International Journal of Computer and Information Sciences, Vol. 12, No. 6, 1983

# Cellular Topology and Its Applications in Image Processing

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Received June 1983; revised October 1983

In this paper the interaction between Minkowski algebra, nondiscrete cellular topologies and some well known basic cellular image processing operations is investigated. It is shown that some useful topological measures can be extracted from these basic image operations and that these operations can be viewed from a nonalgebraic and purely topological point of view.

**KEY WORDS:** Cellular computers; cellular automatons; cellular topology; Minkowski algebra; neighborhood transform; cellular image processing.

## 1. INTRODUCTION

There seems to be little awareness within the mathematical community as to the many diverse applications of topology in the areas of pattern recognition and image processing. In fact, applications of topology outside the realm of mathematics is generally thought to be either rare or nonexisting. We hope to remedy this situation a little by presenting several image processing operations that permit image processing algorithms to be considered from a purely topological point of view. In particular, we use topological notions to define highly parallel neighborhood operations for filtering, data compression and image matching. Some of these operations will be novel, while others date back to Minkowski's development of the geometry of numbers in 1897.

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[3] The author wishes to acknowledge support for this research by the Air Force Office of Scientific Research/AFSC, United States Air Force, under Contract F49620-C-82-0035. The United States Government is authorized to reproduce and distribute reprint for governmental purposes nothwithstanding any copyright notation hereon.



0091-7036/83/1200-0433$03.00/0 

A *digital image* is obtained by sampling brightness, reflectivity or absorption values of photographs, outdoor scenes or real objects and quantizing these values to a finite number of binary places. The resulting digitized image can then be viewed as a subset of Euclidean $n$-space having integer valued coordinates. The elements of these image lattices are called *cells* or *points*.

High speed computers are used to manipulate and transform these digital images in order to compress or smoothen data, identify, classify and/or track objects, or to perform other desired tasks. A typical sequence of these manipulations, or *image processing algorithms*, may consist of noise filtering, thresholding and background removal. This sequence may then be followed by such processes as thinning, edge detection or skeletonizing in order to obtain shape descriptors and/or achieve data compression. The geometric properties inherent in the transformed objects usually serve as a basis for object classification. Of special importance are such topological properties as nearness,(17) connectivity,(2,16,18) path connectivity,(18) genus,(8,11) homotoppy,(13) and dimension.(5) In this paper we are not directly concerned with these notions. However, we would like to note that many of these concepts are either mathematically imprecise or not well defined in much of the current image processing literature. This is partially due to the fact that many researchers view digital images as "subspaces" of $E^n$. In this case a notion such as connectivity for discrete sets is contrary to the standard mathematical definition of connectedness. It will follow from our definition of cellular topologies that "discrete" sets can indeed have connectivity properties relating to the real world.

Current image processing algorithm development is not based on an efficient mathematical structure that is designed specifically for image manipulation, feature measurement extraction and analysis. In general, each researcher develops his own set of ad hoc image processing tools, usually at great expense. Furthermore, many image processing algorithms presently employed are often extensions of one dimensional signal processing algorithms heuristically applied to the $n$-dimensional case. These methods generally require excessive computer time and memory since each computer instruction typically affects only one or two pieces of data of the entire image array. The topological neighborhood approach discussed in this paper is highly parallel in nature and can act on an image as a whole. It thus has real time capabilities and is highly suited to processor size and time constrained applications. Also, the topological interpretation suggested here has the potential of providing a geometric basis for a rigorous mathematical foundation of an image processing algebra.

The approach is based on von Neumann's concept of cellular automata.(3,19) Each point or cell of an image is subjected to a sequence of transformations, where the transformed value of the cell depends on the values of the cells in its neighborhood. The transforms (operations) are defined in terms of the topology of cellular spaces and their origins can be traced to the works of Blaschke,[1] and Minkowski.[6,7,10]

Finally, it is our hope that this paper will catch the attention of mathematicians interested in applying their knowledge to the many problems faced by the image processing and pattern recognition community.

## 2. CELLULAR TOPOLOGIES

The cellular automata of von Neumann[3] and Moore,[12] and digital images share a common framework. Each image point can be viewed as a cell in a given state. By defining neighborhood relationships and cell transition functions on an image in terms of these neighborhood relationships, the configuration of the cell states in an image can be changed to a new configuration.

In this essay we shall consider transition functions that will change images in a predictable and useful fashion. Because of the dependence of these transition functions on the values of cells within the neighborhood of a range cell, these functions are also known as *neighborhood transforms*. Before describing neighborhood relations and operations, we shall define and give examples of topological spaces that provided the underlying geometric foundation for image processing.

Cellular automata and cellular image processing algorithms are based on the notion of neighborhood operations. This suggests that we study the topology of images by first choosing a subset of cells to serve as neighborhoods, and then carry over unaltered all other notions of topology. For this reason topological spaces will be defined in terms of basic open neighborhoods.

Henceforth $R$ and $Z$ will denote the set of real numbers and integers, respectively. Let $Z^k$ denote the Cartesian product of $k$-copies of $Z$. With each $p = (p_1, p_2, \ldots, p_k) \in Z^k$ we associate the cell with center $p$, $c(p)$, defined by

$$c(p) = \{q \mid q = (q_1, q_2, \ldots, q_k),\ q_i \in R,\ |p_i - q_i| \leqslant \tfrac{1}{2} \text{ and } 1 \leqslant i \leqslant k\}$$

The set of all $k$-dimensional cells is denoted by $C^k$ and defined as $C^k = \{c(p) \mid p \in Z^k\}$. Topological spaces obtained by endowing $C^k$ with a topology are called *cellular spaces*.

We now give examples of cellular spaces based on neighborhood configuration most frequently used in present day neighborhood image processing.

### 2.1. The von Neumann Topology

Let $L = \{-1, 0, 1\}$ and $p = (p_1, p_2, ..., p_k) \in Z^k$. The neighborhood $N[c(p)]$—or simply $N(p)$—of $c(p) \in C^k$ is defined by

$$N(p) = \begin{cases} \{c(p)\} \text{ if } \sum_{i=1}^{k} p_i \text{ is odd} \\ \{c(p_1, ..., p_i - 1, p_i + n, p_{i+1}, ..., p_k) \mid 1 \leqslant i \leqslant k \text{ and } n \in L\} \\ \text{otherwise} \end{cases}$$

The collection $N = \{N(p) \mid p \in Z^k\}$ is a neighborhood basis for a topology and the topology thus derived is called the *von Neumann topology* for $C^k$.

Suppose $p = (i, j) \in Z^2$ and $q = (i, j, k) \in Z^3$. Then the possible neighborhoods of $c(p)$ and $c(q)$ are shown in Figs. 1a and 1b, respectively. The cruciform neighborhood shape in Fig. 1a is due to von Neumann,[3] hence the name for this topology.

**Definition.** A cell (point) in the von Neumann space is called *even* if the sum of its coordinates is even, otherwise it is called *odd.*

We note that if $c(p)$ is odd, then the closure of $N(p)$ is a cruciform neighborhood, and if $c(p)$ is even, then $\{c(p)\}$ is closed.

It is also not difficult to show that in the von Neumann topology, sets are connected if and only if they are path-connected. However, we shall not use this fact and therefore omit its proof. As an example, the set $A \subset C^2$ shown in Fig. 2a is connected and hence path-connected. On the other hand, the configuration $C = A \cup B$ shown in Fig. 2b is not connected since $A \cap \mathrm{Cl}(B) = \mathrm{Cl}(A) \cap B = \varnothing$, where Cl denotes closure. This separation due to diagonally adjacent cells has resulted in numerous investigations [see Refs. (15–18)].


Fig. 1. (a) The von Neumann neighborhoods if $k = 2$, and (b) if $k = 3$.

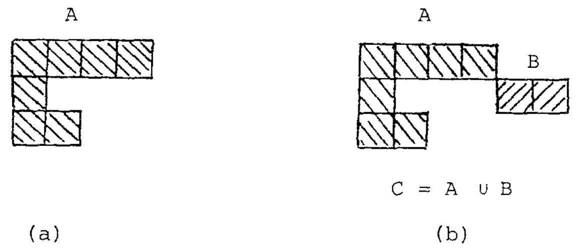



Fig. 2. Connectivity in the von Neumann topology: the set $A$ in (a) is connected and the set $C$ in (b) is separated.

## 2.2. The Moore Topology

The Moore topology for $C^k$ is obtained by defining the neighborhood of a cell $c(p)$, $p=(p_1,\ldots, p_k)$, as follows:

$$N(p)=\begin{cases}\{c(p_1+l_1, p_2+l_2,\ldots, p_k+l_k) \mid l_i \in L\} \text{ if } p_i \text{ is even} \\ \text{for each } i=1, 2,\ldots, k \\ \{c(p)\} \text{ if } p_i \text{ is odd for at least one } i\end{cases}$$

As before, the collection $N=\{N(p) \mid p \in Z^k\}$ forms a neighborhood basis. The topology generated by this neighborhood system is called the *Moore topology.*

The two possible neighborhood configuration of this system for $C^2$ are shown in Fig. 3 where the shaded cell represents $c(p)$.

In a Moore space a point is *even* if each of its coordinates is even, otherwise it is called *odd.* Implications of using the neighborhood configuration $N(p)$ ($p$ even) for cellular automata were first investigated by Moore.[3,12]

Note that in the Moore topology two diagonally adjacent cells can form a connected set.

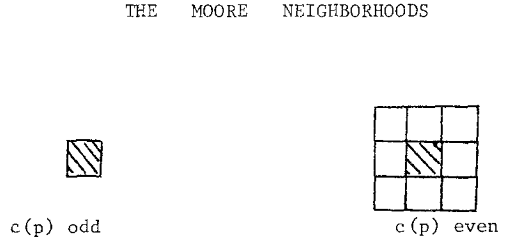



Fig. 3. The Moore neighborhoods if $k=2$.

### 2.3. The $2^k$-Topology

Thus far every topology for $C^k$ was generated by two distinct neighborhood configurations. We now define a more complex topology that is based upon a greater variety of neighborhood configurations.

The set $C^1$ becomes a topological space if for each $p \in Z$ we define neighborhoods in $C^1$ by

$$N(p) = \begin{cases} \{c(p-1), \quad c(p), \quad c(p+1)\} & \text{if } p \text{ is even} \\ \{c(p)\} \quad \text{if } p \text{ is odd} \end{cases}$$

The topology generated is called the two-neighborhood topology for $C^1$.

For $k > 1$ we define the $2^k$-neighborhood topology (or simply $2^k$-topology) for $C^k$ as the product topology of the two-neighborhood space $C^1$. The four different neighborhood configurations for the $2^2$-space $C^2$ are shown in Fig. 4. The shaded cell represents the cell $c(i, j) \in N[c(i, j)]$. In this topology some diagonally adjacent cells are connected.

## 3. NEIGHBORHOOD TRANSFORMS

If $x = c(p)$ and $y = c(q)$ are two points in a cellular space, then their *sum* is the point $x + y$ defined by $x + y = c(p + q)$. The *difference* $x - y$ is defined by $x - y = c(p - q)$, and if $k \in Z$, then the *scalar product* $kx$ is the point $kx = c(kp)$.

**Definition.** Let $X = C^k$ be a cellular space and $W = \{c(p_1, p_2, ..., p_k) \in X \mid p_i \in L\}$. The *window* at $x \in X$ is defined as $W(x) = \{x + y \mid y \in W\}$. The basic neighborhood system associated with $X$ is said to *satisfy the window condition* if it satisfies the following two properties:

(1) $N(x) \subset W(x)$ for every $x \in X$

(2) if $z \in N(x) \cap N(y)$, then $x \in W(z)$ and $y \in W(z)$

Hereafter we shall always assume that $X$ is a cellular space whose neighborhood system satisfies the window condition. Note that the discrete space and our examples of cellular spaces satisfy the window condition.

Fig. 4. The $2^k$ basic neighborhoods if $k = 2$.

**Note.** The power set of $X$ will be denoted by $\mathrm{P}(X)$. If $A \subset X$, then $\mathrm{Int}(A)$ denotes the interior of $A$, $\mathrm{Cl}(A)$ the closure of $A$, and $C(A)$ the complement of $A$.

**Definition.** An *image* is an element of $\mathrm{P}(X)$ and the complement of the image is its *background.* An *image transformation* on $X$ is a function $\mathrm{P}(X) \rightarrow \mathrm{P}(X)$.

Definitions of a set in terms of a given set can be viewed as image transformations. Thus Int, Cl and $C$ can all be thought of as image transformations. For instance, Int: $\mathrm{P}(X) \rightarrow \mathrm{P}(X)$ is the transformation defined by Int: $A \rightarrow \mathrm{Int}(A)$.

Composition of image transformations will be denoted by "·". In particular, $\mathrm{Int} \cdot \mathrm{Cl}(A) = \mathrm{Int}[\mathrm{Cl}(A)]$. If $T_1, \ldots, T_n$ are image transformations, then the composition $T_n \cdot T_{n-1} \cdot, \ldots, \cdot T_1$ will be denoted by $\prod_{i=1}^{n} T_i$. If all the $T_i$'s represent the same transformation $T$, then for $n \geqslant 0$ we define $T^n = \prod_{i=1}^{n} T_i$ with $T^0(A) = A$.

**Definition.** A *window configuration* $w$ is a function $w$: $X \rightarrow P(X)$ such that $x \in w(x) \subset W(x)$.

If $B \subset X$, let $I(w, B)$ denote the set $I(w, B) = \{w(x) \mid w(x) \cap B \neq \varnothing\}$. A *window relation* for $A \subset X$ is a set $R(w, A)$ obtainable from $I(w, A)$ and/or $I[w, C(A)]$ by intersection, union or complementation. Thus, $C(I[w, C(A)]) = \{w(x) \mid w(x) \subset A\}$ and $C[I(w, A)] = \{w(x) \mid w(x) \cap A = \varnothing\}$ are two examples of window relations for $A$. In particular, $I(w, A)$ is a window relation for $A$.

A *local operator* or *window operator* is a function $L$: $I(w, X) \rightarrow \mathrm{P}(X)$ such that $\varnothing \neq L(w(x)) \subset W(x)$.

**Definition.** An *elementary neighborhood transform* $T$ is an image transformation defined locally in terms of a window configuration $w$, a window relations $R(w, \cdot)$ and a local operator $L$. Explicitly, $T(A) = \{b \mid b \in L(w(x)),\ w(x) \in R(w, A)\}$.

A neighborhood transform is an image transformation that can be expressed as a composition of elementary neighborhood transforms.

The importance of neighborhood transforms in image processing is their potential for simultaneous (parallel) application to the neighborhood of every cell of an image. More specifically, elementary neighborhood transforms are the only image transformations that can be performed by a cellular image processing computer with window $W$ as one-step transformations in parallel.

**Example.** Some obvious examples of elementary neighborhood transforms are Int, Cl and $C$. For example, if $L[w(x)] = w(x) = N(x)$, then

$T(A) = \{b \mid b \in L[w(x)],\ w(x) \in C(I[w, C(A)])\} = \{b \mid b \in N(x), N(x) \subset A\} = \mathrm{Int}(A)$. Similarly, let $w(x) = N(x)$ and $L[w(x)] = \{x\}$. Then $\mathrm{Cl}(A) = \{b \mid b \in L[w(x)],\ w(x) \in I(w, A)\}$.

The transform $G(A) = \{b \mid b \in N(a),\ a \in A\}$ is an elementary neighborhood transform. In order to verify this, let $w(x) = \{x\}$, $L[w(x)] = N(x)$ and $R(w, A) = I(w, A)$. Note that $G(A)$ is the smallest open set containing $A$.

The transform $F = C \cdot G \cdot C$ is an example of a neighborhood transform that transforms $A$ into the largest closed set contained in $A$. In particular, we have the relationship

$$F(A) \subset A \subset G(A)$$

We cnclude this section by defining two fundamental neighborhood transforms.

**Definition.** The *dilation transform* $D$ is the image transformation defined by $D(A) = G(A) \cup \mathrm{Cl}(A)$. The *shrinking transform* is the image transformation $S$ defined $S = C \cdot D \cdot C$.

**Theorem 1.** The transforms $D$ and $S$ are elementary neighborhood transforms.

*Proof.* (1) Let $w(x) = \{x\}$, $L[w(x)] = N(x) \cup \{y \mid x \in N(x) \cap N(y)\}$ and $T(A) = \{b \mid b \in L[w(x)],\ w(x) \in I(w, A)\}$. Since $N(x) \subset G(A)$ and $\{y \mid x \in N(x) \cap N(y)\} \subset \mathrm{Cl}(A)$ whenever $w(x) \in I(w, A)$, $T(A) \subset D(A)$.

In order to show that $D(A) \subset T(A)$, let $b \in D(A)$. Then either $b \in G(A)$ or $b \in \mathrm{Cl}(A)$ or both. If $b \in G(A)$, then $b \in N(x)$ for some $x \in A$. But $N(x) \subset L[w(x)] \subset T(A)$ for every $x \in A$. Hence $b \in T(A)$. If $b \in \mathrm{Cl}(A)$, then $N(b) \cap A \neq \varnothing$. Hence there is an $x \in N(b) \cap A$ and, therefore, $x \in N(b) \cap N(x)$ with $\{x\} = w(x) \in I(w, A)$. Thus, $b \in \{y \mid x \in N(y) \cap N(x)\} \subset T(A)$.

(2) To prove that $S$ is an elementary neighborhood transform, reverse the roles of $w$, $L$ and $R$ by letting $w(x) = N(x) \cup \{y \mid x \in N(x) \cap N(y)\}$, $L[w(x)] = \{x\}$ and $T(A) = \{b \mid b \in L[w(x)],\ w(x) \in C(I[w, C(A)])\}$. The argument for showing that $T(A) = S(A)$ is now similar to the argument in part (1). ∎

**Theorem 2.** $a \in D(b)$ if and only if $b \in D(a)$.

*Proof.* If $a = b$, there is nothing to prove. Suppose $b \neq a \in D(b)$. Then either $a \in G(b)$ or $a \in \mathrm{Cl}(b)$. If $a \in G(b)$, then $a \in N(b)$. Hence $b \in \mathrm{Cl}(a) \subset D(a)$. If $a \in \mathrm{Cl}(b)$, then $b \in N(a) \subset D(a)$. Thus $a \in D(b)$ implies that $b \in D(a)$. The proof of the converse is analagous. ∎

The next theorem is a direct consequence of the definition of $D$ and $S$.

**Theorem 3.** Let $A \subset X$. Then:

(1) $S(A) = F(A) \cap \mathrm{Int}(A)$

(2) $S(A) \subset \mathrm{Int}(A) \subset A \subset \mathrm{Cl}(A) \subset D(A)$

(3) If $A$ is open (closed), then $D(A)$ and $S(A)$ are closed (open)

EXAMPLE. Let $X = C^2$ have the von Neumann topology and let $A \subset X$ be the shaded region shown in Fig. 5a. The region $A$ represents a digitized version of a tank. The "holes" (unshaded cells) in the tank and the isolated shaded cell represent "noise" (artifacts) introduced by the digitization process.

Figures 5b and c show the effects of applying $G$ and $D$ to $A$, respectively. Note that $D(A)$ is simply connected (both as a subset of $E^2$ and as a subset of $C^2$!). Roughly speaking, $D$ removes small holes and produces a "smoother" looking version of $A$. On the negative side, the shaded isolated cell has been dialated and the cavity beneath the gun has disappeared.

The shrinking transform $S$ has just the opposite effect (Fig. 5d). The isolated cell has disappeared and cavities have been enlarged. Note also that the gun, which represents a cavity in $C(A)$, has vanished.

The principle behind the above dilation and shrinking transforms is quite old (see Ref. 6) and can be traced to Minkowski's work on the geometry of numbers.[10] To be more explicit, if $A$ and $B$ are two sets in $E^k$, then the Minkowski sum of $A$ and $B$ is defined vectorially as $A \times B = \{a + b \mid a \in A, b \in B\}$ [see Ref. (6), p. 13]. *Minkowski substraction* is defined in terms of complementation [Ref. (6), p. 142] by $A/B = C[C(A) \times (-B)]$, where $-B = \{b \mid -b \in B\}$. The connection between Minkowski's algebra and neighborhood transformation is given by the next theorem.

**Theorem 4.** If $X$ is a von Neumann space, then $D(A) = A \times N(0)$ and $S(A) = A/N(0)$.

*Proof.* Let $w(x) = \{p \mid |p - x| \leqslant 1\}$. If $a \in X$ is even, then $w(a) = N(a)$ and if $a$ is odd, then $w(a) = \mathrm{Cl}[N(a)]$. Thus, $\bigcup_{a \in A} w(a) = G(A) \cup \mathrm{Cl}(A) = D(A)$. On the other hand,

$$A \times N(0) = \{a + b \mid b \in N(0), \quad a \in A\}$$

$$= \{p \mid p \in N(0) \times \{a\}, a \in A\} = \{p \mid p \in w(a), a \in A\} = \bigcup_{a \in A} w(a)$$

Therefore, $A \times N(0) = D(A)$.

Using this fact and complementation, we obtain $A/N(0) = C[C(A) \times N(0)] = C\{D(A)]\} = S(A)$. ∎

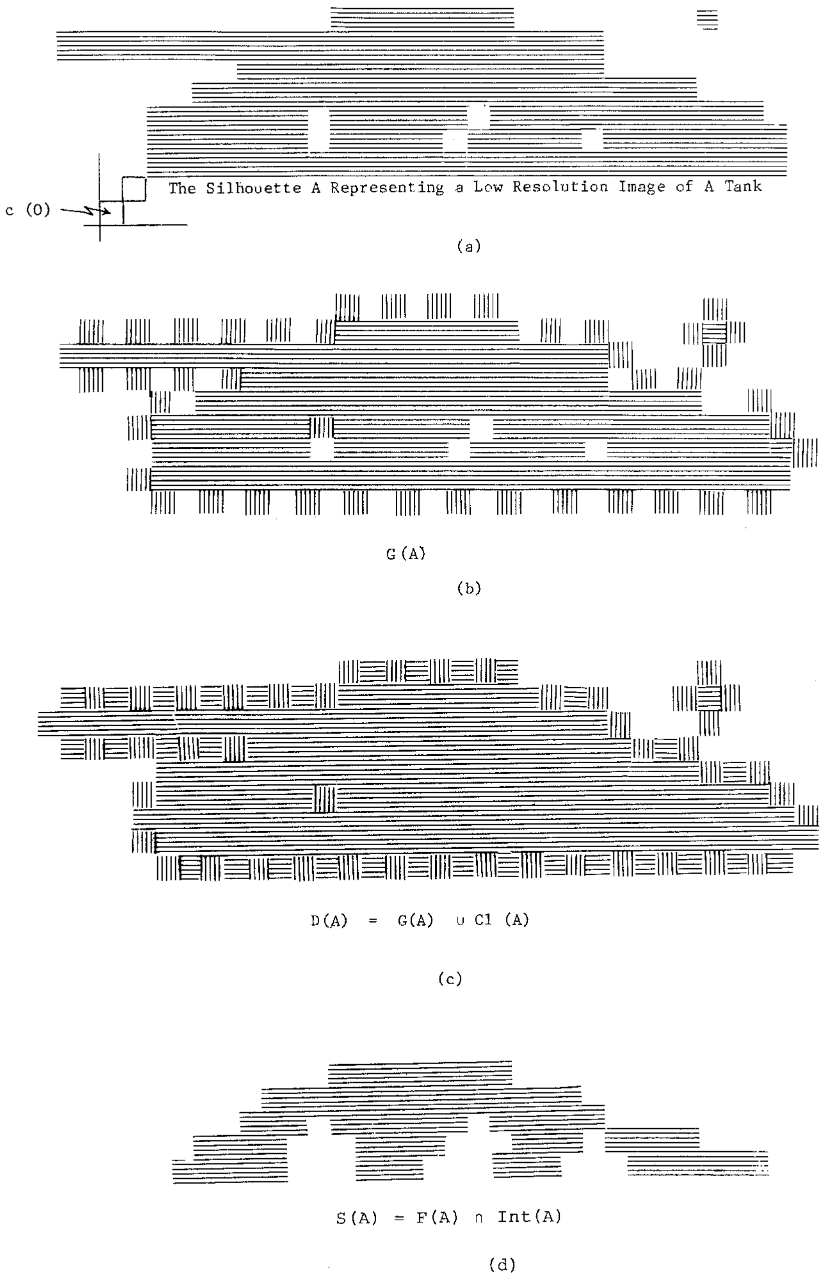



Fig. 5. The effects of the transforms $G$, $D$, and $S$. (a) The original image $A$; (b) the transformed image $G(A)$; (c) the transformed image $D(A)$; and (d) the transformed image $S(A)$.

Obviously, similar results can be obtained for other cellular spaces.

Several theorems in this section classified some specific image transforms as neighborhood transforms. In practice it is often very difficult and tedious to determine whether or not a given image transform is a neighborhood transform. For this reason it would be valuable to have solutions to the following problem:

*Problem 1.* Develop necessary and sufficient conditions for classifying neighborhood transforms.

## 4. TOPOLOGICAL FILTERS

As we observed in the last section, the transform $D$ smoothens images, removes small holes, but enlarges the image and isolated cells beyond their actual size. At the other extreme, $S$ erodes an image and creates large cavities. Thus it becomes natural to consider compositions of these transforms in order to obtain an intermediate effect of these two extremes.

**Definition.** If $A \subset X$, then the *exterior hull* of $A$ is denoted by $E(A)$ and defined by $E(A) = S \cdot D(A)$. The *interior hull* $I(A)$ of $A$ is defined as $I(A) = D \cdot S(A)$.

**Example.** Figure 6a shows the exterior hull of $A$, where $A$ is the image shown in Fig. 5a. The neighborhood transform $E$ is a noise filter since it removes small noisy holes and smoothens the image. Note that $E(A)$ is

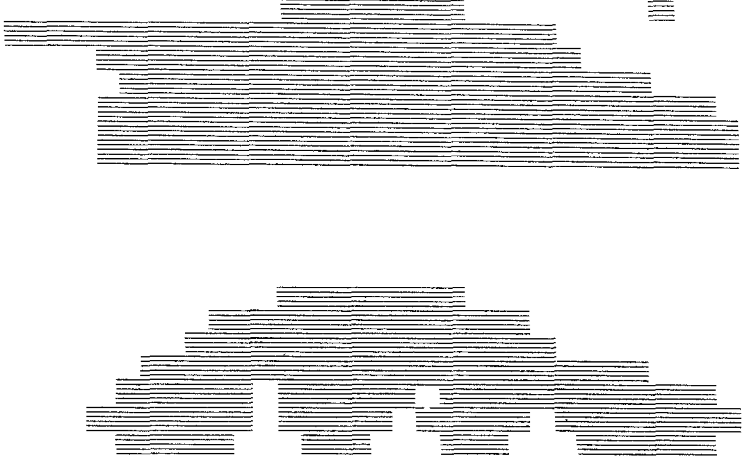


Fig. 6. (a) The exterior filter $E(A)$ and (b) the interior filter $I(A)$.

about the same size as $A$ and that the isolated cell has not been enlarged. Similar observations can be made for the interior hull of $A$ (Fig. 6b). Even though cavities still appear enlarged, erosion is not as severe as for $S(A)$. Observe also that the front and rear end of the tank appear more rounded (smoother).

Because of its ability to remove concavities without reducing convexities, the transform $E$ is a good noise filter in a noisy environment such as real-world active target tracking, where targets often appear as convex objects. Active sensors are generally capable of measuring the intensity of the reflected signal as well as the phase or time delay between the transmitted and received pulses. The intensity (energy) of the return provides a relative measure of the surface reflectivity and the phase can provide either an absolute or relative measure of range. Thus a 3-dimensional image (in $C^3$) can be obtained from actively sensed data with the third coordinate obtained from the range information.

Figure 7a represents an actively sensed image of a tank in a noise free environment. Figure 7b is the thresholded (clipped) version of Fig. 7a at about $\frac{1}{2}$ the height of the tank.

Adding random Gaussian distributed noise to the scence represented in Fig. 7a results in Fig. 8a. The thresholded version of Fig. 8a corresponding to Fig. 7b is shown in Fig. 8b.

Endowing $C^3$ with the von Neumann topology and applying the transform $I$ to Fig. 8a yields the scence shown in Fig. 9a. The tresholded version of Fig. 9a is shown in Fig. 9b. Of special interest are the differences of the planar regions in the threshold planes of Figs. 7b and 9b since these regions could be used for image matching. Note for example that Fig. 9b exhibits a hole in the region determined by the threshold plane.

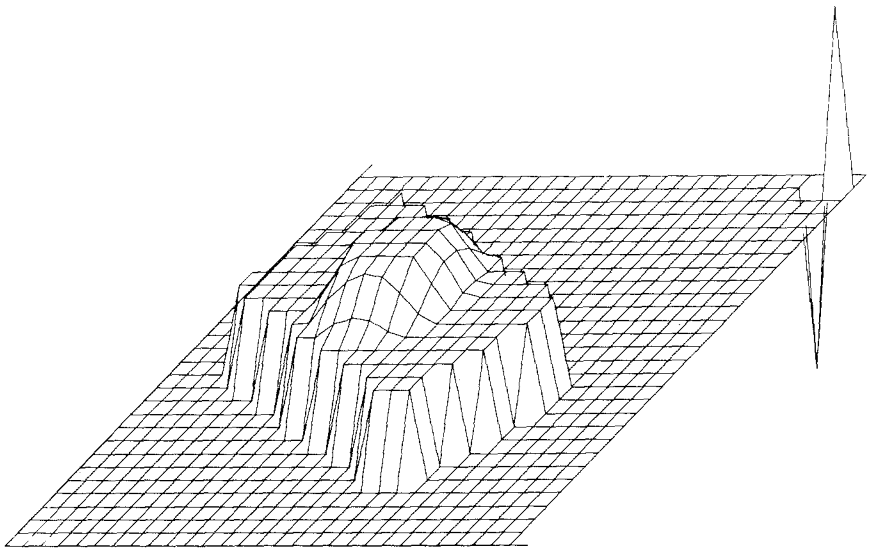
Using the transform $E$ instead of $I$ produces the configurations shown in Figs. 10a and b. Observe the likeness of the images in the threshold planes of Fig. 7b and 10b.

Unlike $E$, such filberts as local averaging do not exhibit the ability to totally remove noise which is either uncorrelated or correlated in one direction (Fig. 11a and b).

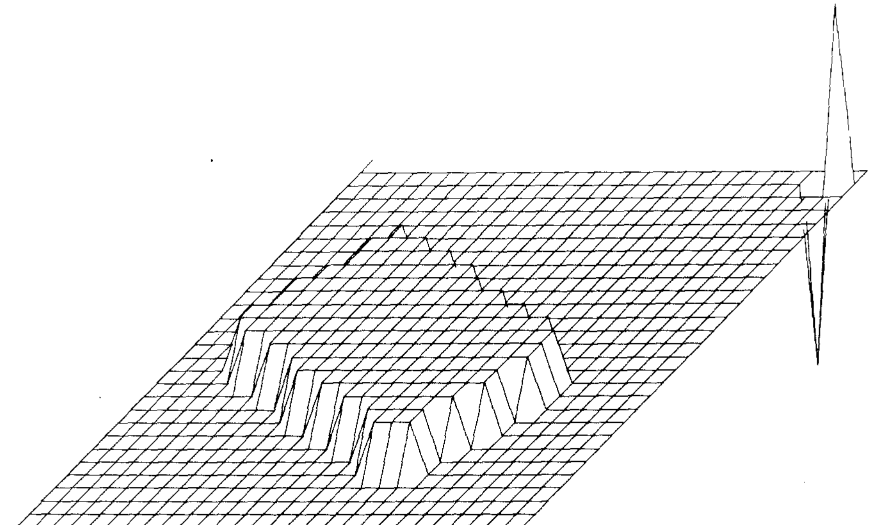
An important and interesting problem is the classification of images that remain invariant under neighborhood transforms. Before considering images that remain invariant under the transforms $E$ and $I$, we recall the definition of regular sets:

**Definition.** An open set $A$ is called *regular* if $A = \mathrm{Int}[\mathrm{Cl}(A)]$. A closed set $A$ is called *regular* if $A = \mathrm{Cl}[\mathrm{Int}(A)]$.

The properties of regular sets summarized in the next theorem are well known [cf. Ref. (4)].



(a)



(b)

Fig. 7. (a) An actively sensed image in a noise free environment and (b) a thresholded version of Fig. 7a.

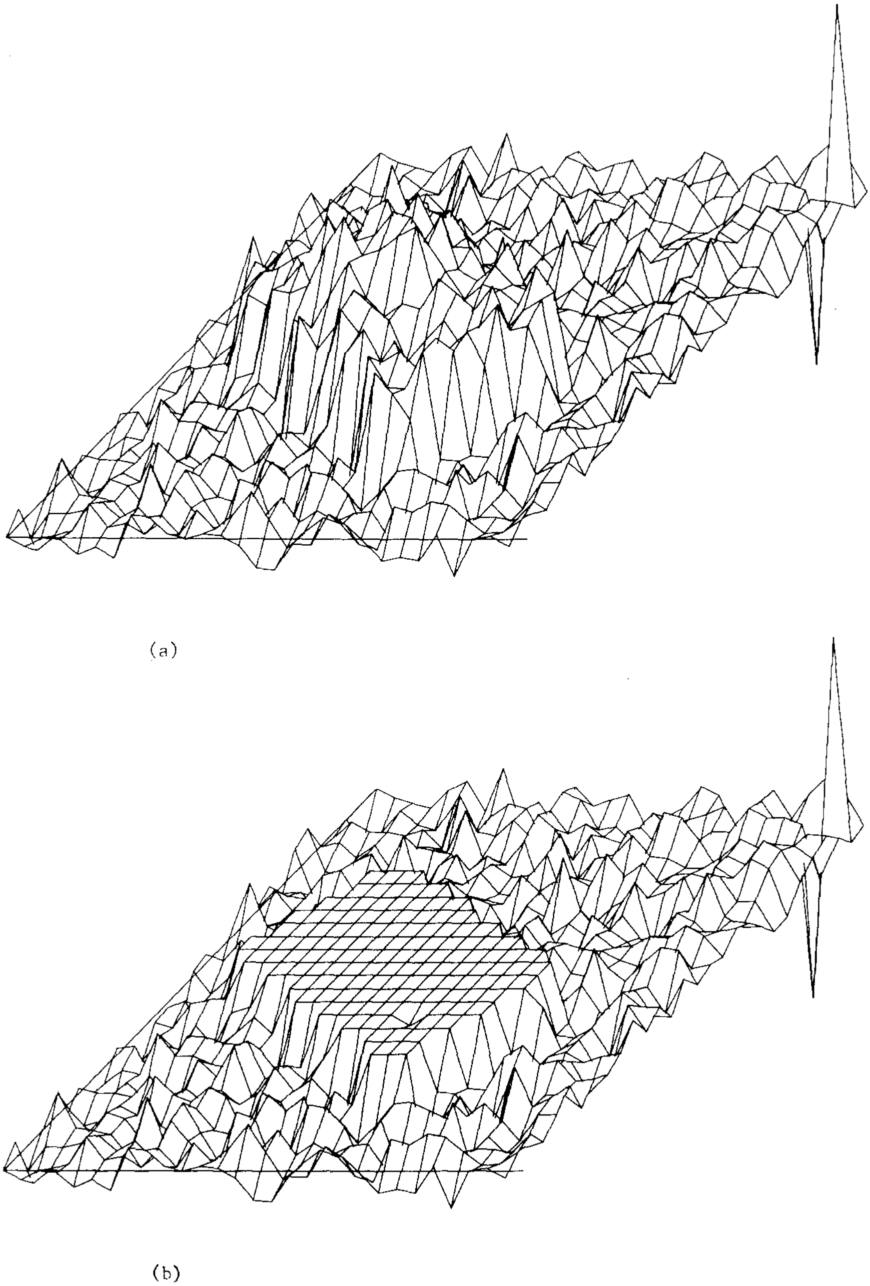
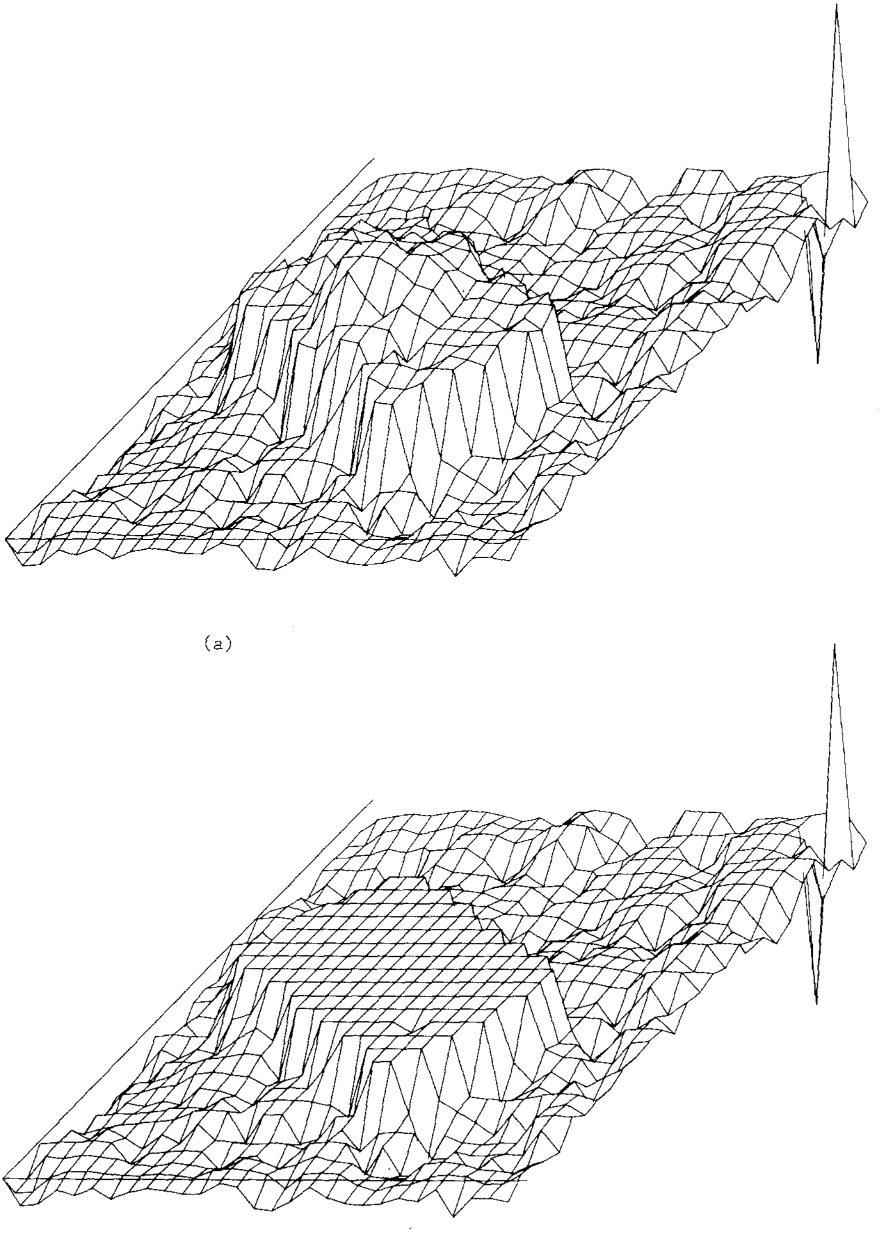


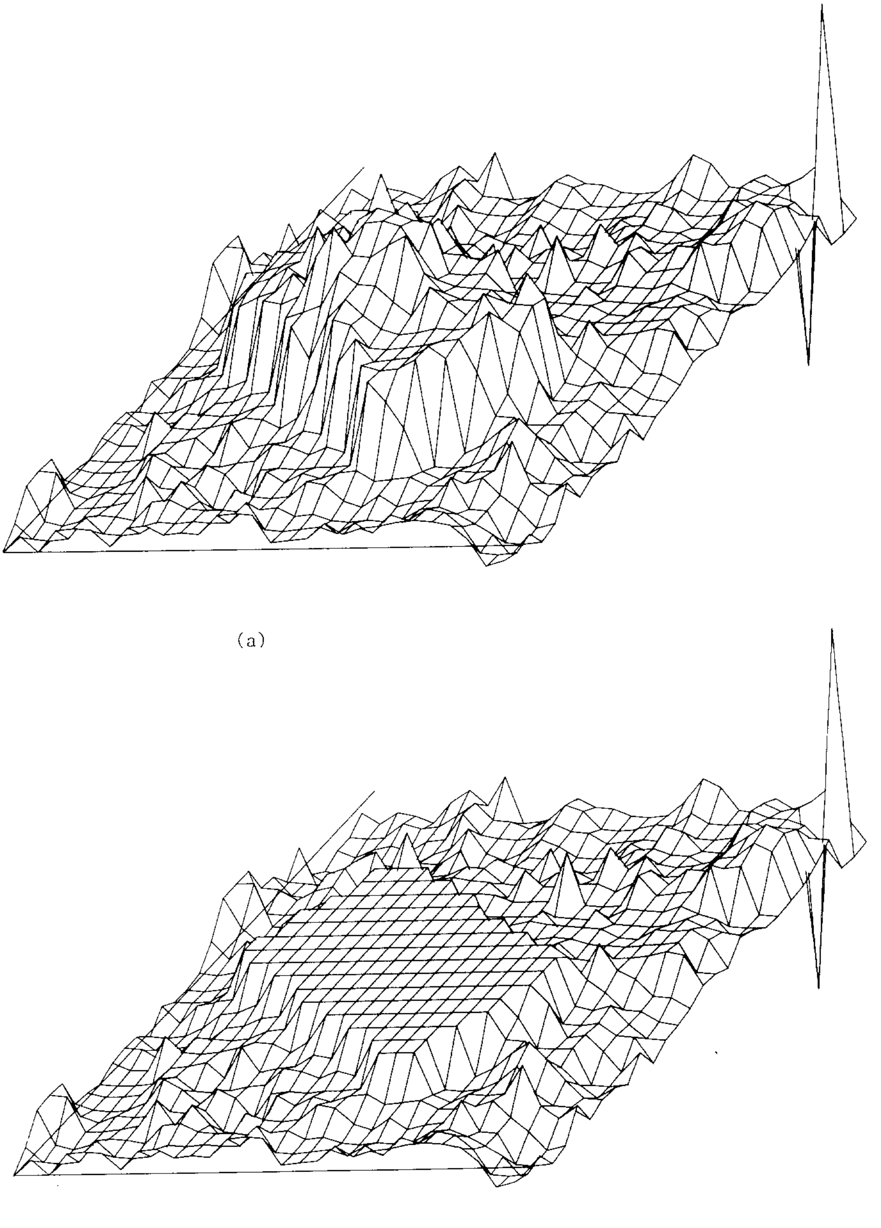

Fig. 8. (a) Gigure 7a in noisy environment and (b) thresholded version of Fig. 8a.



(a)

(b)

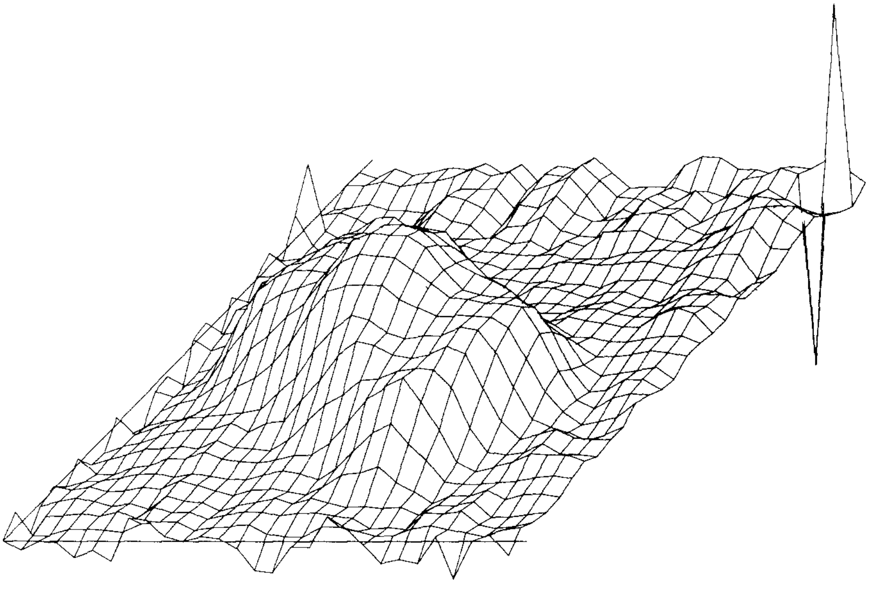
Fig. 9. (a) Figure 8a after applying the filter I and (b) the thresholded version of Fig. 9a.



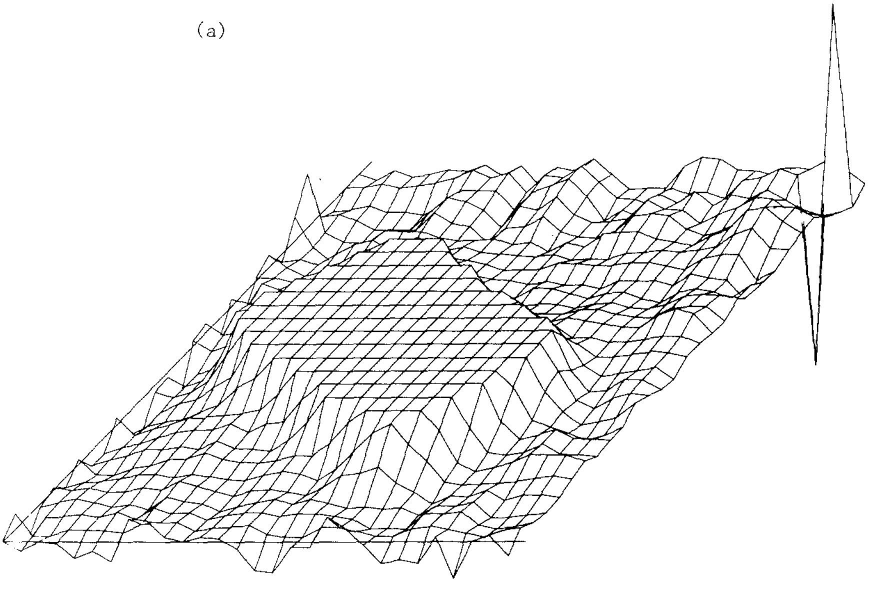
(a)

(b)

Fig. 10. (a) Figure 8a after applying the filter $E$ and (b) The thresholded version of Fig. 10a.



(a)



(b)

Fig. 11. (a) Figure 8a after local averaging and (b) The thresholded version of Fig. 11a.

**Theorem 5.**

(1) If $A$ is closed, then $\mathrm{Int}(A)$ is regular.

(2) If $A$ is open, then $\mathrm{Cl}(A)$ is regular.

(3) If $A$ is open (closed) and regular, then $C(A)$ is closed (open) and regular.

(4) If $A$ and $B$ are open and regular sets, then $A \subset B$ if and only if $\mathrm{Cl}(A) \subset \mathrm{Cl}(B)$.

(5) If $A$ and $B$ are closed and regular sets, then $A \subset B$ if and only if $\mathrm{Int}(A) \subset \mathrm{Int}(B)$.

(6) If $A$ and $B$ are closed and regular sets, then $A \cup B$ is a closed and regular set.

(7) If $A$ and $B$ are open and regular sets, then $A \cap B$ is an open and regular sets.

**Theorem 6.** If $A$ is open and regular, then $E(A) = A$. If $A$ is closed and regular, then $I(A) = A$.

*Proof.* If $A$ is open, then $D(A) = \mathrm{Cl}(A)$. But then $E(A) = S \cdot D(A) = \mathrm{Int}[\mathrm{Cl}(A)]$ and the result follows since $A$ is regular.

If $A$ is closed, then $S(A) = \mathrm{Int}(A)$. Hence $I(A) = D \cdot S(A) = \mathrm{Cl}[\mathrm{Int}(A)]$. Again by regularity, $\mathrm{Cl}[\mathrm{Int}(A)] = A$. This proves Theorem 6.

By combining Theorems 5 and 6 further results may be obtained. For instance, if $A$ is closed, then $\mathrm{Int}(A)$ remains invariant under $E$. Similarly, if $A$ is open, then the interior hull of $\mathrm{Cl}(A)$ is $\mathrm{Cl}(A)$.

The proof of Theorem 6 in conjunction with the next theorem show that $E$ and $I$ map open (closed) sets to open (closed) sets.

**Theorem 7.** If $A$ is closed, then $E(A) = F[G(A)]$ and if $A$ is open, then $I(A) = G[F(A)]$.

*Proof.* If $A$ is closed, then $D(A) = G(A)$. Hence $E(A) = S \cdot D(A) = F[G(A)]$. Similarly, if $A$ is open, then $S(A) = F(A)$ and $I(A) = D \cdot S(A) = G[F(A)]$. This proves Theorem 7.

**Theorem 8.** $S(A) \subset I(A) \subset A \subset E(A) \subset D(A)$.

*Proof.* $E(A) = S \cdot D(A) \subset D(A)$ by Theorem 3. Now let $a \in A$. Then $N(a) \subset D(A)$. Hence $a \in \{a\} = S[N(a)] \subset S \cdot D(A)$. Therefore, $A \subset E(A)$. The proof that $S(A) \subset I(A) \subset A$ is similar and therefore omitted. This establishes Theorem 8.

**Theorem 9.** $E^2(A) = E(A)$ and $I^2(A) = I(A)$.

*Proof.* Obviously $I^2(A) = I[I(A)] \subset I(A)$. On the other hand,

$$I \cdot I(A) = D[S \cdot D[S(A)]) = D\{E[S(A)]\} \supset D[S(A)] = I(A)$$

A similar argument shows that $E^2(A) = A$. This establishes Theorem 9.

**Theorem 10.** $I[D(A)] = D(A)$ and $E[S(A)] = S(A)$.

*Proof.* Clearly, $I[D(A)] \subset D(A)$. But $I[D(A)] = D[S \cdot D(A)] = D[E(A)] \supset D(A)$. Hence $I[D(A)] = D(A)$.

Similarity, $S(A) \subset E[S(A)]$ and $E[S(A)] = S[D \cdot S(A)] = S[I(A)] \subset S(A)$. Therefore $E[S(A)] = S(A)$. This proves Theorem 10.

The above theorems classify many but not all images that remain unchanged by the transforms $E$ and $I$. This leaves us with the following problem.

*Problem 2.* Classify all images that remain invariant under the transformations $E$ and $S$.

## 5. NEIGHBORHOOD REDUCTION AND RECONSTRUCTION

Neighborhood reduction is a data compression device. This technique, defined in terms of neighborhood transforms, reduces the image to a few cells. Reacquisition of the original image from the reduced image can be accomplished via neighborhood reconstruction. Thus, neighborhood reduction compresses data without loss of information. The basic idea of neighborhood reduction and reconstructions stems from Miller's research on image convolutions.[9]

**Definition.** Let $i$ be a nonnegative integer and $A \subset X$. The $i$th reduction of $A$ is defined as $R_i(A) = S^i(A) \cap C \cdot I \cdot S^i(A)$. The *neighborhood reduction* of $A$ is the disjoint union $R(A) = \bigcup R_i(A)$.

Since $S^i(A) = \{a \in A \mid D^i(a) \subset A\}$ and $R_i(A) = S^i(A) \cap C\{D[S^{i+1}(A)]\}$, we can express $R_i(A)$ in set theoretic terms as $R_i(A) = \{a \mid D^i(a) \subset A$ and $D^{i+1}(b) \not\subset A$ whenever $b \in D(a)\}$. This last expression provides a clearer view of the geometric manipulations that are required in order to obtain $R_i(A)$. The next example should elucidate the importance of neighborhood reductions.

**Example.** Let $A$ be the image shown in Fig. 6a with the isolated shaded cell deleted. Figure 12 shows $A$ (dotted outline) superimposed on

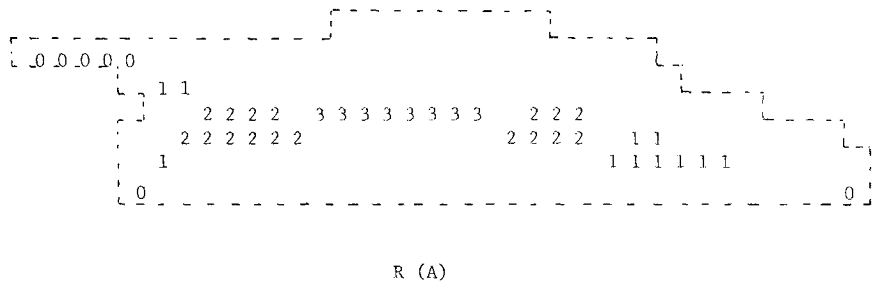



Fig. 12. The neighborhood reduced image $R(A)$.

$R(A)$. In order to show the position of $R_i(A)$ with respect to $A$, we labeled the cells of $R_i(A)$ by $i$ ($i = 0, 1, 2, 3$). Note that $R_i(A) = \varnothing$ for $i > 3$.

An important observation is that the reduction $R(A)$ consists of only 44 cells. This is less than 1/3 of the 159 cells composing the image $A$. Another useful observation is that $R(A)$ can be viewed as a weighted skeleton of $A$, where a cell of the skeleton has weight $i$ if it belongs to $R_i(A)$. As can be inferred from the reconstruction procedure, the largest numbered cells correspond to the bulk of the image, while the cells with small weights correspond to the finer features. In tracking an object such as a moving target, it is often sufficient to only track cells with large weights, thus allowing further data reduction.

**Definition.** The *interior radius* $m(A)$ of $A$ is defined as $m(A) = \max\{i \mid D^i(a) \subset A,\ a \in A\}$ if the maximum exists, otherwise $m(A) = \infty$.

The exterior radius $M(A)$ of $A$ is defined as $M(A) = \min\{i \mid A \subset D^i(a),\ a \in A\}$ if the minimum exists, otherwise $M(A) = \infty$.

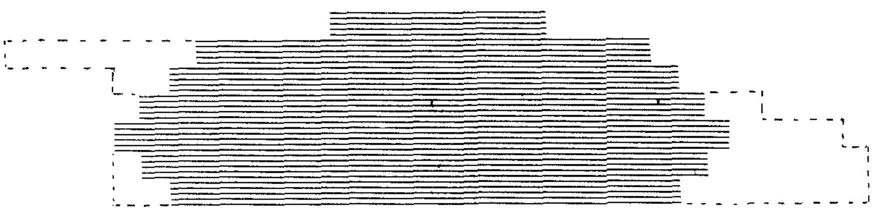
It follows that $0 \leqslant m(A) \leqslant M(A)$. Also, $M(A) < \infty$ if and only if $A$ is finite. However it is possible that $m(A) < \infty$ even though $M(A) = \infty$. Unless otherwise stated, we shall always assume that $m(A) < \infty$. This complies with the spirit of image processing by computer where images are always finite.

**Definition.** Let $A \subset X$ and $0 \leqslant k \leqslant m(A)$. The $k$th partial reconstruction $r_k$ of $A$ from $R(A)$ is defined as

$$r_k(A) = \bigcup_{i=m(A)-k}^{m(A)} D^i[R_i(A)]$$

**Theorem 11.** $r_{m(A)}(A) = A$.

*Proof.* If $p \in R_i(A)$, then $D^i(p) \subset A$. Hence $D^i(R_i(A)) \subset A$ for each $i \geqslant 0$ and, therefore, $r_{m(A)}(A) \subset A$.



$r_1(A)$

Fig. 13. The partially reconstructed image $r_1(A)$.

In order to show that $A \subset r_{m(A)}(A)$, let $A_i = \{a \mid \text{there exists } b \in A \text{ with } a \in D^i(b) \subset A\}$. Since $A = A_0$, $A = \bigcup A_i$. Now let $a \in A$ and $k = \max\{i \mid a \in A_i\}$. Thus $a \in D^k(b) \subset A$ for some $b \in A$. Suppose there is a $c \in D(b)$ such that $D^{k+1}(c) \subset A$. By Theorem 2, $b \in D(c)$. But then $D^k(b) \subset D^{k+1}(c)$. Therefore $a \in D^{k+1}(c) \subset A$ and $a \in A_{k+1}$, contradicting the fact that $k$ was maximal with respect to this property. Hence $b \in R_k(A)$ and $a \in D^k(b) \subset D^k[R_k(A)] \subset r_{m(A)}(A)$. This proves Theorem 11.

It now follows that the definition of $r_k$ provides an algorithmic method of reconstructing $A$—or specific subsets of $A$—knowing only $R(A)$. As an example, Figure 13 shows the first partial reconstruction $r_1(A)$ of $A$, where $R(A)$ is as in the previous example.

## 6. NEIGHBORHOOD METRICS

A *neighborhood metric* is a metric between two images $A$ and $B$ that can be computed in terms of neighborhood transforms. The metric we are about to define was first proposed by Blaschke,[1] and is based on Minkowski's idea of engulfing by dilation.[6,10]

**Definition.** The *Minkowski distance* between two images $A$ and $B$ in a cellular space is defined as

$$d(A, B) = \min\{k \mid A \subset D^k(B) \text{ and } B \subset D^k(A)\}$$

It is interesting to note that Blaschke actually calls this distance the "neighborhood" distance between A and B. Hadwiger[6] observes that the analog of this distance for general metric spaces is the Hausdorff distance between sets, which was first formulated by Pompeju.[14] Hadwiger also shows that the Minkowski distance has some interesting properties with respect to unions and dialations. We summarize some of this results in the next theorem.

**Theorem 12.** Let $Y \subset P(X)$ consist of finite images and let $A, B, C, D \in Y$. Then

(1) $d$ is a metric on $Y$

(2) $d(A \cup B,\ C \cup D) \leqslant \max\{d(A, C), d(B, D)\}$

(3) $d(A \cup B, A) \leqslant d(A, B)$

(4) $d(a \cup B, C) \leqslant d(A, B) + d(B, C)$

(5) $d(A \times B,\ C \times D) \leqslant d(A, C) + d(B, D)$

(6) $d(A, B) = d(A^k, B^k)$

Perturbations affect the Minkowski metric severly. $d(A, B)$ may be large even though $A$ and $B$ represent the same configuration. This is the case when $A$ occupies a different location than $B$. For this reason the Minkowski metric has not been a useful image matching device. However, the Minkowski distance can be employed in defining a more applicable neighborhood distance.

**Definition.** The *Minkowski similarity* $s$ between images is defined by $s(A, B) = \min\{d[A, T_v(B)] \mid v \in X\}$, where $d$ denotes the Minkowski distance and $T_v(B) = A \times \{v\}$.

Since every "vector" $v$ in $C^k$ can be expressed as a finite sum of unit vectors with integral coefficients, $T_v$ is a neighborhood transform. Hence $s$ is a neighborhood distance. Also, since $s[A, T_v(A)] = 0$ and $s(A, B) \leqslant d(A, B)$, $s$ is translation invariant and a more discriminating masure of similarity than $d$.

Although the Minkowski similarity solves the problem caused by translation, it also introduces a new problem, namely the search for an optimal algorithm that will find $v$ such that $d[A, T_v(B)]$ is minimal. Furthermore, $s$ is neither rotation nor reflection invariant. It would therefore be useful to have a solution of the following problem.

*Problem 3.* Are there neighborhood metrics that are rotation, translation and reflection invariant?

Because of its dependence on the elementary neighborhood transform $D$, the Minkowski distance exhibits various interesting properies some of which were summarized in Theorem 12. Some of these proprties are also inherited by $s$. The solution of the next problem would provide further interesting and useful properties.

*Problem 4.* Find relationships between the Minkowski measures and neighborhood transforms whose elementary factorizations involve $D$ and $S$. What are the implications of these relationships?

We conclude this exposition by posing one final problem concerning neighborhood distances. Solutions to this problem could prove extremely useful in pattern recognition.

*Problem 5.* Define optimal neighborhood metrics between images in terms of the weights of neighborhood reduced image skeletons and list their special properties.

This paper provides only an introduction to the study of the interaction of cellular topology and image processing. Further results and topics relating to this subject could have been included. However, we hope that the topics covered and the level of discussion will provide sufficient stimulus for further exploration of this subject by the mathematical community.

## CONCLUSION

We have shown that there is a firm connection between certain basic image operations, special topological spaces and metrics. To what extend knowledge of such connections will prove useful in actual image processing remains to be seen. However, since such connections do exist, we feel confident that further exploration of this subject will indeed provide new insights and techniques.

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*Printed in Belgium*