearest neighbor can be found in constant time and thus the total running time of the algorithm is $0(n)$. ∎

Nor only is this algorithm much simpler than that in Ref. 26 but it works for nonconvex polygons.

For some additional properties of distances in convex polygons relevant to this problem the reader should refer to Refs. 28 and 29.

5. CONCLUDING REMARKS

A list of open problems and some suggestions for further investigation are presented here.

One area for possible investigation concerns the k-modality properties of convex polygons for $k > 1$. For example, are theorems analogous to Theorem 6 possible such that all distance functions $f(p_i, p_j)$ are k-modal where $k \geqslant 2$? Similar questions arise for other distance functions such as $f(p_i \perp L_{j,j+1})$. Other open problems include questions such as: what is the minimum value of n for which a polygon is bimodal? For the function $f(p_i, p_j)$ it has been shown by Olariu[30] that six is the smallest number.

Another topic for further investigation concerns characterizations of unimodal polygons. For example, the converse of Theorem 9 is not true. Are there additional properties that would make the converse true?

In Section 4.1.2 it was shown that the algorithm of Dobkin and Snyder[12] works for unimodal polygons if they are convex. Until recently it was not known whether the algorithm would work for arbitrary unimodal polygons. N. Tsikopoulos has shown that it will not.[31]

For convex polygons it is an open question whether or not one can solve the all-furthest-neighbor problem in $0(n)$ time. If "furthest" is replaced by "nearest," a linear time complexity is achievable.[26]

Finally, no results are yet available concerning the design of efficient algorithms for detecting whether a simple polygon is unimodal in all of the senses considered in this paper. For the function $f(p_i, p_j)$ Aggarwal and Melville[32] have discovered a linear algorithm for determining whether or not a convex polygon is unimodal. They also solve the all-furthest-neighbor problem in $0(m+n)$ time for an m-modal convex $n$gon. Finally, they also present an algorithm to compute the modality of an m-modal simple polygon in $0(n^{1.695}+m)$ time. A notion closely related to unimodality is monotonicity. A simple polygon P is monotone if there exists a straight line l in direction θ such that the boundary of P can be partitioned into two chains C_{ij} and C_{ji} that are monotonic with respect to l. Let ϕ be the direction orthogonal to θ and let (x, y) be two points on the plane such that $L(x, y)$ is a supporting line of P at p_i in direction ϕ. Then clearly if P is monotone in direction θ it follows that $f(L_{x,y} \perp p_k)$, $k=i, i+1,..., j,..., i$, is unimodal. Preparata and Supowit[33] present an $0(n)$ algorithm that obtains all directions with respect to which P is monotone.

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