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SUBMETRIES OF SPACE-FORMS OF NEGATIVE CURVATURE

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INTRODUCTION

A map p of a metric space M into a metric space N is called a submetry if for each point x in M the image of every closed ball centered at x under p is a closed ball of the same radius centered at p(x). The introduction of this notion is justified by the fact that every Riemannian submersion of complete Riemannian manifolds is a submetry.

We establish the following basic results.

THEOREM 1. Every submetry $p\colon M \to N$ of (possibly infinite-dimensional) complete Euclidean spaces M and N can be represented as a composition $p = i_1 \circ p_1$, where p_1 is the orthogonal projection map onto a closed Euclidean subspace M_1 of M and $i_1\colon M_1 \to N$ is an isometry.

THEOREM 2. Every submetry $p\colon M \to N$ of unit spheres in complete Euclidean spaces is an isometry (M and N are endowed with the induced intrinsic metric).

We also note the connection between Riemannian submersion of a special form and the existence of foliations transverse to a given fiber bundle. We prove that the Hopf bundles have no transverse foliations.

1. Submetries of Simply Connected Space-Forms of Nonnegative Curvature

Let M^m, N^n $(m \geqslant n)$ be connected Riemannian C^∞-manifolds. Following [1] we call Riemannian submersion any C^∞-map $p\colon M \to N$ whose differential p_* has constant rank n and preserves the length of the horizontal vectors, i.e., vectors orthogonal to the fibers $p^{-1}(x)$, $x \in N$, of the submersion p. The nonempty fibers $p^{-1}(x)$ give a C^∞-foliation of M of codimension n.

In [2] it is established that if M is a complete space, then such is N, and $p\colon M \to N$ is a locally trivial C^∞-bundle over N. Moreover, for arbitrary points x, y $\in$ N and $\tilde{x}$ in $p^{-1}(x)$ one has the equality

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$$\rho_N(x, y) = \min_{\widetilde{y} \in p^{-1}(y)} \rho_M(\widetilde{x}, \widetilde{y}), \tag{1}$$

where ρ_M and ρ_N are the intrinsic metrics in M and N, respectively. Equality (1) obviously holds if and only if for every point x in M and every positive number r

$$p(B_M(\tilde{x}, r)) = B_N(p(\tilde{x}), r). \tag{2}$$

Here B_M and B_N are closed balls in M and N, respectively. Equality (2) motivates the definition of a submetry given in Introduction.

We remark that if M and N are locally compact complete spaces with intrinsic metric then (2) is equivalent to the (general speaking, weaker) equality

$$p(U_M(\tilde{x}, r)) = U_N(p(\tilde{x}), r), \tag{3}$$

where U_M and U_N are open balls in M and N, respectively.

If m_1, m_2, m_3 are points of the metric space M with metric ρ, the notation $(m_1m_2m_3)$ means that m_2 is different from m_1 and m_3, and $\rho(m_1, m_3) = \rho(m_1, m_2) + \rho(m_2, m_3)$. We say that the space (M, ρ) satisfies the condition of nonoverlapping of shortest paths if from $(m_1m_2m_3)$, $(m_1m_2m_3')$, and $\rho(m_2, m_3) = \rho(m_2, m_3')$ it follows that $m_3' = m_3$.

LEMMA 1. Let p: M → N be a submetry, $m_i \in M$, $n_i \in N$, $p(m_i) = n_i$, i = 1, 2, 3. Suppose $\rho_M(m_1, m_2) = \rho_N(n_1, n_2)$; $\rho_M(m_2, m_3) = \rho_N(n_2, n_3)$, and $(n_1n_2n_3)$. Then $(m_1m_2m_3)$.

Proof. $\rho_N(n_1, n_3) \leqslant \rho_M(m_1, m_3) \leqslant \rho_M(m_1, m_2) + \rho_M(m_2, m_3) = \rho_N(n_1, n_2) + \rho_N(n_2, n_3) = \rho_N(n_1, n_3)$. The first equality follows from the fact that submetries do not increase distance.

Proposition 1. Let p: M → N be a submetry. Then each of the properties of the space M (completeness, intrinsic character of the metric, the fact that two points can be connected by a shortest path, local compactness, nonoverlapping of shortest paths) is inherited by the space N.

Proof. That the first three properties are inherited by N follows from the fact that a submetry does not increase distance and for arbitrary points n_1, n_2 in N and m_1 in $p^{-1}(n_1)$ there is a point m_2 in $p^{-1}(n_2)$ such that $\rho_M(m_1, m_2) = \rho_N(n_1, n_2)$. Next, N inherits the local compactness property since p is both open and closed. Now suppose that $(n_1n_2n_3)$, $(n_1n_2n_3')$, and $\rho_N(n_2, n_3) = \rho_N(n_2, n_3')$. There exist points m_i, i = 1, 2, 3, m_3'' in M such that $p(m_i) = n_i$, $p(m_3') = n_3'$, $\rho_M(m_1, m_2) = \rho_N(n_1, n_2)$, and $\rho_M(m_2, m_3) = \rho_N(n_2, n_3) = \rho_N(n_2, n_3') = \rho_M(m_2, m_3')$. By Lemma 1, $(m_1m_2m_3)$ and $(m_1m_2m_3')$. Since shortest paths in M do not overlap, $m_3' = m_3$, whence $n_3' = p(m_3') = p(m_3) = n_3$.

LEMMA 2. Let M, N be spaces with intrinsic metric, and let p: M → N be a locally isometric covering. If N is locally compact and complete, then such is M, and p is a submetry.

Proof. Since p is locally isometric, the local compactness of N implies the local compactness of M; moreover, every rectifiable curve $\gamma = \gamma(t)$, $0 \leqslant t \leqslant 1$, in M, has length equal to the length of $p \circ \gamma$. Thus, p does not increase distance, because M and N are spaces with intrinsic metric. Consequently, the completeness of N implies that of M. Now since N is locally compact and complete, any two points n_1, n_2 in N can be joined by a shortest path $\bar{\gamma} = \bar{\gamma}(t)$, $0 \leqslant t \leqslant 1$. Let $p(m_1) = n_1$. Then there exists a unique lift $\gamma = \gamma(t)$, $0 \leqslant t \leqslant 1$, of $\bar{\gamma}$ with origin at the point m_1. If $m_2 = \gamma(1)$, then $p(m_2) = n_2$ and, since p does not increase distance, γ is a shortest path joining the points m_1 and m_2. Therefore, $\rho_M(m_1, m_2) = \rho_N(n_1, n_2)$ and p is a submetry, as claimed.

Proposition 2. Let M, N, M_1, N_1 be spaces with intrinsic metric (where M is locally compact and complete) related by a commutative diagram of continuous maps

$$\begin{array}{ccc} M_1 & \xrightarrow{s} & M \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ N_1 & \xrightarrow{q} & N \end{array}$$

where p is a submetry and s, q are locally isometric covering maps. Then N, M_1, N_1 are locally compact complete spaces and s, q, p_1 are submetries.

Proof. By Proposition 1 and Lemma 2, N, M_1, and N_1 are locally compact complete spaces and s and q are submetries.

Let $\gamma(t)$, $0 \leqslant t \leqslant 1$, be a rectifiable curve in M_1. Then the length $(l(p_1 \circ \gamma)$ of the parametrized curve $p_1 \circ \gamma$ is equal to $l(q \circ p_1 \circ \gamma)$ (indeed, q is a covering submetry), i.e., to $l((p \circ s) \circ \gamma)$, and hence it does not exceed $l(\gamma)$ because $p\circ s$ is a submetry. Consequently, p_1 does not increase distance.

Now let $\tilde\gamma=\tilde\gamma(t)$, $0\leqslant t\leqslant a$, be a shortest path in N_1 parametrized by arc-length and joining the points n_1' and n_1. There exists a positive number r such that q maps the ball $B(\tilde\gamma(t), r)$ isometrically onto $B(q\tilde\gamma(t), r)$ for every t, $0\leqslant t\leqslant a$. Further, suppose given a partition $t_0=0<t_1<\ldots<t_n=a$ of the segment $[0,a]$, such that $t_i-t_{i-1}\leqslant r$ and $p_1(m_0)=n_1$. We claim that there exist points $m_0, m_1,\ldots,m_n$ in M_1 such that $p_1(m_i)=\tilde\gamma(t_i)$ and $\rho_{M_1}(m_i, m_{i+1})=\rho_{N_1}(\tilde\gamma(t_i), \tilde\gamma(t_{i+1}))$, $i=0, 1,\ldots, n-1$.

In fact, since $p\circ s$ is an isometry, there is a point m_1 in M_1 such that $(p\circ s)(m_1)=q(\tilde\gamma(t_1))$, $\rho_{M_1}(m_0, m_1)=\rho_N(q(\tilde\gamma(0)), q(\tilde\gamma(t_1)))=t_1$. Then $(p\circ s)(m_1)=q(p_1(m_1))=q(\tilde\gamma(t_1))$. Next, since $\rho_{N_1}(n_1, p_1(m_1))\leqslant\rho_{M_1}(m_0, m_1)=t_1\leqslant r$ and q maps $B(n_1, r)$ isometrically onto $B(q(n_1), r)$, we have $\tilde\gamma(t_1)=p_1(m_1)$. Moreover, $\rho_{M_1}(m_0, m_1)=t_1=\rho_{N_1}(\tilde\gamma(0), \tilde\gamma(t_1))$. In a similar manner we find the points $m_2, m_3,\ldots,m_n$. From the fact that p_1 does not increase distance it follows, using the proof of Lemma 1, that

$$\rho_{M_1}(m_0, m_n)=\sum_{i=0}^{n-1}\rho_{M_1}(m_i, m_{i+1})=\sum_{i=0}^{n-1}(t_{i+1}-t_i)=a=\rho_{N_1}(n_1, n_1').$$

Moreover, $p_1(m_0)=n_1$ and $p_1(m_n)=n_1'$. Hence, p_1 is a submersion, as claimed.

Proposition 3. Let $p\colon M\to N$ be a submetry. Suppose nonoverlapping of shortest paths holds in M. Let $\tilde l=\tilde l(t)$ and $l=l(t)$, $a<t<b$ $(a, b\in\bar R)$, be geodesics in M and N parametrized by arc-length, $p(\tilde l(t_i))=l(t_i)$, $i=1, 2$, $a<t_1<t_2<b$, and let $l|_{[t_1,t_2]}$ be the unique shortest path in N joining the points $l(t_1)$ and $l(t_2)$. Then $p(\tilde l(t))=l(t)$, $a<t<b$.

Proof. For t', t'', where $t_1\leqslant t'<t''\leqslant t_2$, we obtain $t_2-t_1=\rho(l(t_1), l(t_2))\leqslant\rho(p(\tilde l(t_1)), p(\tilde l(t')))+\rho(p(\tilde l(t')), p(\tilde l(t'')))+\rho(p(\tilde l(t'')), p(\tilde l(t_2)))\leqslant\rho(\tilde l(t_1), \tilde l(t'))+\rho(\tilde l(t'), \tilde l(t''))+\rho(\tilde l(t''), \tilde l(t_2))\leqslant(t'-t_1)+(t''-t')+(t_2-t'')=t_2-t_1$. Consequently, all inequalities become equalities and

$$\rho(p(\tilde l(t')), p(\tilde l(t'')))=\rho(\tilde l(t'), \tilde l(t''))=t''-t'.$$

Therefore, $(p\circ\tilde l)(t)$, $t_1\leqslant t\leqslant t_2$, is a shortest path in N, parametrized by arc-length, joining the points $l(t_1)$ and $l(t_2)$. By the uniqueness of the shortest path $l(t)$, $t_1\leqslant t\leqslant t_2$, we get $l(t)=(p\circ\tilde l)(t)$, $t_1\leqslant t\leqslant t_2$. Now using Lemma 1 and Proposition 1 we obtain the needed assertion.

LEMMA 3. Let $p\colon M\to N$ be a submetry where M, N are complete Euclidean spaces or unit spheres in complete Euclidean spaces, and let $m\in M$, $n\in N$, $p(m)=n$. Then for every geodesic $l=l(t)$, $t\in R$, with $l(0)=n$, there is a geodesic $\tilde l=\tilde l(t)$, $t\in R$, with $\tilde l(0)=m$ in N covering l (l, and $\tilde l$ are parametrized by arc-length). Moreover, p maps $\tilde l$ isometrically onto l, and $p^{-1}(n)$ lies in the complete totally geodesic subspace $H_{\tilde l}^{\perp}$ of codimension one in M, orthogonal to $\tilde l$.

Proof. For the proof it suffices to take a point $l(t_0)$, $0<t_0<\pi/2$, on l, then choose a point m_0 in $p^{-1}(l(t_0))$ such that $\rho(m, m_0)=t_0$, and construct the geodesics $\tilde l=\tilde l(t)$, $t\in R$, in M for which $\tilde l(0)=m$, $\tilde l(t_0)=m_0$. By Proposition 3, $\tilde l$ is the sought-for geodesic. Let $H_{\tilde l}^{\perp}$ be the big hypersphere in M (if M is a sphere) or a hyperplane in M (if M is Euclidean space) which passes through m orthogonal to $\tilde l$.

Let M be a sphere, $m_1=\tilde l(\pi/2)$, and $m_{-1}=\tilde l(-\pi/2)$. Then $p^{-1}(n)$ lies outside the open balls $U(m_1, \pi/2)$ and $U(m_{-1}, \pi/2)$ (i.e., in $H_{\tilde l}^{\perp}$); in fact, p is a submetry which maps $\tilde l$ isometrically onto l, and so

$$\rho_M(m, m_1)=\rho_M(m, m_{-1})=\rho_N(n, l(\pi/2))=\rho_N(n, l(-\pi/2)).$$

If M is Euclidean space, then for every real number t, $p^{-1}(n)$ lies outside the open ball $U(\tilde l(t), |t|)$, and hence in $H_{\tilde l}^{\perp}$, as claimed.

LEMMA 4. Let $p\colon M\to N$ be the submetry of Lemma 3. If $m\in M$, $n\in N$, $p(m)=n$, l_1, l_2 are two geodesics in N with origin at n and $\tilde l_1$, $\tilde l_2$ are two geodescis in M with origin at m covering ℓ_1, ℓ_2 (constructed in Lemma 3), then $\angle(l_1, l_2)=\angle(\tilde l_1, \tilde l_2)$. In particular, the geodesic $\tilde l$ with origin at m covering ℓ whose existence is asserted in Lemma 3 is uniquely determined.

Proof. Let $l_i^+=l_i(t)$, $l_i^-=l_i(-t)$, $t\geqslant 0$, $i=1, 2$, be two rays on the geodesic l_i, and $\tilde l_i^+$, $\tilde l_i^-$ the corresponding rays on $\tilde l_i$. Since $\tilde l_i$ is mapped isometrically onto l_i and p does not increase distance,

$$\angle(\tilde l_1^+, \tilde l_2^+)\geqslant\angle(l_1^+, l_2^+)=\pi-\angle(l_1^+, l_2^-)\geqslant\pi-\angle(\tilde l_1^+, \tilde l_2^-)=\angle(\tilde l_1^+, \tilde l_2^+),$$

and so $\angle(\tilde{l}_1^+, \tilde{l}_2^+) = \angle(l_1^+, l_2^+)$. The lemma is proved.

If $\tilde{L} = \{\tilde{l}_\alpha, \alpha \in A\}$ is a family of pairwise orthogonal geodesics $\tilde{l}_\alpha$ in M with origin at m, we let $H_{\tilde{L}}^{\perp}$ and $H_{\tilde{L}}$ denote the intersection of all $H_{\tilde{l}_\alpha}^{\perp}$ (see Lemma 3) and, respectively, the totally geodesic subspace in M passing through m, such that the tangent spaces to $H_{\tilde{L}}^{\perp}$ and $H_{\tilde{L}}$ at m yield a direct sum decomposition of the tangent space to M at m.

LEMMA 5. Let $L = \{l_\alpha, \alpha \in A\}$ be a family of geodesics in N pairwise orthogonal at $n \in N$, such that the closed convex hull of the set L equals N. Let $\tilde{L}$ be the uniquely defined family $\{\tilde{l}_\alpha, \alpha \in A\}$ of geodesics $\tilde{l}_\alpha$ in M with origin at m, with $\tilde{l}_\alpha$ covering l_α. Then p maps $H_{\tilde{L}}$ isometrically onto N, and $H_{\tilde{L}}^{\perp} = p^{-1}(n)$.

Proof. First, let us show that p maps $H_{\tilde{L}}$ onto N. Let $n_1 \in N$, $l = l(t)$, $t \in R$, be the geodesic in N with origin at n passing through n_1, and $\tilde{l}$ the geodesic over l with origin at m. Further, let $e_\alpha = \dot{l}_\alpha(0)$, $\tilde{e}_\alpha = \dot{\tilde{l}}_\alpha(0)$, $\alpha \in A$, and $e = \dot{l}(0)$, $\tilde{e} = \dot{\tilde{l}}(0)$ be the unit tangent vectors to the corresponding geodesics. By Lemma 4, we have the equality of inner products

$$\langle \tilde{e}_\alpha, \tilde{e} \rangle = \langle e_\alpha, e \rangle = a_\alpha, \sum_{\alpha \in A} a_\alpha^2 = 1.$$

Therefore, $\tilde{e} = \sum_{\alpha \in A} a_\alpha \tilde{e}_\alpha$, and so $\tilde{e}$ is tangent to $H_{\tilde{L}}$ at the point m, and $l = \tilde{l}(t)$, $t \in R$ is a geodesic in $H_{\tilde{L}}$. If $l(t) = n_1$ then $p(\tilde{l}(t)) = n_1$.

If n_1, n_2 belong to N and lie on the geodesics $l_1 = l_1(t)$, $l_2 = l_2(t)$, $t \in R$, respectively, then the corresponding geodesics $\tilde{l}_1$, $\tilde{l}_2$ lie in $H_{\tilde{L}}$, and, by Lemma 4, their unit tangent vectors satisfy $\langle e_1, e_2 \rangle = \langle \tilde{e}_1, \tilde{e}_2 \rangle$. Hence, $\rho_M(m_1, m_2) = \rho_N(n_1, n_2)$, if m_1, m_2 are points on $\tilde{l}_1$, $\tilde{l}_2$ such that $p(m_i) = n_i$, $i = 1, 2$.

Now let us show that $p^{-1}(n) = H_{\tilde{L}}^{\perp}$. The inclusion $p^{-1}(n) \subset H_{\tilde{L}}^{\perp}$ follows from Lemma 3. Suppose now that m_1 is not contained in $p^{-1}(n)$, i.e., $p(m_1) = n_1 \neq n$. Let $p(m_2) = n_1$, where m_2 is a point in $H_{\tilde{L}}$. Applying the already proven assertion to the point m_2 instead of m and taking into account that p maps $H_{\tilde{L}}$ isometrically onto N, we conclude that $m_1 \in p^{-1}(n_1) \subset H_{\tilde{L}}^{\perp}(m_2)$, where $H_{\tilde{L}}^{\perp}(m_2)$ intersects $H_{\tilde{L}}$ orthogonally at m_2. We consider two cases.

Let M, N be Euclidean spaces. In this case $H_{\tilde{L}}^{\perp}(m_2)$ and $H_{\tilde{L}}^{\perp}$, being Euclidean subspaces of M and orthogonal complements to one and the same (complete) subspace $H_{\tilde{L}}$, but at different points m and m_2, do not intersect. Hence, m_1 is not contained in $H_{\tilde{L}}^{\perp}$.

Let M and N be unit spheres in Euclidean spaces. We prove that $H_{\tilde{L}} = M$. Assume the contrary. Then $H_{\tilde{L}}^{\perp}(m)$, where m is an arbitrary point in $\tilde{L}$, does not reduce to one point. If m_1 is a point in $H_{\tilde{L}}^{\perp}(m)$ with $\rho_M(m, m_1) < \pi/2$, then m_1 does not belong to $H_{\tilde{L}}^{\perp}(m')$ provided $m' \in H_{\tilde{L}}$, $m' \neq m$, $m' \neq -m$ (where $-m$ designates the point diametrically opposing m). Hence, in view of the already established inclusion $p^{-1}(n) \subset H_{\tilde{L}}^{\perp}$, $p(m_1) = p(m)$ or $p(m_1) = p(-m)$. Obviously, the set

$$\{m_1 \in H_{\tilde{L}}^{\perp}(m) \mid \min(\rho(m_1, m), \rho(m_1, -m)) < \pi/2\}$$

is dense in $H_{\tilde{L}}^{\perp}(m)$. Hence, since the sets $p^{-1}(p(m))$ and $p^{-1}(p(-m))$ are closed,

$$H_{\tilde{L}}^{\perp}(m) = p^{-1}(p(m)) \cup p^{-1}(p(-m)).$$

These sets are closed, nonempty and disjoint. This implies that $H_{\tilde{L}}^{\perp}(m)$ is disconnected, which is impossible. Thus, $p^{-1}(n) = \{m\} = H_{\tilde{L}}^{\perp}$. Lemma 5 is proven.

Theorems 1 and 2 formulated in Introduction are obvious consequences of Lemma 5, more precisely of its proof.

THEOREM 3. Every submetry of complete Riemannian subspaces of equal positive curvature is a Riemannian covering.

Proof. Let p: M → N be the given submetry, s: M_1 → M and q: N_1 → N universal Riemannian coverings. We obtain a commutative diagram of continuous maps

$$\begin{array}{ccc} M_1 & \xrightarrow{s} & M \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p,} \\ N_1 & \xrightarrow{q} & N \end{array} \tag{4}$$

where the existence of p_1 follows from the lifting theorem (see [3]). By Proposition 2, p_1 is a submetry. By Theorem 2, it is an isometry, in particular a Riemannian submersion. Then obviously p is a Riemannian covering, as asserted.

In Sec. 2 we consider submetries of locally Euclidean Riemannian spaces.

2. Category of Riemannian Submersions

The connected complete Riemannian C^∞-manifolds and their Riemannian submersions form a category $\mathscr{R}$. The category of morphisms of $\mathscr{R}$ will be referred to as the category of Riemannian submersions and will be denoted by S. Thus, the objects of S are Riemannian submersions of complete connected Riemannian C^∞-manifolds; a morphism of a submersion $p: M_1 \to M_2$ into a submersion $q: N_1 \to N_2$ is a pair $u = (u_1, u_2)$, where $u_1: M_1 \to N_1$ and $u_2: M_2 \to N_2$ are Riemannian submersions such that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{u_1} & N_1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M_2 & \xrightarrow{u_2} & N_2 \end{array}$$

is commutative.

If u_1 and u_2 are Riemannian submersions and also coverings, we say that submersion p covers submersion q and call $u = (u_1, u_2)$ a covering morphism. The morphism $u = (u_1, u_2)$ is called an equivalence if and only if u_1, u_2 are isometries. Two submersions p and q are equivalent if there exists an equivalence of p into q. The equivalences of p with itself form a group, called the group of self-equivalences of submersion p and denoted $\Gamma(p)$. The sets

$$\Gamma_1(p)=\{u_1|u\in\Gamma(p)\},\quad \Gamma_2(p)=\{u_2|u\in\Gamma(p)\}$$

are subgroups of the groups of motions $I(M_1)$ and $I(M_2)$ of the spaces M_1 and M_2, respectively. If $u\in\Gamma(p)$, then u is a single-valued function $p_+(u_1)$ of u_1. Obviously, p_+ is an epimorphism of the group $\Gamma_1(p)$ onto $\Gamma_2(p)$, and $\Gamma(p)$ is precisely the graph of p_+. The map $u_1\in\Gamma_1(p)\to(u_1, p_+(u_1))$ is a group isomorphism of $\Gamma_1(p)$ onto $\Gamma(p)$. We call the kernel of p_+ the group of sliding motions of submersion p and denote it by $\Sigma(p)$. A motion $\gamma\in I(M_1)$ belongs to $\Gamma_1(p)$ if and only if it maps the fibers of p one into another. Under the natural identification of the set of fibers of p with M_2, γ generates $p_+(\gamma)$.

Remark 1. In the case of a Riemannian submersion which is also a covering our terminology disagrees with that used in [3].

Remark 2. Let $P=M_1\times M_2$ be product of Riemannian spaces M_1 and M_2 and let $p_1: P \to M_1$ be the canonical Riemannian submersion. Then $\Gamma_1(p_1)=I(M_1)\times I(M_2)$ and $(p_1)+(\gamma_1, \gamma_2)=\gamma_1$ whenever $(\gamma_1, \gamma_2)\in\Gamma_1(p_1)$.

Proposition 4. For every Riemannian submersion $q: N_1 \to N_2$ there exist a Riemannian submersion $p: M_1 \to M_2$ and a morphism (u_1, u_2) of p into q such that u_1 and u_2 are universal Riemannian coverings. Moreover, $\Sigma(u_1)\subset\Gamma_1(p)$ and $p_+(\Sigma(u_1))\subset\Sigma(u_2)$, and $\Sigma(u_i)$, i = 1, 2, are isomorphic to the fundamental groups $\pi_1(N_i)$. Under these isomorphisms the restriction of p_+ to $\Sigma(u_1)$ is taken into the homomorphism $q_{\#}: \pi_1(N_1)\to\pi_1(N_2)$ induced by q. Finally, $N_1=M_1/\Sigma(u_1)$, $N_2=M_2/\Sigma(u_2)$, and $q: N_1 \to N_2$ is a factor of p.

Proof. Let $u_i: M_i \to N_i$ be universal Riemannian coverings. By the lifting theorem there exists a continuous map $p: M_1 \to M_2$ such that $u_2\circ p=q\circ u_1$. By Theorem 2, p is a submetry, and it obviously is a Riemannian submersion. Consider the following segment of the homotopy sequence of the C^∞-bundle $p: M_1 \to M_2$ [3]:

$$\ldots\to\pi_1(M_1)\xrightarrow{p_{\#}}\pi_1(M_2)\xrightarrow{\partial}\pi_0(F)\to\pi_0(M_1)\to\pi_0(M_2). \tag{5}$$

Here F is an arbitrary fiber of p and for brevity we omit the base points of the homotopy groups. Since M_1 is connected, M_2 is simply connected, and sequence (5) is exact, F is connected. It follows from the commutativity of the diagram (4) that $u_1\circ\gamma=u_1$ for every γ in $\Sigma(u_1)$, and so $(u_2\circ p)\circ\gamma=(q_0\circ u_1)\circ\gamma=q\circ u_1=u_2\circ p$, i.e., $\gamma\in\Sigma(u_2\circ p)$. For this reason γ permutes the components of the fibers of the submersion $u_2\circ p$ (i.e., the fibers of p) and $\gamma\in\Gamma_1(p)$. From $\gamma\in\Sigma(u_2\circ p)$ and the equality $p_+(\gamma)\circ p=p\circ\gamma$ we conclude that $u_2\circ p_+(\gamma)\circ p=u_2\circ p\circ\gamma=u_2\circ p$. Consequently, $u_2\circ p_+(\gamma)=u_2$ and $p_+(\gamma)\in\Sigma(u_2)$. By Corollary 4 of [3], $\Sigma(u_i)$ is isomorphic to the group $\pi_1(N_i)$, i = 1, 2. The corresponding isomorphism is defined as follows. Let $n_1\in N_1$, $m_1\in u_1^{-1}(n_1)$, and $\gamma\in\Sigma(u_1)$. Then $i_1(\gamma)=[u_1\circ\omega]$, where $[u_1\circ\omega]$ is the element of $\pi_1(N_1, n_1)$ corresponding to the path $u_1\circ\omega$, with ω an arbitrary path in M_1 joining the points m_1 and $\gamma(m_1)\in u_1^{-1}(n_1)$. The homomorphism i_2: $\Sigma(u_2)\to\pi_1(N_2, n_2)$ is defined in a similar manner. We shall assume that $n_2 = q(n_1)$. Set $m_2 = p(m_1)$. For $\gamma\in\Sigma(u_1)$

$$p_+(\gamma)(m_2)=p_+(\gamma)(p(m_1))=p(\gamma(m_1)).$$

If ω joins the points m_1 and $\gamma(m_1)$, then the path $p\circ\omega$ joins the points m_2 and $p_+(\gamma)(m_2)$, and then we have

$$(i_2p_+)(\gamma)=[u_2p\omega]=[qu_1\omega]=q_\#([u_1\omega])=q_\#i_1(\gamma).$$

From Corollary 8 of [3] we obtain the identifications $N_1=M_1/\Sigma(u_1)$ and $N_2=M_2/\Sigma(u_2)$. The discrete subgroups $\Sigma(u_i)$ of $I(M_i)$ act freely on M_i. The submersions u_1 and u_2 can be identified with the corresponding factor maps. It follows from the commutativity of the diagram (4), the equality $p\circ\gamma=p_+(\gamma)\circ p$ for γ in $\Sigma(u_1)$, and the inclusion $p_+(\Sigma(u_1))\subset\Sigma(u_2)$ that q: $N_1\to N_2$ is a factor of p, as claimed.

THEOREM 4. Let p: $M^n\to N^n$ be a submetry of locally Euclidean spaces. Then $M=E^n\times E^{m-n}/\Sigma$, $N=E^n/\Sigma_0$, where $\Sigma\subset I(E^n)\times I(E^{m-n})$ and $\Sigma_0\subset I(E^n)$ are discrete isometry groups acting freely on E^m and E^n, respectively. Moreover, $\Sigma_2=\{\gamma_2|\gamma=(\gamma_1, \gamma_2)\in\Sigma\}\supset\Sigma_0$ and p is a factor of the Riemannian submersion p_1, the projection of $E^n\times E^{m-n}$ onto E^n.

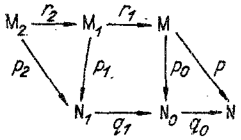
Proof. The submetry p can be incorporated in a commutative diagram

$$\begin{array}{ccc} E^m & \xrightarrow{s} & M^m \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ E^n & \xrightarrow{q} & N^n \end{array}$$

where s, q are Riemannian coverings, and the existence of p_1 is guaranteed by the lifting theorem. By Proposition 2, p_1 is a submetry. Also, by Theorem 1 we may assume that $E^m = E^n\times E^{m-n}$ and p_1 is the orthogonal projection onto E^n; in particular, p_1 is a Riemannian submersion. It remains to use Remark 2 and Proposition 4.

The next theorem supplements Proposition 4.

THEOREM 5. Let p: M → N be a Riemannian submersion. Then p can be incorporated in a commutative diagram of Riemannian submersions with universal Riemannian coverings r_2 and q_1:



Here r_1 is a regular Riemannian covering and q_0 is a Riemannian covering. The fibers of submersion p_0 are connected. The bundle p_1: $M_1\to N_1$ is induced from the bundle p_0: $M\to N_0$ via q_1. Moreover, one has the exact sequence of fundamental groups

$$1\to\pi_1(M_1)\xrightarrow{r_{1\#}}\pi_1(M)\xrightarrow{p_{0\#}}\pi_1(N_0)\to 1. \tag{6}$$

Proof. It follows from Proposition 4 that p is incorporated in a commutative diagram of Riemannian submersions

$$\begin{array}{ccc} M_2 & \xrightarrow{r} & M \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p} \\ N_1 & \xrightarrow{q} & N \end{array}$$

with universal Riemannian coverings r and q. Denote $\Sigma_1=\Sigma(r)\cap\Sigma(p_2)$, $M_1=M_2/\Sigma_1$, r_2: $M_2\to M_1$ the corresponding Riemannian coverings, and r_1: $M_1\to M=M_2/\Sigma(r)$ the Riemannian covering defined by the inclusion $\Sigma_1\subset\Sigma(r)$. As Σ_1 is the kernel of the restriction of the homomorphism $(p_2)_+$ to $\Sigma(r)$, it is a normal subgroup of $\Sigma(r)$. Consequently, the covering r_1 is regular [3] and $\Sigma(r_1)=\Sigma(r)/\Sigma_1$. Since $\Sigma_1\subset\Sigma(p_2)$, for each γ in Σ_1, $p_2\circ\gamma=p_2$ and there exists a unique continuous map p_1: $M_1\to N_1$ such that $p_1\circ r_2=p_2$. As r_2 is a Riemannian covering and p_2 a Riemannian submersion, it is obvious that p_1 is also a Riemannian submersion.

As a consequence of Proposition 4, $\Sigma_0=(p_2)_+(\Sigma(r))\subset\Sigma(q)$. Denote $N_0=N_1/\Sigma_0$, q_1: $N_1\to N_0$ the corresponding universal Riemannian covering, and q_0: $N_0\to N=N_1/\Sigma(q)$ the Riemannian covering defined by the inclusion $\Sigma_0 \subset \Sigma(q)$. Since $(p_2)_+(\Sigma(r)) = \Sigma_0$, there exists a unique map p_0: $M = M_2/\Sigma(r) \to N_0 = N_1/\Sigma_0$, which is a factor of p_2; p_0 is obviously a Riemannian submersion. Next, since p: $M = M_2/\Sigma(r) \to N = N_1/\Sigma(q)$ is also a factor of p_2 (Proposition 4), it follows that $p = q_0 \circ p_0$. It is also clear that $p_0 \circ r = q_1 \circ p_2$. Consequently, $q_1 \circ p_1 \circ r_2 = q_1 \circ p_2 = p_0 \circ r = p_0 \circ r_1 \circ r_2$. Since r_2 is a surjection, $q_1 \circ p_1 = p_0 \circ r_1$, and the diagram of Theorem 5 is commutative.

The groups $\Sigma(r)$, $\Sigma(q)$, Σ_1, and Σ_0 are isomorphic to $\pi_1(M)$, $\pi_1(N)$, $\pi_1(M_1)$, and $\pi_1(N_0)$, respectively. Under these isomorphisms to the homomorphism $p_\#$: $\pi_1(M) \to \pi_1(N)$ these corresponds the homomorphism $(p_2)_+$: $\Sigma(r) \to \Sigma(q)$. Since Σ_1 and Σ_0 are the kernel and, respectively, the image of $(p_2)_+$, sequence (6) is exact.

Examining the segment

$$\ldots \to \pi_1(M) \xrightarrow{p_{0\#}} \pi_1(N_0) \xrightarrow{\partial} \pi_0(p_0^{-1}(n_0)) \to \pi_0(M) \to \ldots$$

of the homotopy sequence of the bundle p_0 [3], where $n_0 \in N_0$, and recalling that (6) is exact, we see that $\partial = 1$. At the same time, ∂ is a surjection (thanks to the connectedness of M). Therefore, $\pi_0(p_0^{-1}(n_0)) = 1$ and the fiber $p_0^{-1}(n_0)$ is connected.

Let p_1': $M_1' \to N_1$ be the bundle induced from the bundle p_0: $M \to N_0$ via the universal covering q_1: $N_1 \to N_0$. By definition, $M_1' = \{(n_1, m) \in N_1 \times M \mid q_1(n_1) = p_0(m)\}$ with the natural differential structure. If $(n_1, m) \in M_1'$, then $p_1'(n_1, m) = n_1$, $r_1'(n_1, m) = m$. Obviously, r_1' is a connected covering of M.

Juxtaposition of the homotopy sequences of the bundles p_0 and p_1' by means of the homomorphisms induced by the maps r_1' and q_1 yields the commutative diagram

$$\begin{array}{ccccccc}
\ldots \to \pi_1(p_1'^{-1}(n_1)) & \xrightarrow{i_{1\#}} & \pi_1(M_1') & \xrightarrow{p'_{1\#}} & \pi_1(N_1)\,(=1) \to \ldots \\
\approx\downarrow & & \downarrow r'_{1\#} & & \downarrow q_{1\#} \\
\ldots \to \pi_1(p_0^{-1}(n_0)) & \xrightarrow{i_{0\#}} & \pi_1(M) & \longrightarrow & \pi_1(N_0) \to \ldots.
\end{array}$$

Here $n_0 = q_1(n_1)$, and i_1, i_0 are the inclusions of the fibers of p_1', p_0 in M_1', M, respectively. Since the bundle p_1': $M_1' \to N_1$ is induced by p_0, it follows that $r_1' \circ i_1$ is a diffeomorphism of the fiber of p_1' onto the fiber of p_0; the left vertical arrow denotes the isomorphism induced by this diffeomorphism. We also have

$$\operatorname{Ker} p_\# = \operatorname{Ker}(q_0 \circ p_0)_\# = \operatorname{Ker} p_{0\#} = (i_0)_\#(\pi_1(p_0^{-1}(n_0))) =$$
$$= (i_0)_\#(\approx(\pi_1(p_1'^{-1}(n_1))) = r'_{1\#}(i_{1\#}(\pi_1(p_1'^{-1}(n_1)))) = r'_{1\#}(\pi_1(M_1')).$$

The last equality follows from the fact that $p'_{1\#} = 1$ and $i_{1\#}$ is an epimorphism. On the other hand, $\operatorname{Ker} p_\# = (r_1)_\#(\pi_1(M_1))$. Therefore, r_1 and r_1' are equivalent in the category of connected coverings of M [3]. From this it readily follows that p_1': $M_1' \to N_1$ and p_1: $M_1 \to N_1$ are equivalent in the category of morphisms over N_1 of the category of C^∞-manifolds and their C^∞-maps. We can therefore assume that p_1 is the bundle induced from p_0 via the map q_1: $N_1 \to N_0$. Theorem 5 is proven.

3. Riemannian Submersions with Integrable Horizontal Distributions

We are interested in Riemannian submersions p: M → N in the case where M and N are spaces of equal constant curvature. It follows from formula 3 of Corollary 1 of [1] that the integrability tensor A of such a submersion p vanishes. This is equivalent to the integrability of the horizontal distribution of p. The present section is devoted to such submersions.

Alongside with A, O'Neill introduces the tensor T, the second fundamental form of the fibers $p^{-1}(x)$, $x \in N$. T vanishes if and only if the fibers of the submersion are totally geodesic. If, in addition, M is complete and connected, then p is a C^∞-bundle whose structure group is the Lie group of isometries of the fiber (see [2]).

The vectors tangent to the fibers of submersion p are called vertical vectors, they form an integrable C^∞-distribution V on M, called the vertical distribution. The distribution H which orthogonally complements V (in TM) is C^∞-differentiable and is termed the horizontal distribution. The orthogonal projection maps of TM onto V and H will be denoted by the same letters. For arbitrary C^∞-vector fields E and F on M, T is defined by the formula

$$T_E F = H\nabla_{VE} VF + V\nabla_{VE} HF. \tag{7}$$

In dual manner one defines the tensor field

$$A_E F = V\nabla_{HE} HF + H\nabla_{HE} VF. \tag{8}$$

Proposition 5. Suppose M is complete and connected and the integrability tensor A of the Riemannian submersion p: M → N vanishes. Then the maximal integral manifolds of distribution H give a totality geodesic foliation $\mathcal{F}_H$ on M. The foliations $\mathcal{F}_H$ and $\mathcal{F}_V$ (where $\mathcal{F}_V$ is formed by the components of the fibers of p) are mutually orthogonal and transverse [4]. If L is a leaf of $\mathcal{F}_H$ and i_L: L → M is the immersion defined by the inclusion $L \subset M$, then $p\circ i_L\colon L \to M$ is a Riemannian covering.

These assertions are essentially contained in [1].

Let X_* be a C^∞-vector field on N. Then X_* has a unique horizontal lift X to M, i.e., a horizontal vector field X on M such that $p_*X = X_*$. Obviously, X is a C^∞-field. Such fields on M are called basic fields [1].

If X, Y, and Z are two vertical and a horizontal vector fields, we denote

$$\nabla^{\tau}_X Y = V\nabla_X Y,\quad \nabla^{\nu}_X Z = H\nabla_X Z. \tag{9}$$

These notations are justified by the following circumstances. Let L be a leaf of the foliation $\mathcal{F}_V$, i.e., a component of a fiber $p^{-1}(x)$, $x\in N$. Let τL and νL denote the tangent and, respectively, the normal vector bundles over L. Then the restrictions X_L and Y_L of the vector fields X and Y to L are sections of τL, whereas Z_L is a section of νL. Moreover, $(\nabla^{\tau}_X Y)_L$ and $(\nabla^{\nu}_X Z)_L$ are the canonical covariant derivatives of the vector fields Y_L and Z_L in the direction of the vector field X_L (in τL and νL, resepctively).

Proposition 6. Let p: M → N be a Riemannian submersion and A the integrability tensor of p. Then A ≡ 0 if and only if for every $x\in N$ and every basic field Z the restriction Z_L of Z to L = $p^{-1}(x)$ is parallel relative to the canonical connection in νL.

Proof. Let X be a vertical vector field on M and Z a basic field. Then [X, Z] is vertical (see [1]) and

$$\nabla_Z X = H\nabla_Z X + V\nabla_Z X = A_Z X + V\nabla_Z X,$$

$$\nabla_Z X = \nabla_X Z + [Z, X] = H\nabla_X Z + V\nabla_X Z + [Z, X] = \nabla^{\nu}_X Z + T_X Z + [Z, X].$$

Equating the horizontal and vertical components we get the equalities

$$\nabla^{\nu}_X Z = A_Z X,\quad T_X Z = V\nabla_Z X - [Z, X].$$

Thus, $\nabla^{\nu}_X Z = 0$ if and only if $A_Z X = 0$. The last condition is equivalent to A ≡ 0 thanks to the arbitrariness of the basic field Z and the vertical field X and the skew-symmetry of A_Z [1]. The proposition is proven.

LEMMA 6. Let p: M → N be a Riemannian submersion with M complete, and let $H\subset TM$ denote the horizontal distribution of p on M. Then one has the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{p_*} & TN \\ \downarrow{\scriptstyle \exp_M} & & \downarrow{\scriptstyle \exp_N} \\ M & \xrightarrow{p} & N \end{array}.$$

Proof. Let $\tilde v\in M_{\tilde x}\cap H$, $x = p(\tilde x)$, $v = p_*(\tilde v)\in N_x$; $\gamma(t) = \exp_N(tv)$, $0\leqslant t\leqslant 1$, connects the points x and y = $\gamma(1)$. By equality (1), there exists a shortest path $\tilde\gamma(t) = \exp_M(t\tilde v_1)$, $0\leqslant t\leqslant 1$, with $\tilde v_1\in M_{\tilde x}$, which connects the points $\tilde x$ and $p^{-1}(y)$, and whose length equals $l(\tilde\gamma) = \rho_N(x, y)$ Obviously, $\tilde v_1\in H$.

If $\gamma = \gamma(t)$ is the unique shortest path, parametrized by reduced arc-length, which joins the points x and y, then it follows from Proposition 3 that $p\circ\tilde\gamma = \gamma$ and $v = \dot\gamma(0) = p_*(\dot{\tilde\gamma}(0)) = p_*(\tilde v_1)$. Hence, $\tilde v_1 = v$. The general case is reduced to the one just examined by partitioning of $\gamma = \gamma(t)$, $0\leqslant t\leqslant 1$, into sufficiently small pieces. The lemma is proven.

LEMMA 7. Let p: M → N be a Riemannian submersion with M complete, let M, $x, y\in N$, $v\in N_x$, y = $\exp_N(v)$, and let V be the lift of the vector v to a horizontal vector field along the fiber F = $p^{-1}(x)$. Then the map $\varphi_V\colon F\to M$ defined by the relation $\varphi_V(f) = \exp_M(V(f))$ is a diffeomorphism of F onto $p^{-1}(y)$.

Proof. Obviously, φ_V is a C^∞-map. Also, by Lemma 6,

$$p(\varphi_V(f)) = p(\exp_M(V(f))) = \exp_N(p_*(V(f))) = \exp_N(v) = y,$$

that is, φ_V maps F into $p^{-1}(y)$.

If $\gamma(1) = -v_1 \in N_y$, then $x = \exp_N(v_1)$. Let V_1 be the vector field along $F_1 = p^{-1}(y)$, which is the horizontal lift of the vector v_1. Then by the foregoing discussion φ_{V_1} is a C^∞-map of F_1 into F. Moreover, the proof of Lemma 6 shows that $u_1 = \widehat{\exp}_M(1V(f))$ is horizontal and $p_*(u_1) = -v_1$. Consequently, $V_1(\varphi_V(f)) = -u_1$ and $(\varphi_{V_1} \circ \varphi_V)(f) = f$, $f \in F$. Similarly, $(\varphi_V \circ \varphi_{V_1}) \times (f_1) = f_1$, $f_1 \in F_1$. Therefore, φ_V is a diffeomorphism of F onto F_1. Lemma 7 is proven.

<u>Proposition 7.</u> Let p: M → N be a Riemannian submersion and A its integrability tensor. Suppose A ≡ 0, M is complete and connected, and N is simply connected. Then every fiber $F = p^{-1}(x)$, $x \in N$, is connected, and M, regarded as the total space of the C^∞-bundle p: M → N, is isomorphic to the trivial bundle N × F over N. If, in addition, T ≡ 0, then M is isometric to N × F.

<u>Proof.</u> The map p is a locally trivial bundle. It follows from the homotopy sequence of the bundle p (see [3]) and the simple-connectedness of N that $p^{-1}(x)$ is connected.

We define a map φ: $N \times F \to M$ as follows. If $v \in N_x$, $n = \exp_N(v)$, and V is the same field as in Lemma 7, we put $\varphi(n, f) = \varphi_V(f)$. First, let us show that this definition is correct. Let $v_1 \in N_x$, and $n = \exp_N(v_1)$. Then, by Lemma 6,

$$p(\varphi_{V_1}(f)) = p(\exp_M(V_1(f))) = \exp_N(p_*(V_1(f))) = \exp_N(v_1) = n = p(\varphi_V(f)).$$

Moreover, the tangent vectors to the curves $\tilde{\gamma}_1(t) = \exp_M(tV_1(f))$ and $\tilde{\gamma}(t) = \exp_M(tV(f))$ are horizontal, as follows from the proof of Lemma 6. If A ≡ 0, then the horizontal distribution H is integrable and so $\varphi_{V_1}(f)$ and $\varphi_V(f)$ lie in the same leaf L (passing through the point f) of the foliation $\mathcal{F}_H$. Moreover, $p \circ i_L$, being a C^∞-covering of the simply-connected space N (Proposition 5), is a diffeomorphism. In particular, $\varphi_V(f) = \varphi_{V_1}(f)$, since they both lie in the fiber $p^{-1}(n)$. This proves the correctness of the definition of φ.

We already saw that $\varphi(n, f)$ lies in the fiber over the point n. If we establish that φ is a diffeomorphism, this will imply that φ is an isomorphism of the bundle under consideration and N × F → N.

The map φ is one-to-one, since $p(\varphi(n, f)) = n$, and, for fixed n, $\varphi(n, \cdot) = \varphi_V$ is a diffeomorphism of F onto $p^{-1}(n)$. This implies that φ is a surjection.

Suppose that there is a $v \in N_x$ such that $\exp_N(v) = n$ and $\exp_{N_x}$: $N_x \to N$ has rank dim N at the point v, i.e., the points x and n are not conjugate along the geodesic $\gamma = \gamma(t) = \exp_N(tv)$, $0 \leqslant t \leqslant 1$. By the inverse function theorem, there exists a neighborhood U of v such that $\psi = \exp_{N_x}|_U$ is a diffeomorphism onto a neighborhood W of n. The correspondence g: $v \to V$ [$v \in N_x$, V the horizontal lift of the vector v along $p^{-1}(x)$] is a linear isomorphism, and the map α: (v, f) → (V(f)) is differentiable. Obviously, $\varphi|_{W\times F} = \exp_M \circ \alpha \circ (\psi^{-1} \times \mathrm{id}_F)$ and the maps $\exp_M$, α, and $\psi^{-1} \times \mathrm{id}_F$ are all differentiable. Consequently, φ is differentiable on W × F.

To show that the map φ is differentiable everywhere, we remark that φ depends in fact on the choice of the point $x \in N$ and should be denoted φ_x. Let y be another point of N. Arguing as above, we find that if $y = \exp_N(v) = \exp_N(v_1)$, where $v, v_1 \in N_x$, then for the corresponding vector fields V, V_1 along $p^{-1}(x)$ the diffeomorphisms $\varphi_V, \varphi_{V_1}: p^{-1}(x) \to p^{-1}(y)$ coincide. Thus, there is a uniquely determined diffeomorphism ψ_{yx} of the fiber $p^{-1}(x)$ onto $p^{-1}(y)$. Obviously, $\psi_{zx} = \psi_{zy} \circ \psi_{yx}$ [it suffices to remark that $f \in p^{-1}(x)$ and $\psi_{yx}(f)$ lie in the same leaf L of the foliation $\mathcal{F}_H$, and L intersects each fiber $p^{-1}(z), z \in N$, at exactly one point].

It is readily seen that $\varphi_x = \varphi_y \circ (\mathrm{id}_N \times \psi_{yx})$ and that $\mathrm{id}_N \times \psi_{yx}$ is a diffeomorphism of N × $p^{-1}(x)$ onto N × $p^{-1}(y)$. For each point n in N one can find a point y in N such that y and n are not conjugate along some geodesic connecting them. As we showed above, φ_y is differentiable on W × $p^{-1}(y)$, where W is a neighborhood of the point $n \in N$. Then φ_x is differentiable on W × $p^{-1}(x)$. Hence, $\varphi = \varphi_x$ is differentiable.

For fixed $n \in N$ $\varphi(n, \cdot)$ is a diffeomorphism of F onto $p^{-1}(n)$. For fixed f, $p \circ (\varphi(\cdot, f)) = \mathrm{id}_N$, $\varphi(n', f)$ lies in the same leaf L (passing through f) of the foliation $\mathcal{F}_H$, and $p \circ i_L$ is a diffeomorphism of L onto N. Finally, L and $p^{-1}(n)$ intersect transversely at the point $\varphi(n, f)$. Consequently, φ has a nonzero Jacobian at the point (n, f) and is a diffeomorphism of N × F onto M.

Let g denote the metric tensor on N × F defined by the equality $g = \varphi^* g_M$ (where g_M is the metric tensor on M). Then p_1: $(N \times F, g) \to (N, g_N)$, the projection on the first factor, is a

Riemannian submersion. Let $g_{F_n} = i^*_{F_n} g$, where $F_n = p_1^{-1}(n)$ and i_{F_n} denotes the inclusion of F_n in $N \times F$. We may assume that $g_{F_n}, n \in N$, is a family of metric tensors g_n on F which depends differentiably on $n \in N$. If $p_2\colon N \times F \to F$ denotes the projection onto the second factor, then $g(n, f) = p_1^* g_N + p_2^* g_n$. Conversely, for a metric tensor of this kind the projection $p_1\colon (N \times F, g) \to (N, g_N)$ is a Riemannian submersion with integrable horizontal distribution.

Finally, suppose $T \equiv 0$. Then all the fibers $p^{-1}(n)$, $n \in N$, are totally geodesic, and it is readily verified that all diffeomorphisms ψ_{yx} are isometries of the fibers. We remarked above that $\varphi_x = \varphi_y \circ (\mathrm{id}_N \circ \psi_{yx})$. This implies that the metric tensor g on $N \times F$ can be written in the form $g = p_1^* g_N + p_2^* g_F$ and that M is isometric to the product $N \times F$. Proposition 7 is proved.

4. Transverse Foliations

It is well known that every C^1-foliation $\mathcal{F}$ of codimension one on an n-dimensional C^1-manifold admits a transverse foliation of codimension $n - 1$ [4, Theorem 4.2]. In the addendum of D. V. Anosov to this theorem (see [4]) it is asserted that it is not known whether there always exists a foliation transverse to $\mathcal{F}$ when $\mathcal{F}$ has codimension higher than one. Here we give examples of foliations (in fact, even fiber bundles) which admit no transverse foliations (in the terminology of I. Tamura).

THEOREM 6. Let $p\colon M \to N$ be a C^∞-submersion of a closed connected C^∞-manifold M onto a closed C^∞-manifold N. Suppose that the foliation $\mathcal{F}$ on M given by the components of the fibers $p^{-1}(n)$, $n \in N$, admits a transverse C^1-foliation $\mathcal{F}'$. Then p is a C^∞-bundle. If N is simply connected, then the fibers of p are connected, and the bundle $p\colon M \to N$ is topologically trivial. If M is simply connected, then the bundle $p\colon M \to N$ lifts to a bundle $p_1\colon M \to N_1$ whose fibers are the leaves of the foliation $\mathcal{F}$, where $q\colon N_1 \to N$ is the universal covering. The bundle p_1 is topologically trivial.

Proof. The first assertion is proved, under weaker conditions, in [4, Theorem 4.3].

Let g_N, g_M be arbitrary metric C^∞-tensors on N and M, respectively. We introduce a new (generally speaking, only continuous) metric tensor g'_M on M, such that $p\colon (M, g'_M) \to (N, g_N)$ is a Riemannian C^1-submersion. Since the foliations $\mathcal{F}$ and $\mathcal{F}'$ are transverse, every continuous vector field X on M admits a unique representation as a sum $X = VX + HX$, where VX and HX are continuous vector fields on M tangent to the foliations $\mathcal{F}$ and $\mathcal{F}'$, respectively. Then obviously

$$g'_M = V^* g_M + H^* p^* g_N$$

is the sought-for metric tensor. It is clear that $\mathcal{F}'$ is the horizontal foliation of the Riemannian submersion $p\colon (M, g'_M) \to (N, g_N)$, and that H and V are the projectors onto the horizontal and vertical distributions of this submersion.

By analogy with Proposition 7 one establishes that the bundle $p\colon M \to N$ is topologically trivial whenever N is simply connected. Here the proof is in fact simpler, since it is not necessary to show that φ is a diffeomorphism.

The last assertion follows from the result just proved, the lifting theorem [3], and Proposition 2.

COROLLARY 1. Under the assumptions of Theorem 6, if p is a principal C^∞-bundle, then p or p_1 is a trivial C^∞-bundle.

COROLLARY 2. Let $p\colon M \to N$ be a topologically nontrivial C^∞-bundle, where M and N are closed and connected, and N is simply connected. Then the foliation $\mathcal{F} = \{p^{-1}(n),\ n \in N\}$ on M admits no transverse C_1-foliations.

COROLLARY 3. Let $p\colon M^m \to N^n$, $m > n$, be a C^∞-bundle, where M and N are closed and connected, and M or N is simply connected; also, the n- or $(m - n)$-dimensional Betti number of M over Z_2 vanishes. Then the foliation $\mathcal{F}$ on M given by the components of the fibers of p admits no transverse C^1-foliations.

Remark. If $\mathcal{F}'$ is a C^2-foliation then (under the assumptions of Theorem 6) using a result of [5] one can show that p (or p_1) is a C^∞-trivial bundle.

THEOREM 7. The Hopf bundles (more precisely, the corresponding foliations on the total spaces of these bundles)

$$S^{2n+1} \to PC^n, \quad S^{4n+3} \to PH^n, \quad S^{15} \to S^8$$

admit no transverse C^1-foliations.

Let us prove Corollary 3 and Theorem 7. The bundle p (or p_1) has a simply connected base N (respectively, N_1). Suppose the foliation $\mathcal{F}$ on N admits a transverse C^1-foliation $\mathcal{F}'$. Then the bundle p (or p_1) over N (respectively, N_1) is topologically trivial, as follows from Theorem 6. In particular, M is homeomorphic to N × F (respectively, N_1 × F), where F is a leaf of $\mathcal{F}$. We remark that N_1 is closed, and hence for the Betti numbers over Z_2, $\beta_n(N_1)\neq 0$, $\beta_{m-n}(F)\neq 0$ [3]. Then, by Kunneth's theorem [3], $\beta_n(M)\neq 0$, $\beta_{m-n}(M)\neq 0$. This proves Corollary 3. Theorem 7 is a straightforward consequence of Corollary 3, since the bases of the Hopf bundles are simply connected, and $\beta_m(S^n) = 0$ if $m \neq 0$, $m \neq n$.

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UNIQUE (UP TO ORIENTATION) DEFINABILITY OF TIME BY MEANS OF THE SPATIAL STRUCTURE OF A SET OF EVENTS

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1. Contents of the Article

In [1] a motivated definition of a kinematic on a set of events is given explicitly (Definition 1). If $K = (\tau, T \to B_T)$ is a kinematic on set M, then τ is a set of partitions of M which are endowed with the structure of three-dimensional Euclidean space and called relative spaces. Mapping $T \to B_T$ maps each relative space $T\in\tau$ to a partition B_T of set M, which is endowed with the structure of one-dimensional oriented Euclidean space and called local time determined by T. We call the set τ, taking into consideration the Euclidean structure of partitions $T\in\tau$, the spatial structure of set M with kinematic K, and we denote it by the symbol τ_K. Correspondingly, we shall call the mapping $T\mapsto B_T$, $T\in\tau$, taking into consideration the structure of one-dimensional oriented Euclidean space on each of the partitions B_T, the time on the set of events M with kinematic K, and we denote it by symbol B_K. If $K = (\tau, T \to B_T)$, $\tilde{K} = (\tau, \tilde{T} \to B_T)$ are kinematics on M with the same set of relative spaces, i.e., those satisfying condition $\tau_{\tilde{K}} = \tau_K$, then notation $B_{\tilde{K}} = B_{\bar{K}}$ means that for every $T\in\tau$, we have that $\tilde{B}_T$ is obtained from B_T by a change of orientation.

Postulate I of [1], which is the kinematic form of the law of inertia, expresses a definite connection between spatial structure τ_K and time B_K on a set of events with kinematic K. In a kinematic K which also satisfies Postulate II of [1] (the kinematic form of the principle of Galilean relativity), the connection between τ_K and B_K proves to be very rigid, namely: time B_K is determined by the spatial structure τ_K of a set of events uniquely up to a possible change of orientation of local time B_T for all $T\in\tau$. In other words, we have the

<u>THEOREM.</u> Let $K = (\tau, T \to B_T)$ be a kinematic on M satisfying Postulates I and II of [1]. The following assertions are true:

1. If $\tilde{K}$ is a kinematic on M that also satisfies Postulates I and II, and $\tau_{\tilde{K}} = \tau_K$, then either $B_{\tilde{K}} = B_K$ or $B_{\tilde{K}} = B_{\bar{K}}$.
2. There exists a kinematic $\tilde{K}$ on M satisfying Postulates I and II such that $\tau_{\tilde{K}} = \tau_K$, $B_{\tilde{K}} = B_{\bar{K}}$.

A proof of the formulated theorem is presented in Sec. 3, and auxiliary propositions on which the proof is based are presented in Sec. 2. In Sec. 4 we state some corollaries of

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