A UNIQUENESS THEOREM FOR A SURFACE WITH PRINCIPAL CURVATURES CONNECTED BY THE RELATION $(1-k_1d)(1-k_2d)=-1$

V. A. Toponogov

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Let F be an oriented complete regular surface in three-dimensional Euclidean space E^3. Denote by $k_1(p)$ and $k_2(p)$ $\left(k_1(p) \leq k_2(p)\right)$ the principal curvatures of F at a point p, whose signs are determined by some continuous field $n(p)$ of normals to F. Denote by $K(p)$ the Gaussian curvature of F at p. In the article we prove the following

Theorem. *If the surface F is analytic and there exists a continuous field $n(p)$ of normals to F such that the principal curvatures $k_1(p)$ and $k_2(p)$ with signs determined by the field $n(p)$ obey the condition $(1-k_1d)(1-k_2d)=-1$, then F is a circular cylinder of radius $d/2$.*

The proof of the theorem leans on two lemmas.

Lemma 1. *If under the conditions of Theorem 1 there exists a regular curve on F such that the Gaussian curvature $K(p)$ of F vanishes along the curve, then $K(p) \equiv 0$.*

The proof of the lemma is presented in [1, Lemma 10].

Lemma 2. *If F is a convex surface whose Gaussian curvature $K(p)$ is not identical zero, then*

$$0 < \sup_{p \in F} k_1(p) \geq \inf_{p \in F} k_2(p).$$

PROOF. If F is a closed convex surface, then it is homeomorphic to a sphere and thus there exists at least one umbilical point p_0 at which $k_1(p_0)=k_2(p_0)$; in this case the claim of Lemma 2 is obvious. It remains to consider the case in which F is a complete open convex surface. In this case the surface F is homeomorphic to a plane by virtue of the assumptions of the lemma. We prove the claim of Lemma 2 by way of contradiction. Assume that

$$0 < k_1^0 = \sup_{p \in F} k_1(p) < \inf_{p \in F} k_2(p) = k_2^0. \tag{1}$$

From (1) it follows that there is a number d for which

$$1/k_2^0 < d < 1/k_1^0. \tag{2}$$

The surface F partitions E^3 into domains $D_1(F)$ and $D_2(F)$, one of which is convex. Assume that this is the domain $D_1(F)$. Assign

$$B_2 = \{p \in D_2(F) \mid \rho(p_1F) > d\} \tag{3}$$

and let $B_1 = E^3 \setminus B_2$. Denote by F_d the surface that is obtained from F by indention along the inner normal $n(p)$ of F to the distance d, and denote by φ_d the map $F \to F_d$ that results from the procedure. Then F_d satisfies the following properties:

(α) $F_d < B_1$;
(β) F_d is a smooth regular surface;
(γ) the Gaussian curvature $\widetilde{K}(q)$ of F_d is nonpositive; i.e., $\widetilde{K}(q) \leq 0$ for all $q \in F_d$.

The validity of (α) follows from the definitions of the surface F_d and the domain B_1. Prove (β). It follows from (1) that there are no umbilical points on the surface F. Therefore, in some neighborhood

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of an arbitrary point $p \in F$ we can choose a parameterization $\bar{r} = \bar{r}(u, v)$ of F whose coordinate lines coincide with the curvature lines on F. Let $q \in F_d$, $q = \varphi_d(p)$. In a neighborhood of q the equation of the surface F_d can be written in the form $\tilde{r} = \bar{r}(u, v) + dn(u, v)$. From the Rodriguez formula it follows that

$$\widetilde{r_u} = \overline{r_u} - k_1 d\overline{r_u} = (1 - k_1 d)\overline{r_u}, \quad \widetilde{r_v} = \overline{r_v} - k_2 d\overline{r_v} = (1 - k_2 d)\overline{r_v}. \tag{4}$$

Therefore, $\widetilde{r_u} \times \widetilde{r_v} = (1 - k_1 d)(1 - k_2 d)(\overline{r_u} \times \overline{r_v}) \neq 0$ by virtue of (2); and item (β) is proved.

Now, we let $R_1(p)$ and $R_2(p)$ denote the principal curvature radii of F at the point p, and we denote by $R_1^d(q)$ and $R_2^d(q)$ the principal curvature radii of F_d at the point $q = \varphi_d(p)$. Then the following relations hold

$$R_1^d(q) = R_1(p) - d, \quad R_2^d(q) = R_2(p) - d. \tag{5}$$

From (5) it follows that

$$\widetilde{K}(q) = \frac{1}{R_1(p) - d} \cdot \frac{1}{R_2(p) - d} = \frac{k_1(p)k_2(p)}{[1 - k_1(p)d][1 - k_2(p)d]} \leq 0,$$

since $k_1(p)k_2(p) \geq 0$, $(1 - dk_1) > 0$, and $(1 - dk_2) < 0$. Thus (γ) is proved.

Now, let Φ be a regular surface meeting the following conditions:

(1) the Gaussian curvature $K_\Phi(p)$ of the surface Φ is strictly positive at every point $p \in \Phi$;

(2) Φ is homeomorphic to a plane;

(3) $B_1 \subset D_1(\Phi)$, where $D_1(\Phi)$ is the convex domain bounded by Φ.

We prove existence for such a surface. Since the Gaussian curvature of F is not identically equal to zero, there exists a point $p \in \partial B_2$ at which the Gaussian curvature is strictly positive. Let $q \in B_2$ be a point for which qp is orthogonal to ∂B_2 at p. Denote by C the cone that is generated by all rays issuing from q and tangent to ∂B_2. If the length of the segment qp is sufficiently small, the cone C is strictly convex at the vertex q; i.e., we can draw a plane through q which has no common points with C except the point q. However, then there exists a circular cone C_1, with vertex q and axis qp, containing the cone C and, consequently, the whole surface ∂B_2. Now, introduce a Cartesian coordinate system with origin q such that the positive direction of the axis z coincides with the ray qp. Assume the equation of the cone to be given by $x^2 + y^2 - \operatorname{tg}^2 \alpha \cdot z^2 = 0$ in this coordinate system. Take the upper sheet of the hyperboloid of two sheets defined by the equation $-\frac{x^2+y^2}{a^2} + \frac{(z+b)^2}{b^2} = 1$, where $b/a = \operatorname{tg} \alpha$, and denote it by Φ. The surface Φ possesses properties (1)–(3). Since $F_d \subset B_1$, we have $F_d \in D_1(\Phi)$ and $F_d \cap \Phi = \varnothing$. Let q_1 be a point in F_d and let p_1 be a point in Φ. Move the surface Φ as a rigid body in the direction of the vector $\overline{p_1 q_1}$ until the first moment at which the intersection of the surfaces F_d and Φ becomes nonvoid. The surfaces F_d and Φ are tangent to each other at any point q_0 of the intersection and F_d lies entirely in Φ. But in this case the Gaussian curvature of F_d at the point q_0 is not less than the Gaussian curvature of Φ at the same point q_0; i.e., it is strictly positive. The contradiction obtained proves Lemma 2.

PROOF OF THE THEOREM. If the Gaussian curvature $K(p)$ of the surface F takes values of distinct signs, then by virtue of analyticity of F there exists a regular curve γ along which $K(p)$ vanishes, and Lemma 1 implies the equality $K(p) \equiv 0$. From the preceding equality and the assumptions of the theorem we obtain $k_1(p) \equiv 0$ and $k_2(p) = 2/d$; whence the assertion of the theorem ensues. It remains to eliminate the next two cases:

(1) $K(p) \geq 0$ for all $p \in F$ and $K(p) \equiv 0$;

(2) $K(p) \leq 0$ for all $p \in F$ and $K(p) \equiv 0$.

In the first case, the inequality $k_1(p) \geq 0$ and the equality $(1 - k_1 d)(1 - k_2 d) = -1$ yield

$$0 \leq k_1 < 1/d < 2/d < k_2. \tag{6}$$

From (6) we infer $0 < \sup_{p \in F} k_1(p) < \inf_{p \in F} k_2(p)$, which contradicts Lemma 2. Consider the second case. Let F_d be the surface constructed from F in the same way as in Lemma 2. Define the normal

$\tilde{n}(q)$ to F_d at the point $q = \varphi_d(p)$ by the equality $\tilde{n}(q) = n(p)$. Denote by $\tilde{k}_1(q)$ and $\tilde{k}_2(q)$ the principal curvatures of F_d at the point q. Then from (5) and the choice of the normal $\tilde{n}(q)$ we infer

$$\tilde{k}_1(q) = \frac{-k_1(p)}{1 - k_1 d}, \quad \tilde{k}_2(q) = \frac{-k_2(p)}{1 - k_2 d}, \tag{7}$$

where $q = \varphi_d(p)$. From (7) we have

$$\widetilde{K}(q) = \frac{-k_1(p)}{1 - k_1 d} \cdot \frac{-k_2(p)}{1 - k_2 d} = -k_1(p)k_2(p) = -K(p) \geq 0 \tag{8}$$

and

$$(1 - k_1(q)d)(1 - k_2(q)d) = \left(1 + \frac{k_1(p)d}{1 - k_1 d}\right)\left(1 + \frac{k_2(p)d}{1 - k_2 d}\right) = \frac{1}{1 - k_1 d} \cdot \frac{1}{1 - k_2 d} = -1. \tag{9}$$

From (8) and Theorem 9 of [2, p. 742], it follows that F_d is a convex surface. At the same time equality (9) shows that F_d satisfies the conditions of the theorem. Therefore, case (2) reduces to case (1) that has been excluded earlier. The theorem is proved.

References

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