PROPERTIES OF MULTIPLE ITÔ INTEGRALS

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Introduction and Notation

In the study of nonlinear transformations of stationary Gaussian processes a considerable space is given to Itô's multiple integrals (m. i.) (cf. [1-4, 15]). One of the fundamental methods of studying this question is the method of moments or semiinvariants [3, 4, 10, 11], since it is difficult to apply the method of characteristic functions due to the absence of the assumption of sufficiently weak dependence in the original Gaussian process. In the present paper we shall give some formulas for calculating moments and semiinvariants of Itô m. i. From exact formulas or from the intermediate equations obtained in proving them in the present paper, one gets estimates of the semiinvariants of the m. i. of such a form that one could apply the results of [5, 8, 13] and get, as a consequence, some properties of the distributions of Itô m. i., the central limit theorem, the rate of convergence to a Gaussian law. The question of large deviations in such a formulation is considered in [9, 10]. We note also that the fundamental results of the present paper are formulated without proofs in [12].

We introduce some notation. By $I^{(m)}(\varphi)$, following [1, 2], we denote the m-fold integral with respect to the random Gaussian measure

$$I^{(m)}(\varphi)=\underbrace{\int\ldots\int}_{m}\varphi(\lambda_1,\ldots,\lambda_m)\beta(d\lambda_1)\ldots\beta(d\lambda_m).$$

Here β is a complex Gaussian orthogonal measure on the line with the usual properties

1) $\mathbf{E}\beta(\Lambda)=0$;

2) $\overline{\beta(\Lambda)}=\beta(-\Lambda)$;

3) $\mathbf{E}\beta(\Lambda_1)\overline{\beta(\Lambda_2)}=F(\Lambda_1\cap\Lambda_2)$;

Λ, Λ_1, Λ_2 are measurable sets on the line, F is the spectral measure of the measure β. Further, it will be assumed that the measure F is continuous and $F(R^1)=1$. We note that the finiteness of the measure F is not used in Lemma 1, and some other results can be obtained in a more general formulation. With respect to the integrand φ it will be assumed that it belongs to the space

$$L_2(R^m,F)=\left\{\varphi(\lambda_1,\ldots,\lambda_m):\|\varphi\|^2=\int_{R^m}|\varphi(\lambda_1,\ldots,\lambda_m)|^2\prod_{j=1}^{m}F(d\lambda_j)<\infty,\ \varphi(-\lambda_1,\ldots,-\lambda_m)=\overline{\varphi(\lambda_1,\ldots,\lambda_m)}\right\}.$$

Let Λ_j, $j=\pm1,\pm2,\ldots$, be some symmetric with respect to the point 0, finite partition of the line R^1 and $-\Lambda_j=\Lambda_{-j}$. For simplicity, it will be assumed that $\Lambda_j\subset[0,\infty)$ for $j>0$ and $\Lambda_j\subset(-\infty,0]$ for $j<0$. We define the space $S^{(m)}$ of step-functions of the following form:

$$\varphi(\lambda_1,\ldots,\lambda_m)=\begin{cases}a_{j_1\ldots j_m}, & \text{if } (\lambda_1,\ldots,\lambda_m)\in\Lambda_{j_1}\ldots\Lambda_{j_m} \text{ and } j_k\neq j_l \text{ for } k\neq l,\\ 0 & \text{otherwise.}\end{cases}$$

By $\Gamma_k\{\xi\}$ and $\Gamma\{\xi_1,\ldots,\xi_k\}$ we denote the k-th semiinvariant of the r.v. ξ and the simple semiinvariant of the random vector $(\xi_1,\ldots,\xi_k)$, respectively.

1. Semiinvariants $I^{(2)}(\varphi)$

We shall prove one simple auxiliary proposition.

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LEMMA 1. Let $\varphi_j \in L_2(R^m, F)$, $j=1, \ldots, k$, the number $k \cdot m$ be even, $r = k \cdot m/2$. Then

$$\left| \int_{R^r} \prod_{j=1}^{k}{}^{*} \varphi_j (\lambda_{j_1}, \ldots, \lambda_{j_m}) \prod_{s=1}^{r} F(d\lambda_s) \right| \leqslant \prod_{j=1}^{k} \|\varphi_j\|.$$

Here the product Π^* is taken over collections of different indices $(j_1, \ldots, j_m)$, $1 \le j_s \le r$ and each index j_s is repeated in two different collections.

Proof. We take from the product Π^* any two functions $\varphi_i(\lambda_{i_1}, \ldots, \lambda_{i_m})$, $\varphi_j(\lambda_{j_1}, \ldots, \lambda_{j_m})$. We consider the collections of indices $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$. Two cases are possible.

1. There are no equal indices in the indicated collections.

2. Some of the indices coincide. Without loss of generality, we shall assume in this case that the first s indices coincide.

We define the function:

$$\psi_1(\lambda_{i_{s+1}}, \ldots, \lambda_{i_m}, \lambda_{j_{s+1}}, \ldots, \lambda_{j_m}) = \begin{cases} \varphi_i \cdot \varphi_j & \text{in case } 1, \\ \int_{R^s} \varphi_i \varphi_j \prod_{\nu=1}^{s} F(d\lambda_{i_\nu}) & \text{in case } 2. \end{cases}$$

We note that by definition of the product $\prod^*$ the product $\prod^*_{\nu \neq i, j} \varphi_\nu$ is independent of the variables with indices $i_1, \ldots, i_s$ in Case 2. It is easy to verify that the function ψ_1 is square integrable with respect to the corresponding product of the measures of F

$$\|\psi_1\|^2 = \int_{R^{2(m-s)}} |\psi_1|^2 \prod_{p=s+1}^{m} F(d\lambda_{i_p}) \prod_{q=s+1}^{m} F(d\lambda_{j_q})$$

$$\leqslant \int_{R^{2(m-s)}} \left[\left(\int_{R^s} |\varphi_i|^2 \prod_{\nu=1}^{s} F(d\lambda_{i_\nu}) \right)^{1/2} \left(\int_{R^s} |\varphi_j|^2 \prod_{\nu=1}^{s} F(d\lambda_{i_\nu}) \right)^{1/2} \right]^2 \prod_{\nu=s+1}^{m} F(d\lambda_{i_\nu}) F(d\lambda_{j_\nu}) = \|\varphi_i\|^2 \|\varphi_j\|^2 < \infty.$$

Now we choose another factor $\varphi_l(\lambda_{l_1}, \ldots, \lambda_{l_m})$, $l \neq i, j$ and we define a function ψ_2 by integrating the product $\varphi_l \psi_1$ with respect to the variables whose indices are contained in both collections $(l_1, \ldots, l_m)$ and $(i_{s+1}, \ldots, i_m, j_{s+1}, \ldots, j_m)$. If in the indicated collections there are no equal indices, then we set $\varphi_2 = \varphi_l \psi_1$. Further analogously we see that

$$\|\psi_2\| \leqslant \|\varphi_l\| \|\psi_1\| \leqslant \|\varphi_l\| \|\varphi_i\| \|\varphi_j\|.$$

Continuing the indicated procedure, we get the assertion of the lemma.

LEMMA 2. Let $\varphi \in L_2(R^m, F)$ be symmetric. Then

$$\Gamma_k\{I^{(2)}(\varphi)\} = 2^{k-1}(k-1)! \int_{R^k} \varphi(\lambda_1, -\lambda_2) \varphi(\lambda_2, -\lambda_3) \ldots \varphi(\lambda_{k-1}, -\lambda_k)$$

$$\times \varphi(\lambda_k, -\lambda_1) \prod_{\nu=1}^{k} F(d\lambda_\nu) \overset{df}{=} 2^{k-1}(k-1)!\, m_k(\varphi), \quad k = 2, 3, \ldots$$

Proof. First we prove the assertion of the lemma for $\varphi \in S^{(2)}$. Using elementary properties of semiinvariants (cf., e.g., [9]), we have

$$\Gamma_k\{I^{(2)}\varphi\} = \Gamma_k\left\{ \sum_{j_1, j_2} a_{j_1 j_2} \beta(\Lambda_{j_1}) \beta(\Lambda_{j_2}) \right\} = \Gamma\{\Sigma a_{j_1 j_2} \beta(\Lambda_{j_1}) \beta(\Lambda_{j_2}), \ldots, \Sigma a_{j_1 j_2} \beta(\Lambda_{j_1}) \beta(\Lambda_{j_2})\}$$

$$= \Sigma a_{j_1 j_2} a_{j_3 j_4} \ldots a_{j_{2k-1} j_{2k}} \Gamma\{\beta(\Lambda_{j_1}) \beta(\Lambda_{j_2}), \ldots, \beta(\Lambda_{j_{2k-1}}) \beta(\Lambda_{j_{2k}})\}. \tag{1}$$

For calculating the last semiinvariant, we shall use formula IV.d of [9]. Since the r.v. $\beta(\lambda)$ is Gaussian, the indicated semiinvariant will be expressible in terms of a certain sum of products of second semiinvariants. We note that the semiinvariant $\Gamma\{., \ldots, .\}$ from (1) can be nonzero only when the collection of indices $(j_1, \ldots, j_{2k})$ satisfies the following conditions:

1) for any index j_p one can find an index j_q such that $j_p = -j_q$;

2) the product $a_{j_1 j_2} \ldots a_{j_{2k-1} j_{2k}}$ cannot be divided into two parts, having no indices which are equal in modulus.

Now, using the symmetry of the function φ and the evenness of the measure F, we get from (1) $2^{k-1}(k-1)!$ identical integral sums. It remains to see the validity of the lemma for any symmetric function φ from $L_2(R^2, F)$. We compare the function φ with a fundamental sequence $\{\varphi_n\}$ from $S^{(2)}$, which converges to φ in $L_2(R^2, F)$. This means that $I^{(2)}(\varphi_n) \xrightarrow{n\to\infty} I^{(2)}(\varphi)$ in mean square. Using Lemma 1, we have

$$|\Gamma_k\{I^{(2)}(\varphi_n)-I^{(2)}(\varphi_m)\}|=|\Gamma_k\{I^{(2)}(\varphi_n-\varphi_m)\}|\leqslant 2^{k-1}(k-1)!\,\|\varphi_n-\varphi_m\|^k, \quad k=2,\ 3,\ \ldots$$

This is equivalent to

$$\mathbf{E}\,|I^{(2)}(\varphi_n)-I^{(2)}(\varphi_m)|^k\leqslant C\,\|\varphi_n-\varphi_m\|^k, \quad k=2,\ 3,\ \ldots,\ C=C(k)>0.$$

Whence it follows that $\mathbf{E}(I^{(2)}(\varphi_n))^k \to \mathbf{E}(I^{(2)}(\varphi))^k$, $n\to\infty$, i.e., $\Gamma_k\{I^{(2)}(\varphi_n)\} \to \Gamma_k\{I^{(2)}(\varphi)\}$, $n\to\infty$. Noting that $m_k(\varphi_n) \to m_k(\varphi)$, $n\to\infty$ also, we conclude that the lemma is proved.

A symmetric and almost everywhere with respect to the measure F nonzero function φ from $L_2(R^2, F)$ defines by the equation

$$\psi_1(\cdot)=\int_{R^1}\psi_2(\lambda_2)\,\varphi(\cdot,\ \lambda_2)\,F(d\lambda_2),\quad \psi_2(\cdot)\in L_2(R^1,\ F),$$

a self-adjoint operator from $L_2(R^1, F)$ to $L_2(R^1, F)$, having nonzero eigenvalues $\{\mu_j\}$. If $\{\psi_j\}$ are eigenfunctions corresponding to these eigenvalues, then it is known that one has the following decomposition:

$$\varphi(\lambda_1,\ \lambda_2)=\sum_j \mu_j\,\psi_j(\lambda_1)\,\psi_j(\lambda_2).$$

Using this decomposition, it is easy to verify that

$$m_k(\varphi)=\sum_j \mu_j^k.$$

Thus, Lemma 2 assumes the following form.

LEMMA 2'. Under the hypotheses of Lemma 2 one has

$$\Gamma_k\{I^{(2)}(\varphi)\}=2^{k-1}(k-1)!\sum_j \mu_j^k,\quad k=2,\ 3,\ \ldots$$

From Lemmas 1 and 2 we get the following corollary.

COROLLARY 1. Let $\varphi\in L_2(R^2,\ F)$ be symmetric. Then

$$|\Gamma_k\{I^{(2)}(\varphi)\}|\leqslant 2^{k-1}(k-1)!\,\|\varphi\|^k,\quad k=2,\ 3,\ \ldots$$

The equality sign is achieved for φ = const.

This estimate in using the Carleman test of the moment problem allows one to assert that the distribution $I^{(2)}(\varphi)$ is completely determined by its moments. Whence we deduce the following proposition.

LEMMA 3.* Let $\varphi\in L_2(R^2,\ F)$ be symmetric. The random variable $I^{(2)}(\varphi)$ is distributed just like $\sum_j \mu_j(X_j^2-1)$, where X_j are independent standard Gaussian variables.

To prove Lemma 3 it suffices to see the equality of all moments or semiinvariants of the r.v. $I^{(2)}(\varphi)$ and the r.v. $\sum_j \mu_j(X_j^2-1)$. From Lemma 3 it follows that the characteristic function h(t) of the r.v. $I^{(2)}(\varphi)$ has the form

$$h(t)=\prod_k (1-2i\mu_k t)^{-1/2}e^{-it\mu_k},\quad i=\sqrt{-1}\,.$$

We give a formula for calculating the moments of $I^{(2)}(\varphi)$.

*This result, apparently, is known to specialists, but the author has not been able to find it in the literature. After [12] was in press, Dobrushin and Major [15] appeared, in which this assertion is also given.

LEMMA 4. Let $\varphi \in L_2(R^2, F)$ be symmetric. Then

$$\mathbf{E}\left(I^{(2)}(\varphi)\right)^k = k! \sum_{q=1}^{k} \sum_{\sum_{j=1}^{q} k_j = k} \frac{2^{k-q}}{q!} \prod_{j=1}^{q} \frac{m_{k_j}}{k_j},$$

where $m_{k_j} = m_{k_j}(\varphi)$, $k_j \geq 1$, $m_1 = 0$. The inner sum is taken over all positive integral solutions of the equation $\sum_{j=1}^{q} k_j = k$.

The proof follows from Lemma 2 and the connection formulas between moments and semiinvariants [9]. One can also derive Lemma 4 directly for functions from $S^{(2)}$, as in Lemma 2, but this method is more cumbersome.

2. Semiinvariants $I^{(m)}(\varphi)$, $m = 2, 3, \ldots$

Suppose we have some set of pairs of indices of the following form:

$$D = \begin{pmatrix} (1, 1), (1, 2), \ldots, (1, m) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (k, 1), (k, 2), \ldots, (k, m) \end{pmatrix},$$

i.e., the set D has m columns and k rows, and its elements are pairs (i, j), $1 \leq i \leq k$, $1 \leq j \leq m$. A partition of the set $D = D' \cup D''$ will be called a row partition if any row from D belongs either to D' or D''. Further we shall consider only such sets D when the number $k \cdot m$ is even. We write $r = k \cdot m/2$ and we introduce the following definition.

Definition. A partition $D = \bigcup_{j=1}^{r} D_j$ will be called indecomposable if:

a) there exists no row partition $D = D' \cup D''$ for which any D_j belongs either to D' or D'';

b) all D_j contain at least two pairs of indices from different rows.

We note that in our definition of indecomposable partition, requirement b) differs from the definition used in [9].

LEMMA 5. Let $\varphi \in L_2(R^m, F)$, $m = 2, 3, \ldots$ Then, if the number $k \cdot m$ is even and $r = k \cdot m/2$, then

$$\Gamma_k\{I^{(m)}(\varphi)\} = \Sigma^* \int_{R^r} \prod_{j=1}^{k} \varphi(\lambda_{j,1}, \ldots, \lambda_{j,m}) \prod_{\nu=1}^{r} F(d\lambda_\nu) \tag{2}$$

and $\Gamma_k\{I^{(m)}(\varphi)\} = 0$, if $k \cdot m$ is odd. The product $\prod_{j=1}^{k} \varphi(\lambda_{j,1}, \ldots, \lambda_{j,m})$ in (2) defines an indecomposable partition $D = \bigcup_{j=1}^{r} D_j$ as follows: if $D_j = \{(p, q), (s, t)\}$, then one should set $\lambda_{p,q} = \lambda_j$, $\lambda_{s,t} = -\lambda_j$. The sum Σ^* denotes summation over all indecomposable partitions of the set D.

The proof of (2) is in principle analogous to the proof of Lemma 2, so we omit it.

We denote by $H_m(x) = (-1)^m e^{x^2/2}(d^m/dx^m)e^{-x^2/2}$ the Hermite polynomial of degree m. From Lemma 5, using Lemma 1, it is easy to get the following estimate.

LEMMA 6. Let $\varphi \in L_2(R^m, F)$. Then

$$|\Gamma_k\{I^{(m)}(\varphi)\}| \leqslant M(k, m)\|\varphi\|^k, \quad k = 2, 3, \ldots \tag{3}$$

Here $M(k, m) = \Gamma_k\{H_m(X)\}$, X is a standard Gaussian r.v. The equality sign is achieved for $\varphi = \text{const}$.

We note that M(k, m) is the number of indecomposable partitions of the set D. We have already seen that $M(k, 2) = 2^{k-1}(k-1)!$ Using Stirling's formula for factorials, one can estimate the constant M(k, m) from above so that the estimate of the semiinvariants (3) assumes a form allowing the use of Lemma 2.1 of [8]. We get

$$|\Gamma_k\{I^{(m)}(\varphi)\}| \leqslant (m^{m/2}\|\varphi\|)^k (k!)^{m/2}, \quad k = 2, 3, \ldots \tag{4}$$

Since

$$\sigma_m^2 \overset{df}{=} \mathbf{E}\left(I^{(m)}(\varphi)\right)^2 = m! \, \| \varphi \|^2,$$

it follows from (4) that

$$\left| \Gamma_k \left\{ \frac{1}{\sigma_m} I^{(m)}(\varphi) \right\} \right| \leqslant (\pi m)^{-1/2} \left(\frac{2}{3} e^2\right)^{m/2} [(2\pi m)^{1/4} e^{-m/2}]^{2-k} \left(\frac{k!}{2}\right)^{m/2}.$$

Now it remains to apply Lemma 2.1 of [8].

THEOREM 1. For any $\varphi \in L_2(R^m, F)$ one has

$$P\{I^{(m)}(\varphi) \geqslant \sigma_m x\} \leqslant \exp\left\{ \frac{-x^2}{H + \Delta x^{2-2/m}} \right\}, \quad x > 0.$$

In particular,

$$P\{I^{(m)}(\varphi) \geqslant \sigma_m x\} \leqslant \begin{cases} \exp\left\{ \dfrac{-x^2}{2H} \right\} & \text{for } x \leqslant A_m; \\ \exp\left\{ \dfrac{-x^{2/m}}{2\Delta} \right\} & \text{for } x > A_m. \end{cases}$$

Here

$$H = H(m) = 2(\pi m)^{-1/2} \left(\frac{2}{3} e^2\right)^{m/2}, \quad \Delta = \Delta(m) = 2e(2\pi m)^{-1/2m},$$

$$A = A(m) = (\pi \cdot m)^{-1/4} \, 2^{\frac{m^2+1}{4(m-1)}} \, 3^{\frac{m^2}{4(1-m)}} \, e^{m/2}.$$

3. Semiinvariants $\int_0^T I_t^{(m)}(\varphi)\, dt$

We consider the stationary Gaussian process

$$x_t = \int_{R^1} e^{it\lambda} \beta(d\lambda), \quad t \in R^1.$$

Further it will be assumed that the process X_t has spectral density $f(\lambda)$, i.e., $F(d\lambda) = f(\lambda)/d\lambda$. By $I_t^{(m)}(\varphi)$ we shall denote the m-fold Itô integral

$$I_t^{(m)}(\varphi) = \underbrace{\int \cdots \int}_{m} e^{it(\lambda_1 + \cdots + \lambda_m)} \varphi(\lambda_1, \ldots, \lambda_m) \beta(d\lambda_1) \ldots \beta(d\lambda_m).$$

We note that $I_t^{(m)}(\varphi)$ is the result of applying the operation of translation generated by the process X_t to the r.v. $I^{(m)}(\varphi) \overset{df}{=} I_0^{(m)}(\varphi)$. In [10, 11] the probability is studied of large deviations for the quantities

$$Y_t^{(m)} = \int_0^T I_t^{(m)}(\varphi)\, dt, \quad T > 0.$$

The exponential expression for these probabilities contain semiinvariants of the quantities $Y_T^{(m)}$, in terms of which one can express the coefficients of the Cramer series. Hence there is interest in exact formulas for the semiinvariants of the r.v. $Y_T^{(m)}$. These formulas will have the form of some integral of the product of two functions, one of which is independent of T, and the other of φ. We define the first of them. We take a function $\varphi \in L_2(R^m, F)$ and we fix some indecomposable partition of the set $D = \bigcup_{j=1}^{r} D_j$, $r = k \cdot m/2$. (We assume that the number $k \cdot m$ is even for us.) We form the function

$$\varphi^*(\lambda_1, \ldots, \lambda_r) = \prod_{n=1}^{k} \varphi(\lambda_{n,1}, \ldots, \lambda_{n,m}).$$

Here on the right side of the equation for $D_j = \{(p, q), (s, t)\}$ we set $\lambda_{p,q} = \lambda_j$, $\lambda_{s,t} = -\lambda_j$. We define a nondegenerate transformation of variables of the form

$$\sum_{\nu=1}^{m} \lambda_{n,\nu} = x_n, \quad n = 1, \ldots, k-1,$$

$$\lambda_{j_k} = x_k,$$

$$\cdots\cdots$$

$$\lambda_{j_r} = x_r.$$

The first k − 1 equations are linearly independent by virtue of the indecomposability of the partition of the set D. The remaining r − k + 1 equations we choose in the simplest way and so that the transformation will be nondegenerate. Now we take the function

$$\varphi^*(\lambda_1, \ldots, \lambda_r) \prod_{j=1}^{r} f(\lambda_j)$$

and we make the indicated change of variables. We get some function $\varphi_1^*(x_1, \ldots, x_r)$ and finally we set

$$\Phi_{k,m}(x_1, \ldots, x_{k-1}) = \Phi_{k,m}(x) = \Sigma^* \int_{R^{r-k+1}} \varphi_1^*(x_1, \ldots, x_r)\, dx_k\, dx_{k+1} \ldots dx_r;$$

Σ^* denotes summation over all indecomposable partitions of the set D. We write further

$$\Psi_T^{(n)}(x) = 2\pi^{-n} T^{-1} \prod_{j=1}^{n+1} x_j^{-1} \sin \frac{1}{2} T x_j,$$

$$x = (x_1, \ldots, x_n),\ x_{n+1} = \sum_{j=1}^{n} x_j,\ T > 0,\ n = 1, 2, \ldots$$

The function $\Psi_T^{(1)}(x)$ is usually called the Fejer kernel. The functions $\Psi_T^{(n)}$, $n \geq 2$, we introduced by Bentkus in [6, 7]. Their most important properties were investigated there too.

LEMMA 7. Let $\varphi \in L_2(R^m, F)$. Then

$$\Gamma_k\{Y_T^{(m)}\} = (2\pi)^{k-1} T \int_{R^{k-1}} \Psi_T^{(k-1)}(x)\, \Phi_{k,m}(x)\, dx, \quad k = 2, 3, \ldots$$

Proof. We set $\varphi_N \in S^{(m)}$, i.e.,

$$\varphi_N(\lambda_1, \ldots, \lambda_m) = \begin{cases} a_{j_1 \ldots j_m} \overset{df}{=} a_{(j)}, & \text{if } (\lambda_1, \ldots, \lambda_m) \in \Lambda_{j_1} \times \ldots \times \Lambda_{j_m} \overset{df}{=} \Lambda_{(j)} \\ & \text{and } j_k \neq j_l \text{ for } k \neq l, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(j)_s = (j_1^s, \ldots, j_m^s)$ denote different collections of m different indices, each of which varies within certain finite limits $|i_j| \leq N$. Then

$$I_t^{(m)}(\varphi_N) = \sum_{(j)} a_{(j)} \int \ldots \int_{\Lambda_{(j)}} e^{it(\lambda_1 + \ldots + \lambda_m)} \beta(d\lambda_1) \ldots \beta(d\lambda_m) = \sum_{(j)} a_{(j)} \prod_{n=1}^{m} \int_{\Lambda_{j_n}} e^{it\lambda} \beta(d\lambda) \overset{df}{=} \sum_{(j)} a_{(j)} \prod_{(j)} Y(t).$$

Now it is easy to calculate that

$$\Gamma_k\left\{\int_0^T I_t^{(m)}(\varphi_N)\, dt\right\} = \int_{[0,T]^k} \sum_{(j)_1, \ldots, (j)_k} \prod_{s=1}^{k} a_{(j)_s} \Gamma\left\{\prod_{(j)_1} Y(t_1), \ldots, \prod_{(j)_k} Y(t_k)\right\} \overline{dt}, \tag{5}$$

$$\overline{dt} = dt_1 \ldots dt_k.$$

Further, to conclude the proof of the lemma it is necessary to apply formula IV.d of [9] and to integrate with respect to $\overline{dt}$. The validity of the lemma for any $\varphi \in L_2(R^m, F)$ is proved analogously to the corresponding place in the proof of Lemma 2.

4. Estimates of the Semiinvariants of the r.v. $Y_T^{(m)}$

We denote by $R(t) = EX_0X_t$ the correlation function of our original Gaussian process. One has the following estimate of the semiinvariants of the r.v. $Y_T^{(m)}$.

LEMMA 8. Suppose one has

$$\text{ess sup}\, |\varphi(\lambda_1, \ldots, \lambda_m)| = A_\varphi < \infty.$$

Then

$$|\Gamma_k\{Y_T^{(m)}\}| \leq A_\varphi^k M(k, m) T v_T^{k-2} \int_{[-T,T]} |R(t)|^2 dt,$$

where $v_T = \int_{[-T,T]} |R(t)|\, dt.$

Remarks on the Proof. To prove this lemma one can use (5). From the right side of (5) we take out A_φ^k, and to estimate the remaining semiinvariant, which, as we saw in the proof of Lemma 7, can be expressed through the sum Σ^* (sum over indecomposable partitions), we estimate each of the summands and multiply by the number of summands M(k, m).

Let $\sigma_m^2(T) = E(Y_T^{(m)})^2$. Maruyama in [2, 3], without proof there is formulated the central limit theorem for the r.v. $Z_T^{(m)} = \sigma_m^{-1}(T)Y_T^{(m)}$. We prove this theorem under some other assumptions.

THEOREM 2. Suppose one has the following conditions:

a) $\operatorname{ess\,sup} |\varphi(\lambda_1, \ldots, \lambda_m)| = A_\varphi < \infty$,

b) $\int_{R^1} |R(t)|^2 dt = A_R < \infty$,

c) there exists a constant $C_1 > 0$ such that

$$\sigma_m^2(T) \geqslant C_1^2 T, \quad \forall T > 0.$$

Then

$$P\{Z_T^{(m)} < x\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy \stackrel{df}{=} \Phi(x), \quad T \to \infty.$$

Proof. Since the hypotheses of Lemma 8 hold, one has

$$|\Gamma_k\{Z_T^{(m)}\}| \leqslant \frac{A_\varphi^k A_R M(k, m)}{C_1^k} \left(\frac{v_T}{\sqrt{T}}\right)^{k-2}, \quad k = 3, 4, \ldots \tag{6}$$

It follows from (b) that $v_T = o(\sqrt{T})$, $T \to \infty$. Consequently, (6) gives the convergence of all semiinvariants, starting with the third of the r.v. $Z_T^{(m)}$ to zero, which proves our theorem.

We note that a) can be weakened, but we shall not get this here.

Using the results of Saulis [14] and Lemma 8, one can get a uniform estimate of the rate of convergence in the central limit theorem for $Z_T^{(m)}$.

THEOREM 3. Suppose one has a) and c) of Theorem 2 and the condition

$$\lim_{T\to\infty} v_T = C_R < \infty.$$

Then

$$\sup_x |P\{Z_T^{(m)} < x\} - \Phi(x)| \leqslant CT^{1/2(1-m)}.$$

Here C depends on C_1, C_R, A_φ and it can be estimated from above easily.

In conclusion, we add that under the hypotheses of Theorem 3 one can get the ratios of large deviations for the r.v. $Z_T^{(m)}$ also. Here the hypotheses of Theorem 3 differ from the hypotheses under which the ratios mentioned are obtained in [10, 11].

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