ESTIMATES OF SEMIINVARIANTS AND CENTERED MOMENTS OF STOCHASTIC PROCESSES WITH MIXING. I

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1. Introduction and Basic Results

Let X_t, $t = 1, 2, \dots$, be a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and $\{\mathcal{F}_s^t,\ 1 \leqslant s \leqslant t < \infty\}$ be a family of σ-algebras such that

1) $\mathcal{F}_s^t \subset \mathcal{F}\ \ \forall s, t$;
2) $\mathcal{F}_{s_1}^{t_1} \subset \mathcal{F}_{s_2}^{t_2}\ \ \forall [s_1, t_1] \subset [s_2, t_2]$;
3) $\mathcal{F}_s^t \supset \sigma\{X_u,\ s \leqslant u \leqslant t\}$.

We introduce, as usual, the α-mixing, φ-mixing, and ψ-mixing functions by the following relations:

$$\alpha(s, t) = \sup_{A \in \mathcal{F}_1^s,\ B \in \mathcal{F}_t^\infty} |\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)|,$$

$$\varphi(s, t) = \sup_{\substack{A \in \mathcal{F}_1^s,\ B \in \mathcal{F}_t^\infty \\ \mathbf{P}(A) > 0}} \left| \frac{\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)}{\mathbf{P}(A)} \right|,$$

$$\psi(s, t) = \sup_{\substack{A \in \mathcal{F}_1^s,\ B \in \mathcal{F}_t^\infty \\ \mathbf{P}(A) > 0,\ \mathbf{P}(B) > 0}} \left| \frac{\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)}{\mathbf{P}(A)\mathbf{P}(B)} \right|.$$

Let $\mathbf{E}|X_t|^k < \infty$ for some $k \geq 2$ and $\Gamma(X_{t_1}, \dots, X_{t_k})$ be the k-th order correlation function of the stochastic process X_t, i.e., the mixed semiinvariant of the process X_t

$$\Gamma(X_{t_1}, \dots, X_{t_k}) = \frac{1}{i^k} \frac{\partial^k}{\partial u_1 \dots \partial u_k} \ln \mathbf{E}\, e^{i(u, X)} \Big|_{u=0},$$

where $u \in R^k$, $X = (X_{t_1}, \dots, X_{t_k})$ [2].

The centered moment $\widehat{\mathbf{E}} X_{t_1} \dots X_{t_k}$ is defined as follows:

$$\widehat{\mathbf{E}} X_{t_1} \dots X_{t_k} = \mathbf{E} X_{t_1} \widehat{X_{t_2} \cdots \widehat{X_{t_{k-1}} \widehat{X}_{t_k}}}.$$

The symbol " $\frown$ " over a random variable means that it is centered by its expectation $\widehat{\xi} = \xi - \mathbf{E}\xi$ (cf. [4-6]).

This paper is devoted to upper estimates of the function $\widehat{\mathbf{E}} X_{t_1}, \dots, X_{t_k}$, and with their help estimating the mixed semiinvariant $\Gamma(X_{t_1}, \dots, X_{t_k})$ and k-th order semiinvariant $\Gamma_k(S_n)$ of the sum $S_n = X_1 + \dots + X_n$ since

$$\Gamma_k(S_n) = \sum_{1 \leq t_1, \dots, t_k \leq n} \Gamma(X_{t_1}, \dots, X_{t_k}). \tag{1.1}$$

The estimates are found in terms of one of the mixing functions α, φ, and ψ and the moment $\mathbf{E}|X_t|^k, l \leqslant k$. As is known ([4, 5], cf. also [3]) $\Gamma(X_{t_1}, \dots, X_{t_k})$ can be expressed in terms of the function $\widehat{\mathbf{E}} X_{t_1} \dots X_{t_k}$ by the following formula:

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$$\Gamma(X_{t_1}, \ldots, X_{t_k}) = \sum_{\nu=1}^{k} (-1)^{\nu} \sum_{\bigcup_{p=1}^{\nu} I_p = I} N_{\nu}(I_1, \ldots, I_{\nu}) \prod_{p=1}^{\nu} \widehat{\mathbf{E}} X_{I_p}, \tag{1.2}$$

where $\sum\limits_{\bigcup_{p=1}^{\nu} I_p=I}$ denotes summation over all partitions of the set of indices $I=\{t_1, \ldots, t_k\}$ into unordered subsets $I_p \subset I$. Further, if $I_p=\{t_1^{(p)}, \ldots, t_{k_p}^{(p)}\}$, then by definition we set $X_{I_p} := X_{t_1^{(p)}} \ldots X_{t_{k_p}^{(p)}}$. The integers $0 \leqslant N_{\nu}(I_1, \ldots, I_{\nu}) \leqslant (\nu-1)!$ depend only on the partition $\{I_1, \ldots, I_{\nu}\}$. If $N_{\nu}(I_1, \ldots, I_{\nu}) > 0$, then

$$\sum_{p=1}^{\nu} \max_{t_i, t_j \in I_p} (t_j - t_i) \geqslant \max_{1 \leqslant i, j \leqslant k} (t_j - t_i). \tag{1.3}$$

An exact formula for $N_{\nu}(I_1, \ldots, I_{\nu})$ is given in Sec. 3.

We give the basic results. For simplicity of notation, we shall assume everywhere that $t_1 \le \ldots \le t_k$.

<u>THEOREM 1.</u> 1) if $|X_t| \le C$ with probability 1, $t = t_1, \ldots, t_k$, then for all i, $1 \le i < k$,

$$\text{a)} \quad |\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant 2^k C^k \alpha(t_i, t_{i+1}),$$

$$\text{b)} \quad |\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant 2^{k-1} C^k \varphi(t_i, t_{i+1}),$$

$$\text{c)} \quad |\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant 2^{k-2} C^k \psi(t_i, t_{i+1});$$

2) if for some integer $k \ge 2$ and $\delta > 0$, $\mathbf{E}\,|X_t|^{(1+\delta)k} < \infty$, $t=t_1, \ldots, t_k$, then for all i, $1 \le i < k$,

$$|\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant 3 \cdot 2^{k-1} \alpha^{\delta/(1+\delta)}(t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{1/(1+\delta)k} |X_{t_j}|^{(1+\delta)k};$$

3) if for some collection $p_i \geqslant 1$, $1 \leqslant j \leqslant k$ with $\sum_{j=1}^{k} 1/p_j = 1$ exist $\mathbf{E}|X_{t_j}|^{p_j}$, $1 \leqslant j \leqslant k$, then for all i, $1 \le j < k$,

$$\text{a)} \quad |\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant 2^{k-1} \varphi^{\sum_{j=1}^{i} 1/p_j}(t_i, t_{i+1}) \prod_{j=1}^{k} \mathbf{E}^{1/p_j} |X_{t_j}|^{p_j},$$

$$\text{b)} \quad |\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant 2^{k-2} \psi(t_i, t_{i+1}) \sum_{j=1}^{k} \mathbf{E}^{1/k} |X_{t_j}|^{k};$$

4) if $\exists\, a_1 > 0$, $C_1 \geqslant 1 : \mathbf{E} e^{a_1 |X_t|} \leqslant C_1$, $t=t_1, \ldots, t_k$, then for all i, $1 \le j < k$, and any $\delta > 0$

$$\text{a)} \quad |\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant 3k!\, 2^{k-1} C_1 \big((\widehat{1+\delta})/a_1\big)^k \alpha^{\delta/(1+\delta)}(t_i, t_{i+1}),$$

$$\text{b)} \quad |\widehat{\mathbf{E}} X_{t_1} \ldots X_{t_k}| \leqslant \prod_{j=1}^{k} \hat{p}_j 2^{k-1} C_1 (1/a_1)^k \varphi^{\sum_{j=1}^{i} 1/p_j}(t_i, t_{i+1})$$

for any collection $p_j \geqslant 1$, $1 \leqslant j \leqslant k$, with $\sum_{j=1}^{k} 1/p_j = 1$, where $\hat{u} = \min\{v \geqslant u | v$ is an integer$\}$.

We consider the case when the variables X_t are connected in a Markov chain ξ_t (i.e., $X_t = g_t(\xi_t)$, where $g_t(x)$ is a measurable function for each t) with transition probabilities

$P_{st}(x, A)=\mathbf{P}\{\xi_t\in A\mid\xi_s=x\}$ and $P_t(A)=\mathbf{P}\{\xi_t\in A\}$. In this case we set $\mathcal{F}_s^t=\sigma\{\xi_u, s\leqslant u\leqslant t\}$. Then $\varphi(s, t)=\sup_{x, A\in\mathcal{F}_t^t}|P_{st}(x, A)-P_t(A)|\leqslant 1-\alpha_{st}$, where α_{st} is the coefficient of ergodicity of the function P_{st}

$$\alpha_{st}=1-\sup_{x, y, A\in\mathcal{F}_t^t}|P_{st}(x, A)-P_{st}(y, A)|.$$

If $\alpha^{(n)}=\min_{1\leqslant s<n}\alpha_{s, s+1}$ is the coefficient of ergodicity of the chain, then it is known that $1-\alpha_{st}\leqslant(1-\alpha^{(n)})^{t-s}\leqslant e^{-\alpha^{(n)}(t-s)}$ for all $1 \le s \le t \le n$.

Let $l_1, \ldots, l_r$ be growth points of the sequence $t_1, \ldots, t_k, m_1+\ldots+m_j=\max\{i: t_i=l_j\}, 1\leqslant j\leqslant r$.

THEOREM 2. Let the variables X_t be connected in the Markov chain ξ_t;

1) if $|X_t| \le C$ with probability 1, $t = t_1, \ldots, t_k$, then

$$\text{a)}\quad |\widehat{\mathbf{E}}X_{t_1}\ldots X_{t_k}|\leqslant 2^{k-1}C^k\sum_{j=1}^{r-1}\varphi(l_j, l_{j+1}),$$

$$\text{b)}\quad |\widehat{\mathbf{E}}X_{t_1}\ldots X_{t_k}|\leqslant 2^{k-r}C^k\sum_{j=1}^{r-1}\psi(l_j, l_{j+1});$$

2) if for some collection $q_j\geqslant 1, 1\leqslant j\leqslant r$, with $\sum_{j=1}^{r}1/q_j=1$ the $\mathbf{E}|X_{l_j}|^{m_jq_j}, 1\leqslant j\leqslant r$, exist, then

$$\text{a)}\quad |\widehat{\mathbf{E}}X_{t_1}\ldots X_{t_k}|\leqslant 2^{k-1}\prod_{j=1}^{r-1}\varphi^{\sum_{i=1}^{j}1/q_i}(l_j, l_{j+1})\prod_{j=1}^{r}\mathbf{E}^{1/q_j}|X_{l_j}|^{m_jq_j},$$

$$\text{b)}\quad |\widehat{\mathbf{E}}X_{t_1}\ldots X_{t_k}|\leqslant 2^{k-r}\prod_{j=1}^{r-1}\psi(l_j, l_{j+1})\prod_{j=1}^{r}\mathbf{E}|X_{l_j}|^{m_j}.$$

Having the estimates of Theorems 1 and 2, from (1.2), considering the behavior of N_ν we get an estimate for $\Gamma(X_{t_1}, \ldots, X_{t_k})$.

THEOREM 3. 1) if $|X_t| \le C$ with probability 1, $t = t_1, \ldots, t_k$, then for all i, $1 \le i < k$

$$\text{a)}\quad |\Gamma(X_{t_1}, \ldots, X_{t_k})|\leqslant(k-1)!\,2^kC^k\alpha(t_i, t_{i+1}),$$

$$\text{b)}\quad |\Gamma(X_{t_1}, \ldots, X_{t_k})|\leqslant(k-1)!\,2^{k-1}C^k\varphi(t_i, t_{i+1}),$$

$$\text{c)}\quad |\Gamma(X_{t_1}, \ldots, X_{t_k})|\leqslant(k-1)!\,2^{k-2}C^k\varphi(t_i, t_{i+1});$$

2) if for some integer $k \ge 2$ and $\delta > 0$ the $\mathbf{E}|X_t|^{(1+\delta)k}$, $t=t_1,\ldots, t_k$, exist, then for all i, $1 \le i < k$,

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})|\leqslant 3(k-1)!\,2^{k-1}\alpha^{\delta/(1+\delta)}(t_i, t_{i+1})\prod_{j=1}^{k}\mathbf{E}^{1/(1+\delta)k}|X_{t_j}|^{(1+\delta)k};$$

3) if for some collection $p_j\geqslant 1, 1\leqslant j\leqslant k$, with $\sum_{i=1}^{k}1/p_j=1$ the $\mathbf{E}|X_{t_j}|^{p_j}$ exist, then for all i, $1 \le i < k$,

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})|\leqslant(k-1)!\,2^{k-1}\varphi^{\sum_{j=1}^{i}1/p_j}(t_i, t_{i+1})\prod_{j=1}^{k}\mathbf{E}^{1/p_j}|X_{t_j}|^{p_j};$$

4) if $\exists a_1>0, C_1\geqslant 1$; $\mathbf{E}e^{a_1|X_t|}\leqslant C_1$, $t=t_1,\ldots, t_k$, then for all i, $1 \le i < k$, and any $\delta > 0$,

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})|\leqslant 3k!\,4^{k-1}\left(C_1(1\widehat{+}\delta)/a_1\right)^k\alpha^{\delta/(1+\delta)}(t_i, t_{i+1}).$$

THEOREM 4. Let the random variables X_t be connected in a Markov chain ξ_t (as in Theorem 2):

1) if $|X_t|\leqslant C$, $t=t_1,\ldots, t_k$, then for all $k \ge 2$

a) $|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)!\, 2^{k-1} C^k \prod_{j=1}^{r-1} \varphi(l_j, l_{j+1}),$

b) $|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)!\, 2^{k-r} C^k \prod_{j=1}^{r-1} \psi(l_j, l_{j+1});$

2) if for some collection $q_j \geqslant 1,\ 1 \leqslant j \leqslant r,$ with $\sum_{j=1}^{r} 1/q_j = 1$ the $\mathbf{E}|X_{t_j}|^{m_j q_j},\ j=1,\ldots,r,$ exist, then

a) $|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)!\, 2^{k-1} \prod_{j=1}^{r-1} \varphi^{\sum_{i=1}^{j} 1/q_i}(l_j, l_{j+1}) \prod_{j=1}^{r} \mathbf{E}^{1/q_j}|X_{t_j}|^{q_j m_j},$

b) $|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant (k-1)!\, 2^{k-r} \prod_{j=1}^{r-1} \psi(l_j, l_{j+1}) \prod_{j=1}^{r} \mathbf{E}|X_{t_j}|^{m_j};$

3) if $\exists a_1 > 0,\ C_1 \geqslant 1:\ \mathbf{E}e^{a_1|X_t|} \leqslant C_1,\ t = t_1, \ldots, t_k,$ then for all k ≥ 2 and δ < 0

$$|\Gamma(X_{t_1}, \ldots, X_{t_k})| \leqslant k!\, 4^{k-1} (C_1 (\widehat{1+\delta})/a_1)^k \prod_{j=1}^{r-1} \varphi^{\delta/(1+\delta)}(l_j, l_{j+1}).$$

Having the results of Theorems 3 and 4, we can quickly get estimates for $\Gamma_k(S_n)$ from (1.1) under different mixing conditions and different restrictions on the behavior of the moments $E|X_t|^k$. However, they will not be optimal with respect to the order of the semiinvariant k, since for this, as we see in the proof of the theorems, it is necessary to study and estimate the depenedence of $\Gamma(X_{t_1}, \ldots, X_{t_k})$ on $t_1, \ldots, t_k$ more closely. This is done with the help of (1.1), (1.2), and the estimates for $\widehat{\mathbf{E}}X_{t_1}\ldots X_{t_k}$, which are given in Theorems 1 and 2.

We set

$$\Lambda_n(f, u) = \max\left\{1,\ \max_{1 \leqslant s \leqslant n} \sum_{t=s}^{n} f^{1/u}(s, t)\right\}$$

for any function f ≥ 0 and u > 0.

<u>THEOREM 5.</u> 1) if $|X_t| \le C$ with probability 1, t = 1, ..., n, then for all k ≥ 2, $\beta > 0$, $\delta > 0$

a) $|\Gamma_k(S_n)| \leqslant 2k!\, 8^{k-1} C^k \Lambda_n^{k-1}(\alpha, k-1)\, n,$

b) $|\Gamma_k(S_n)| \leqslant k!\, 8^{k-1} C^{k-2} \Lambda_n^{k-2}(\varphi, (1+\beta)(1+1/\delta)(k-2)) \times$
$\times \sum_{1 \leqslant s \leqslant t \leqslant n} \varphi^{\beta\delta/(1+\beta)(1+\delta)}(s, t)\, \mathbf{E}^{\delta/(1+\delta)}|X_s|^{1+1/\delta}\, \mathbf{E}^{1/(1+\delta)}|X_t|^{1+\delta};$

2) if for some k ≥ 2 and δ > 0, $\mathbf{E}|X_t|^{(1+\delta)k} < \infty,\ t = 1, \ldots, n,$ then for all $\beta > 0$

a) $|\Gamma_k(S_n)| \leqslant 2k!\, 12^{k-1} \Lambda_n^{k-1}(\alpha, (1+1/\delta)(k-1))\, n \max_{1 \leqslant t \leqslant n} \mathbf{E}^{1/(1+\delta)}|X_t|^{(1+\delta)k},$

b) $|\Gamma_k(S_n)| \leqslant$
$\leqslant k!\, 8^{k-1} \Lambda_n^{k-2}(\varphi, (1+\beta)(1+1/\delta)(k-2)) \max_{1 \leqslant t \leqslant n} \mathbf{E}^{(k-2)/(1+\delta)k}|X_t|^{(1+\delta)k} \times$
$\times \sum_{1 \leqslant s \leqslant t \leqslant n} \varphi^{\beta\delta/(1+\beta)(1+\delta)}(s, t)\, \mathbf{E}^{1/(1+\delta)k}|X_s|^{(1+\delta)k}\, \mathbf{E}^{1/(1+\delta)k}|X_t|^{(1+\delta)k};$

3) if $\exists a_1 > 0,\ C_1 \geqslant 1: \mathbf{E}e^{a_1|X_t|} \leqslant C_1,\ t = 1, \ldots, n,$ then for all k ≥ 2 and δ > 0,

$$|\Gamma_k(S_n)| \leqslant 2(k!)^2\, 12^{k-1} (C_1(\widehat{1+\delta})/a_1)^k \Lambda_n^{k-1}(\alpha, (1+1/\delta)(k-1))\, n.$$

<u>THEOREM 6.</u> Let the random variables X_t be connected in a Markov chain ξ_t (as in Theorem 2);

1) if $|X_t| \le C$ with probability 1, t = 1,..., n, then for all k ≥ 2 and δ > 0,

a) $|\Gamma_k(S_n)| \leqslant k!\,8^{k-1} C^k \Lambda_n^{k-1}(\varphi,\ 1)\,n,$

b) $|\Gamma_k(S_n)| \leqslant k!\,8^{k-1} C^{k-2} \Lambda_n^{k-2}(\varphi,\ 1+1/\delta) \sum_{1\leqslant s\leqslant t\leqslant n} \varphi^{\delta/(1+\delta)}(s,\ t)\,\mathbf{E}^{\delta/(1+\delta)}|X_s|^{1+1/\delta}\,\mathbf{E}^{1/(1+\delta)}|X_t|^{1+\delta};$

2) if for some $k \geq 2$ and $\delta > 0$, $\mathbf{E}|X_t|^{(1+\delta)k} < \infty,\ t=1,\ldots,n$, then

a) $|\Gamma_k(S_n)| \leqslant k!\,8^{k-1}\Lambda_n^{k-1}(\varphi,\ 1+1/\delta)\,n \max_{1\leqslant t\leqslant n} \mathbf{E}^{1/(1+\delta)}|X_t|^{(1+\delta)k},$

b) $|\Gamma_k(S_n)| \leqslant k!\,8^{k-1}\Lambda_n^{k-2}(\varphi,\ 1+1/\delta) \max_{1\leqslant t\leqslant n} \mathbf{E}^{(k-2)/(1+\delta)k}|X_t|^{(1+\delta)k}\times$

$$\times \sum_{1\leqslant s\leqslant t\leqslant n} \varphi^{\delta/(1+\delta)}(s,\ t)\,\mathbf{E}^{1/(1+\delta)k}|X_s|^{(1+\delta)k}\,\mathbf{E}^{1/(1+\delta)k}|X_t|^{(1+\delta)k};$$

3) if $\exists a_1>0,\ C_1\geqslant 1: \mathbf{E}e^{a_1|X_t|}\leqslant C_1,\ t=1,\ldots,n,$ then for all $k \geq 2$ and $\delta > 0$

$$|\Gamma_k(S_n)| \leqslant (k!)^2\,8^{k-1}\big(C_1(1\hat{+}\delta)/a_1\big)^k \Lambda_n^{k-1}(\varphi,\ 1+1/\delta)\,n;$$

4) if $\exists a_1>0,\ C_1\geqslant 1: \mathbf{E}e^{a_1|X_t|}\leqslant C_1,\ t=1,\ldots,n,$ then for all $k \geq 2$,

$$|\Gamma_k(S_n)| \leqslant k!\,16^{k-1}(C_1/a_1)^k \Lambda_n^{k-1}(\psi,\ 1)\,n.$$

The estimates for $\Gamma_k(S_n)$ found in Theorems 5 and 6 let us get theorems on large deviations for the distribution $P(Z_n < x)$ of the normalized sum $Z_n=S_n/B_n,\ B_n^2=\mathbf{E}S_n^2$ (we shall assume everywhere that $\mathbf{E}X_t=0,\ t=1,\ldots,n$). For this we shall use the following lemma.

LEMMA. If for the random variable Z with $\mathbf{E}Z=0,\ \mathbf{E}Z^2=1$, there exist $\gamma \geq 0$ and $\Delta > 0$ such that for the k-th-order semiinvariant one has

$$|\Gamma_k(Z)| \leqslant (k!)^{1+\gamma}/\Delta^{k-2},\quad k=3,\ 4,\ \ldots, \tag{1.4}$$

then in the interval $0 \leq x < \Delta_\gamma$

$$\Delta_\gamma = c_\gamma \Delta^{1/(1+2\gamma)},\quad c_\gamma = 1/6(\sqrt{2}/6)^{1/(1+2\gamma)}$$

one has (cf. [8])

$$\frac{\mathbf{P}(Z>x)}{1-\Phi(x)} = \exp\{L_\gamma(x)\}\left(1+\theta_1 f(x)\,\frac{x+1}{\Delta_\gamma}\right),$$
$$\frac{\mathbf{P}(Z<-x)}{\Phi(-x)} = \exp\{L_\gamma(-x)\}\left(1+\theta_2 f(x)\,\frac{x+1}{\Delta_\gamma}\right), \tag{1.5}$$

where

$$L_\gamma(x) = \sum_{3\leqslant k<p} c_k x^k,\quad p=\begin{cases} 1/\gamma+2, & \gamma>0,\\ \infty, & \gamma=0,\end{cases}$$

$$|c_k| \leqslant 2/k\,(16/\Delta)^{k-2}\big((k+1)!\big)^\gamma,\quad k=3,\ 4,\ \ldots,\quad |\theta_i|\leqslant 1,\quad i=1,\ 2,$$
$$f(x) = \frac{60\left(1+10\Delta_\gamma^2 \exp\{-(1-x/\Delta_\gamma)\sqrt{\Delta_\gamma}\}\right)\left(1+2(x/\Delta_\gamma)^2\right)}{1-x/\Delta_\gamma}.$$

Moreover (cf. [9]), if

$$|\Gamma_k(Z)| \leqslant (k!/2)^{1+\gamma} H/\bar{\Delta}^{k-2},\quad k=2,\ 3,\ \ldots,\quad \bar{\Delta}>0,\quad H>0,$$

then for all $x \geq 0$,

$$\mathbf{P}(\pm Z\geqslant x) \leqslant \exp\left\{-\frac{1}{2}\,\frac{x^2}{H+(x/\bar{\Delta}^{1/(1+2\gamma)})^{(1+2\gamma)/(1+\gamma)}}\right\}. \tag{1.6}$$

From this

$$\mathbf{P}(\pm Z\geqslant x) \leqslant \begin{cases} \exp\{-x^2/4H\}, & x\leqslant (H^{1+\gamma}\bar{\Delta})^{1/(1+2\gamma)},\\ \exp\{-(x\bar{\Delta})^{1/(1+\gamma)}/4\}, & x\geqslant (H^{1+\gamma}\bar{\Delta})^{1/(1+2\gamma)}.\end{cases}$$

Under the condition (1.4),

$$\sup_x |\mathbf{P}(Z<x)-\Phi(x)| \leqslant 18/\Delta_\gamma, \tag{1.7}$$

and if in addition we know an upper bound for $|f_Z(u)-e^{-u^2/2}|$, $f_Z(u)=\mathbf{E}e^{uZ}$ outside the interval $[0, \Delta_\gamma]$, then the following relation holds [10]:

$$\sup_x |\mathbf{P}(Z<x)-\Phi(x)| \leqslant \frac{3}{\sqrt{2\pi}}\left\{\frac{3\cdot 6^\gamma}{\Delta}+100\Delta_\gamma e^{-3\sqrt{\Delta_\gamma}/2}+\frac{3}{2T}+\frac{1}{\sqrt{2\pi}}\int_{\Delta_\gamma}^{T}|f_Z(u)-e^{-u^2/2}|\frac{du}{u}\right\} \tag{1.8}$$

for any $T \geq \Delta_\gamma$.

In order not to complicate the proof too much, we give the theorems of large deviations here for $P(Z_n < x)$ only for the case of a stationary sequence X_t, $t = 1, 2, \ldots$. In the case of a general nonstationary sequence, it is better to express the estimates of Theorems 5 and 6 for $\Gamma_k(Z_n)$ in terms of $\Lambda_n^{k-2}(\ldots)L_{kn}$ instead of $\Lambda_n^{k-2}(\ldots)n\max_{1\leqslant t\leqslant n}\mathbf{E}|X_t|^k/B_n^k$, where $L_{kn}=\sum_{t=1}^{n}\mathbf{E}|X_t|^k/B_n^k$. For this it is necessary to express the sum S_n in terms of new consolidated summands and to study the behavior of B_n^2. This will be done in a subsequent paper, published in Lietuvos Matematikos Rinkinys. Estimates for $\Gamma_k(S_n)$ in terms of the conditional moments $\mathbf{E}(|X_t|^k\mathscr{F}_1^{t-1})$ will also be given.

Thus, in the following theorems we consider a stationary sequence X_t with $\mathbf{E}X_1=0$, $\mathbf{E}X_1^2=1$, $B_n^2=\mathbf{E}S_n^2\geqslant\sigma_0^2 n$. We set $Z_n=S_n/B_n$.

<u>THEOREM 7.</u> 1) if $|X_1| \leq C$, with probability 1, $\alpha(s,t)\leqslant K_1 e^{-b_1(t-s)}$, then

$$|\Gamma_k(Z_n)| \leqslant (k!)^2\frac{8C^2K}{b_1\sigma_0^2}e^{1+b_1}\left(\frac{8Ce}{b_1B_n}\right)^{k-2},$$

$k > 2$, $K = \max(1, K_1)$, i.e., for $Z = Z_n$ the large deviation relations (1.5) and (1.6) hold as well as the estimates (1.7) and (1.8) for

$$\gamma=1,\quad \Delta_\gamma=c_\gamma(B_n/H_0)^{1/3},$$

where

$$H_0=\frac{8eC}{b_1}\max\left\{1,\ \frac{8KC^2}{b_1\sigma_0^2}e^{1+b_1}\right\},$$

$$\bar\Delta=\frac{B_nb_1}{8eC},\quad H=\frac{32KC^2}{\sigma_0^2b_1}e^{1+b_1};$$

2) if $\exists a_1>0,\ C_1\geqslant 1:\mathbf{E}e^{a_1|X_1|}\leqslant C_1$, $\alpha(s,t)\leqslant K_1e^{-b_1(t-s)}$, then

$$|\Gamma_k(Z_n)|\leqslant(k!)^3\frac{96\sqrt{K}C_1^2}{a_1^2b_1\sigma_0^2}e^{1+b_1/2}\left(\frac{48eC_1}{a_1b_1B_n}\right)^{k-2},$$

$k>2$, $K=\max(1, K_1)$, i.e., for $Z = Z_n$ the large deviation relations (1.5) and (1.6) as well as the estimates (1.7) and (1.8) for

$$\gamma=2,\quad \Delta_\gamma=c_\gamma(B_n/H_0)^{1/5},$$

where

$$H_0=\frac{48eC_1}{a_1b_1}\max\left\{1,\ \frac{96\sqrt{K}C_1^2}{a_1^2b_1\sigma_0^2}e^{1+b_1/2}\right\},$$

$$\bar\Delta=\frac{a_1b_1B_n}{48eC_1},\quad H=\frac{768\sqrt{K}C_1^2}{a_1^2b_1\sigma_0^2}e^{1+b_1/2};$$

3) if $\exists a_1>0,\ C_1\geqslant 1:\mathbf{E}e^{a_1|X_1|}\leqslant C_1$, $\psi(s,t)\leqslant K_3e^{-b_3(t-s)}$, then

$$|\Gamma_k(Z_n)|\leqslant(k!)^2\frac{2KC_1^2}{a_1^2b_3\sigma_0^2}e^{1+b_3}\left(\frac{4eKC_1}{a_1b_3B_n}\right)^{k-2},$$

$k > 2$, $K = \max(1, K_3)$, i.e., for $Z = Z_n$ the large deviation relations (1.5) and (1.6) hold as well as the estimates (1.7) and (1.8) for

$$\gamma=1,\quad \Delta_\gamma=c_\gamma(B_n/H_0)^{1/3},$$

where

$$H_0=\frac{4eKC_1}{a_1b_3}\max\left\{1,\ \frac{2KC_1^2}{a_1^2b_3\sigma_0^2}e^{1+b_3}\right\},$$

$$\bar{\Delta}=\frac{a_1b_3B_n}{4eKC_1},\quad H=\frac{8KC_1^2}{a_1^2b_3\sigma_0^2}e^{1+b_3}.$$

THEOREM 8. Let the random variables X_t be connected in a Markov chain ξ_t; 1) if $|X_1| \le C$ with probability 1, $\varphi(s,t)\leqslant e^{-b_2(t-s)}$, then

$$|\Gamma_k(Z_n)|\leqslant k!\ \frac{8(1+b_2)C^2}{b_2\sigma_0^2}\left(\frac{8(1+b_2)C}{b_2B_n}\right)^{k-2},$$

$k > 2$, i.e., for $Z = Z_n$ the large deviation relations (1.5) and (1.6) hold as well as the estimates (1.7) and (1.8) for

$$\gamma=0,\quad \Delta_\gamma=c_\gamma B_n/H_0,$$

where

$$H_0=\frac{8(1+b_2)C}{b_2}\max\left\{1,\ \frac{8(1+b_2)C^2}{b_2\sigma_0^2}\right\},$$

$$\bar{\Delta}=\frac{b_2B_n}{8(1+b_2)C},\quad H=\frac{16(1+b_2)C^2}{b_2\sigma_0^2}$$

(cf. also [4]);

2) if $\exists a_1>0,\ C_1\geqslant 1: \mathbf{E}e^{a_1|X_1|}\leqslant C_1,\ \varphi(s,t)\leqslant e^{-b_2(t-s)}$, then

$$|\Gamma_k(Z_n)|\leqslant (k!)^2\ \frac{32(2+b_2)C_1^2}{a_1^2b_2\sigma_0^2}\left(\frac{16(2+b_2)C_1}{a_1b_2B_n}\right)^{k-2},$$

$k > 2$, i.e., for $Z = Z_n$ the large deviation relations (1.5) and (1.6) hold as well as the estimates (1.7) and (1.8) for

$$\gamma=1,\quad \Delta_\gamma=c_\gamma(B_n/H_0)^{1/3},$$

where

$$H_0=\frac{16(2+b_2)C_1}{a_1b_2}\max\left\{1,\ \frac{32(2+b_2)C_1^2}{a_1^2b_2\sigma_0^2}\right\},$$

$$\bar{\Delta}=\frac{a_1b_2B_n}{16(2+b_2)C_1},\quad H=\frac{128(2+b_2)C_1^2}{a_1^2b_2\sigma_0^2}$$

(cf. also [33, 34]);

3) if $\exists a_1>0,\ C_1\geqslant 1: \mathbf{E}e^{a_1|X_1|}\leqslant C_1,\ \psi(s,t)\leqslant K_3e^{-b_3(t-s)}$, then

$$|\Gamma_k(Z_n)|\leqslant k!\ \frac{16(1+b_3)KC_1^2}{a_1^2b_3\sigma_0^2}\left(\frac{16(1+b_3)KC_1}{a_1b_3B_n}\right)^{k-2},$$

$k>2,\ K=\max(1,K_3)$, i.e., for $Z = Z_n$ the large deviation relations (1.5) and (1.6) hold as well as the estimates (1.7) and (1.8) for

$$\gamma=0,\quad \Delta_\gamma=c_\gamma B_n/H_0,$$

where

$$H_0=\frac{16(1+b_3)KC_1}{a_1b_3}\max\left\{1,\ \frac{16(1+b_3)KC_1^2}{a_1^2b_3\sigma_0^2}\right\},$$

$$\bar{\Delta}=\frac{a_1b_3B_n}{16(1+b_3)KC_1},\quad H=\frac{32(1+b_3)KC_1^2}{a_1^2b_2\sigma_0^2}.$$

THEOREM 9. Let the random variables X_t be m-dependent;

1) if $|X|_1 \le C$ with probability 1, then

$$|\Gamma_k(Z_n)|\leqslant k!\ \frac{16(m+1)C^2}{\sigma_0^2}\left(\frac{8C(m+1)}{B_n}\right)^{k-2},$$

$k > 2$, i.e., for $Z = Z_n$ the large deviation relations (1.5) and (1.6) hold as well as the estimates (1.7) and (1.8) for

$$\gamma=0,\quad \Delta_\gamma=c_\gamma B_n/H_0,$$

where

$$H_0 = 8(m+1)C\max\left\{1, \frac{16(m+1)C^2}{\sigma_0^2}\right\},$$
$$\bar{\Delta} = \frac{B_n}{8(m+1)C}, \quad H = \frac{32(m+1)C^2}{\sigma_0^2};$$

2) if $\exists a_1 > 0,\ C_1 \geqslant 1 : \mathbf{E}\, e^{a_1 |X_1|} \leqslant C_1$, then

$$|\Gamma_k(Z_n)| \leqslant (k!)^2 \frac{96(m+1)C_1^2}{a_1^2\sigma_0^2}\left(\frac{16(m+1)C_1}{a_1 B_n}\right)^{k-2},$$

$k > 2$, i.e., for $Z = Z_n$ the large deviation relations (1.5) and (1.6) hold as well as the estimates (1.7) and (1.8) for

$$\gamma = 1, \quad \Delta_\gamma = c_\gamma (B_n/H_0)^{1/3},$$

where

$$H_0 = \frac{16(m+1)C_1}{a_1}\max\left\{1, \frac{96(m+1)C_1^2}{a_1^2\sigma_0^2}\right\},$$
$$\bar{\Delta} = \frac{a_1 B_n}{16(m+1)C_1}, \quad H = \frac{384(m+1)C_1^2}{a_1^2\sigma_0^2}.$$

Some more partial results were published by the authors previously (cf. [3-7]), some results are cited in conference reports (cf. [11-21]) without proof. The papers of I. G. Zhurbenko [22-24], A. V. Bulinskii [25], N. M. Zuev [26, 27], P. M. Lappo [29], and of other authors are devoted to estimates of $\Gamma(X_{t1}, \ldots, X_{tk})$.

Using a different approach, the authors of the present paper gave sharper estimates for $\hat{\mathbf{E}} X_{t_1} \ldots X_{t_k}$, $\Gamma(X_{t_1}, \ldots, X_{t_k})$, and $\Gamma_k(S_n)$ with respect to $k!$, α, φ, ψ, n, and the moments $\mathbf{E}|X_t|^k$ with numerical constants.

2. Notation

For the convenience of the reader and more clarity in the paper we use standard notation.

1. Constants. We shall denote positive constants by the letters C, K, a, and b with or without indices.

2. Sets (the definitions are taken from [32], Chap. I). Finite sets will be denoted by the letters $\mathscr{A}$, $\mathscr{B}$, $\mathfrak{N}$, I, J, possibly in square brackets [], with or without indices, and it will be assumed that the sets below have the same structure throughout the entire paper:

$$\mathscr{A} = \{a_1, \ldots, a_s\}, \quad a_1 \leqslant \ldots \leqslant a_s,$$
$$\mathfrak{N} = \{1, \ldots, n\},$$
$$I = \{t_1, \ldots, t_k : t_j \in \mathfrak{N}\}, \quad t_1 \leqslant \ldots \leqslant t_k,$$
$$J = \{l_1, \ldots, l_r\}, \quad l_1 < \ldots < l_r,$$

J is the support of the k-set I,

$\{I_1, \ldots, I_\nu\}$ — is a partition of the set I,

$$I_p = \{t_1^{(p)}, \ldots, t_{k_p}^{(p)}\}, \quad t_1^{(p)} \leqslant \ldots \leqslant t_{k_p}^{(p)}, \quad 1 \leqslant p \leqslant \nu,$$
$$k_1 + \ldots + k_\nu = k,$$

$\{J_1, \ldots, J_\nu\}$ is the partition of the set J, corresponding to the partition $\{I_1, \ldots, I_\nu\}$ of the set I, such that J_p is the support of the k_p-set I_p,

$$J_p = \{l_1^{(p)}, \ldots, l_{r_p}^{(p)}\}, \quad l_1^{(p)} < \ldots < l_{r_p}^{(p)}, \quad 1 \leqslant p \leqslant \nu,$$
$$r_1 + \ldots + r_\nu \geqslant r.$$

$(m_1, \ldots, m_r)$ is the vector of exponents of the primary specification of the set I, generated by the set J, $m_1 + \ldots + m_r = k$,

$(m_1^{(p)}, \ldots, m_{r_p}^{(p)})$ is the vector of exponents of primary specification of the set I_p generated by the set J_p,

$$m_1^{(p)} + \ldots + m_{r_p}^{(p)} = k_p, \quad 1 \leqslant p \leqslant \nu,$$
$$[\mathscr{A}] = [\mathscr{A}]_I = \{t \in I \mid a_1 \leqslant t \leqslant a_s\}.$$

Obviously

$$[I]=[J]=I,$$
$$[I_p]=[J_p].$$

We introduce three more notations for sets

$$\mathcal{N}_u^v=\{u,\ u+1,\ \ldots,\ v\},\quad u,\ v\in\mathcal{N},\quad u\leqslant v\leqslant n,$$
$$\mathcal{N}_u=\{u,\ u+1,\ \ldots,\ n\},\quad u\in\mathcal{N},\quad u\leqslant n,$$
$$\mathscr{A}^{(i)}=\mathscr{A}\setminus\{a_1,\ \ldots,\ a_i\},\quad i\leqslant s.$$

The symbol $|\mathscr{A}|$ will, as usual, denote the number of elements of the set $\mathscr{A}$.

3. Sums. We shall denote the s-Cartesian power of the set $\mathscr{A}$ by ${}_{(s)}\mathfrak{A}$.

$${}_{(s)}\mathfrak{A}=\underbrace{\mathscr{A}\times\ldots\times\mathscr{A}}_{s\ \text{times}}.$$

Indices are preserved upon passing from a set to its Cartesian power, for example,

$${}_{(s)}[\mathfrak{A}]^{(i)}=\underbrace{[\mathscr{A}]^{(i)}\times\ldots\times[\mathscr{A}]^{(i)}}_{s\ \text{times}},$$
$${}_{(s)}\mathfrak{N}_u^v=\underbrace{\mathcal{N}_u^v\times\ldots\times\mathcal{N}_u^v}_{s\ \text{times}}.$$

In accord with the notation introduced for Cartesian powers, we denote by

$\sum\limits_{\mathfrak{a}\in{}_{(s)}\mathfrak{A}}$ — is the sum over all $=(\mathfrak{a}_1,\ \ldots,\ \mathfrak{a}_s)\in{}_{(s)}\mathfrak{A}$,

$\sum\limits_{\mathfrak{a}\in{}_{(s)}\mathfrak{A}}^{(\leqslant)}$ — is the sum over all $\mathfrak{a}\in{}_{(s)}\mathfrak{A}$ such that $\mathfrak{a}_1\leqslant\ldots\leqslant\mathfrak{a}_s$,

$\sum\limits_{\mathfrak{a}\in{}_{(s)}\mathfrak{A}}^{(<)}$ — is the sum over all $\mathfrak{a}\in{}_{(s)}\mathfrak{A}$ such that $\mathfrak{a}_1<\ldots<\mathfrak{a}_s$.

Sometimes, instead of ${}_{(s)}\mathfrak{A}$ we shall simply write $\mathfrak{A}$, provided the number of coordinates of the vector $\mathfrak{a}$, which determines the value of the index (s), is known.

If it will be convenient for us, as the notation for the domain of summation of the vector $\mathfrak{a}$ we shall use the set $\mathscr{B}$; in this case the summation will be over elements of the set $\mathscr{B}$ with values from $\mathscr{A}$ for each of them, and the value of the index (s) will be equal to the cardinality of the set $\mathscr{B}$; $\sum\limits_{\bigcup_{p=1}^{\nu} I_p=I}$ will denote the sum over all ν-block partitions $\{I_1,\ \ldots,\ I_\nu\}$ of the set I.

Following [31], by a partition of the natural number k we mean any finite nonincreasing sequence of natural numbers $\lambda_1,\ \ldots,\ \lambda_\nu$ such that $\sum\limits_{p=1}^{\nu}\lambda_p=k$; we call the λ_p the parts of the partition $\{\lambda_1,\ \ldots,\ \lambda_\nu\}$.

By a composition of the natural number k we mean any finite sequence of natural numbers $\lambda_1,\ \ldots,\ \lambda_\nu$ such that $\sum\limits_{p=1}^{\nu}\lambda_p=k$.

We denote by $p(k,\nu)$ the number of partitions of the number k into ν parts, and by $c(k,\nu)$ the number of compositions of the number k into ν parts. Then

$$c(k,\ \nu)=\binom{k-1}{\nu-1}=\frac{(k-1)!}{(\nu-1)!\,(k-\nu)!}\,, \tag{2.1}$$
$$p(k,\ \nu)\leqslant c(k,\ \nu),$$

$$\sum_{\{\lambda_1,\ldots,\lambda_\nu\}\vdash k}$$ is the sum over all partitions of the number k into ν parts,

$$\sum_{\lambda_1+\ldots+\lambda_\nu=k}$$ is the sum over all compositions of the number k into ν parts.

We need some facts from the theory of partitions [30, 32].

By the Stirling number of the second kind $s(k, \nu)$ we mean the number of ways of partitioning a k-element set into ν nonempty subsets. In our standard notation this definition can be written in one of the following ways:

$$s(k, \nu)=|\{\{I_1, \ldots, I_\nu\}\}|, \tag{2.2}$$

$$s(k, \nu)=\sum_{\bigcup_{p=1}^{\nu} I_p=I} 1, \tag{2.3}$$

$$s(k, \nu)=\sum_{k_1+\ldots+k_\nu=k} \frac{k!}{k_1!\ldots k_\nu!\,\nu!}. \tag{2.4}$$

The following part of the paper contains the proofs of the results. It is divided into sections. The numbering of formulas and also of auxiliary assertions supplementary to the basic results and their corollaries is done as follows: we indicate the number of the section and the number of the formula (assertion) with respect to the beginning of the section.

We shall refer to the basic results analogously, i.e., write, for example, "Theorem 3.1.b" instead of "part b of Assertion 1 of Theorem 3," etc.

3. Decomposition of the Mixed Semiinvariant with Respect to Centered Moments

Following [4, 5], we define a function $N_\nu(I_1, \ldots, I_\nu)$ with nonnegative integral values, on the set of all ν-block partitions $\{I_1, \ldots, I_\nu\}$ of the set I. Its explicit form is given by (11) of [5]. Here we give an equivalent representation.

Let $\{\mathscr{A}_1,\ldots,\mathscr{A}_\mu\}$ be a system of subsets of the set I.

We shall say that $\{\mathscr{A}_1,\ldots,\mathscr{A}_\mu\}$ covers $t \in I$ if

$$\{q : t\in[\mathscr{A}_q\backslash\{t\}],\ 1\leqslant q\leqslant\mu\}\neq\varnothing. \tag{3.1}$$

We call the number

$$n_t(\mathscr{A}_1, \ldots, \mathscr{A}_\mu)=|\{q\mid t\in[\mathscr{A}_q\backslash\{t\}],\ 1\leqslant q\leqslant\mu\}|$$

the maximal covering number of the point t by the system $\{\mathscr{A}_1,\ldots,\mathscr{A}_\mu\}$.

We set

$$N_1(I)=1,$$

$$N_\nu(I_1, \ldots, I_\nu)=\prod_{j=2}^{\nu} n_{t_1^{(j)}}(I_1, \ldots, I_\nu),\quad 2\leqslant\nu\leqslant k. \tag{3.2}$$

Following [22, 23], we make correspond to the set I and to each block I_p of a partition $\{I_1, \ldots, I_\nu\}$ the vectors of exponents of primary specification $\mu=(m_1,\ldots, m_r)$ and $\mu^{(p)}=(\mu_1^{(p)}, \ldots, \mu_r^{(p)})$ with respect to the set J. Obviously in $\mu^{(p)}$ there are exactly r_p nonzero components, equal to $m_1^{(p)}, \ldots, m_{r_p}^{(p)}$ and located in $\mu^{(p)}$ at those places at which $l_1^{(p)}, \ldots, l_{r_p}^{(p)}$ are situated among the numbers $\ell_1, \ldots, \ell_r$.

In our notation

$$t_{\min}(\mu^{(p)})=\min_{1\leqslant j\leqslant r}\{j\mid\mu_j^{(p)}\neq 0\},$$

$$t_{\max}(\mu^{(p)})=\max_{1\leqslant j\leqslant r}\{j\mid\mu_j^{(p)}\neq 0\}.$$

The "reflexivity" property of a sequence of vectors $\mu^{(1)},\ldots,\mu^{(\nu)}:\mu^{(1)}+\ldots+\mu^{(\nu)}=\mu$ (cf. [22, 23]) is defined by the inequalities

$$t_{\max}(\mu^{(p)})\geqslant t_{\min}(\mu^{(p+1)}),\quad 1\leqslant p<\nu,$$
$$t_{\max}(\mu^{(\nu)})\geqslant t_{\min}(\mu^{(1)}).$$

The concept of "indecomposability" is introduced in [1]. It turns out that the vectors of an "indecomposable" sequence $\lambda^{(1)},\ldots,\lambda^{(\nu)}$ such that $\lambda^{(1)}+\ldots+\lambda^{(\nu)}=\lambda$, as well as $\mu^{(1)},\ldots,\mu^{(\nu)}$, are the vectors of corresponding exponents of primary specification of the sets $I_1,\ldots,I_\nu$ with respect to J, and one has the following relations:

$$\{\{I_1,\ldots,I_\nu\}\mid N_\nu(I_1,\ldots,I_\nu)>0\}=$$
$$=\{\{I_1,\ldots,I_\nu\}\mid \mu^{(1)},\ldots,\mu^{(\nu)} - \text{is "reflective"}\}\supset$$
$$\supset\{\{I_1,\ldots,I_\nu\}\mid \lambda^{(1)},\ldots,\lambda^{(\nu)} - \text{is "indecomposable"}\}.$$

The mixed semiinvariant can be decomposed with respect to moments in the following ways:

$$\Gamma(X_{t_1},\ldots,X_{t_k})=\sum_{\nu=1}^{k}(-1)^{\nu-1}(\nu-1)!\sum_{\bigcup_{p=1}^{\nu}I_p=I}\prod_{p=1}^{\nu}\mathbf{E}X_{I_p},\tag{3.3}$$

$$\Gamma(X_{t_1},\ldots,X_{t_k})=\sum_{\nu=1}^{k}(-1)^{\nu-1}\sum_{\bigcup_{p=1}^{\nu}I_p=I}N_\nu(I_1,\ldots,I_\nu)\prod_{p=1}^{\nu}\widehat{\mathbf{E}}X_{I_p},\tag{3.4}$$

$$\Gamma(X_{t_1},\ldots,X_{t_k})=\sum_{\nu=1}^{k}(-1)^{\nu-1}\sum_{\bigcup_{p=1}^{\nu}I_p=I}N_\nu^{(2)}(I_1,\ldots,I_\nu)\prod_{p=1}^{\nu}\widehat{\mathbf{E}}^{(2)}X_{I_p},\tag{3.5}$$

where

$$\mathbf{E}X_{I_p}=\mathbf{E}X_{t_1^{(p)}}\ldots X_{t_{k_p}^{(p)}}=\mathbf{E}X_{l_1^{(p)}}^{m_1^{(p)}}\ldots X_{l_{r_p}^{(p)}}^{m_{r_p}^{(p)}},$$

$$\widehat{\mathbf{E}}X_{I_p}=\mathbf{E}\,\widehat{X_{t_1^{(p)}}\widehat{X_{t_2^{(p)}}\ldots X_{t_{k_p-1}^{(p)}}\widehat{X}_{t_{k_p}^{(p)}}}},$$

$$\widehat{\mathbf{E}}^{(2)}X_{I_p}=\mathbf{E}\,\widehat{X_{l_1^{(p)}}^{m_1^{(p)}}\widehat{X_{l_2^{(p)}}^{m_2^{(p)}}\ldots X_{l_{r_p-1}^{(p)}}^{m_{r_p-1}^{(p)}}\widehat{X_{l_{r_p}^{(p)}}^{m_{r_p}^{(p)}}}}},$$

$$\widehat{\xi}=\xi-\mathbf{E}\xi.$$

(3.3) is proved in [1]; (3.4) in [4, 5], although the idea of decomposition with respect to centered moments is already realized in [3] (cf. Lemma 6). (3.5) is a modification of a formula of [22, 23]. Since the kernels of the maps N_ν and $N_\nu^{(2)}$ coincide, passage from the method of centered $\widehat{\mathbf{E}}X_{I_p}$ to $\widehat{\mathbf{E}}^{(2)}X_{I_p}$ only replaces N_ν by $N_\nu^{(2)}$.

We shall use (3.4), since the number $N_\nu(I_1,\ldots,I_\nu)$ has a number of good properties, one of which is the estimate

$$N_\nu(I_1,\ldots,I_\nu)\leqslant(\nu-1)!,$$

and another is proved below.

<u>LEMMA 3.1.</u>

$$\sum_{\bigcup_{p=1}^{\nu}I_p=I}N_\nu(I_1,\ldots,I_\nu)=\sum_{j=0}^{\nu-1}(-1)^j\binom{k-\nu+j}{k-\nu}(\nu-j-1)!\,s(k,\nu-j).\tag{3.6}$$

Proof. Direct comparison of the right sides of (3.3) and (3.4) leads to the relations

$$s(k,\ \nu)=\frac{1}{(\nu-1)!}\sum_{j=0}^{\nu-1}\binom{k-\nu+j}{k-\nu}N(k,\ \nu-j),\quad 1\leqslant\nu\leqslant k, \tag{3.7}$$

where

$$N(k,\ \nu)=\sum_{\bigcup_{p=1}^{\nu} I_p=I} N_\nu(I_1,\ \ldots,\ I_\nu).$$

We note that N(k, ν) is the number of terms of the form $\prod_{p=1}^{\nu}\hat{\mathbf{E}}X_{I_p}$, which appear in the decomposition of the mixed semiinvariant by (3.4).

For example, for k = 6,

$$0!\,s(6,\ 1)=\binom{5}{5}N(6,\ 1),$$

$$1!\,s(6,\ 2)=\binom{5}{4}N(6,\ 1)+\binom{4}{4}N(6,\ 2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$5!\,s(6,\ 6)=\binom{5}{0}N(6,\ 1)+\binom{4}{0}N(6,\ 2)+\ldots+\binom{0}{0}N(6,\ 6).$$

From this,

$$N(6,\ 1)=1,\quad N(6,\ 2)=26,\quad N(6,\ 3)=66,$$
$$N(6,\ 4)=26,\quad N(6,\ 5)=1,\quad N(6,\ 6)=0.$$

By substitution of (3.7) into (3.6), we get

$$N(k,\ \nu)=\sum_{j=0}^{\nu-1}(-1)^j\binom{k-\nu+j}{k-\nu}(\nu-j-1)!\,s(k,\ \nu-j)=\sum_{j=0}^{\nu-1}(-1)^j\sum_{i=0}^{\nu-j-1}\frac{(k-\nu+i+j)!}{(k-\nu)!\,i!\,j!}N(k,\ \nu-i-j)=$$

$$=\sum_{\alpha=0}^{\nu-1}\sum_{\substack{i+j=\alpha\\ i,\,j\geqslant 0}}\frac{(k-\nu+\alpha)!}{(k-\nu)!\,i!\,j!}(-1)^jN(k,\ \nu-\alpha)=N(k,\ \nu)+\sum_{\alpha=1}^{\nu-1}\frac{(k-\nu+\alpha)!}{(k-\nu)!\,\alpha!}N(k,\ \nu-\alpha)\sum_{\substack{i+j=\alpha\\ i,\,j\geqslant 0}}(-1)^j\frac{\alpha!}{i!\,j!}.$$

Since $\sum_{\substack{i+j=\alpha\\ i,\,j\geqslant 0}}(-1)^j\frac{\alpha!}{i!\,j!}=0\ \ \forall\alpha\geqslant 1$, (3.6) is proved.

COROLLARIES of Lemma 3.1.

$$1)\quad N(k,\ \nu)=\sum_{j=0}^{\nu-1}(-1)^{\nu-j+1}\binom{k-j-1}{k-\nu}j!\,s(k,\ j+1); \tag{3.8}$$

$$2)\quad \sum_{\nu=1}^{k}N(k,\ \nu)=(k-1)! \tag{3.9}$$

Proof. (3.8) follows quickly from (3.7) if one changes the direction of summation from ν - 1 to 0. (3.9) follows from the equations

$$\sum_{\nu=1}^{k}N(k,\ \nu)=\sum_{\nu=1}^{k}\sum_{j=0}^{\nu-1}(-1)^{\nu-j+1}\binom{k-j-1}{k-\nu}j!\,s(k,\ j+1)=$$

$$=\sum_{j=0}^{k-1}\sum_{\nu=j+1}^{k}(-1)^{\nu-j+1}\binom{k-j-1}{k-\nu}j!\,s(k,\ j+1)=\sum_{j=0}^{k-1}\sum_{\nu=0}^{k-j-1}(-1)^{\nu}\binom{k-j-1}{k-j-\nu-1}j!\,s(k,\ j+1)=$$

$$=\sum_{j=0}^{k-2}j!\,s(k,\ j+1)\sum_{\nu=0}^{k-j-1}(-1)^{\nu}\binom{k-j-1}{k-j-\nu-1}+(k-1)!\,s(k,\ k).$$

Since s(k, k) = 1, and the left term of the last equation vanishes, (3.9) is proved.

LITERATURE CITED

1. V. P. Leonov and A. N. Shiryaev, "Technique of calculating semiinvariants," Teor. Veroyatn. Primen., 4, No. 3, 342-355 (1959).
2. A. N. Shiryaev, "Questions of the spectral theory of highest moments. I," Teor. Veroyatn. Primen., 5, No. 3, 293-313 (1960).
3. V. A. Statulyavichus, "Local limit theorems and asymptotic decompositions for inhomogeneous Markov chains," Liet. Mat. Rinkinys, 1, No. 1-2, 231-314 (1961).
4. V. A. Statulyavichus, "Limit theorems for sums of random variables, connected in a Markov chain. I-III," Liet. Mat. Rinkinys, 9, No. 2, 345-362 (1969); No. 3, 635-672 (1969); 10, No. 1, 161-169 (1970).
5. V. A. Statulyavichus, "Limit theorems for stochastic functions. I," Liet. Mat. Rinkinys, 10, No. 3, 583-592 (1970).
6. V. A. Statulyavichus, "Theorems of large deviations for sums of independent random variables. I," Liet. Mat. Rinkinys, 19, No, 2, 199-208 (1979).
7. D. A. Yakimavichyus, "Estimates of correlation functions of higher order for random variables connected in an inhomogeneous Markov chain," Liet. Mat. Rinkinys, 27, No. 1, 195-202 (1987).
8. R. Rudzkis, L. Saulis, and V. Statulyavichus, "General lemma on probabilities of large deviations," Liet. Mat. Rinkinys, 18, No. 2, 99-116 (1978).
9. R. Yu. Bentkus and R. A. Rudzkis, "Exponential estimates of the distribution of random variables," Liet. Mat. Rinkinys, 20, No. 1, 15-30 (1980).
10. L. Saulis, "General lemmas on approximations by normal distribution," Liet. Mat. Rinkinys, 21, No. 2, 175-189 (1981).
11. V. A. Statulevicius, "Limit theorems for dependent random variables under various regularity conditions," in: Proc. Intern. Congress Math., Vancouver (1974), pp. 173-181.
12. V. A. Statulevicius, "Application of semiinvariants to asymptotic analysis of distributions of random processes," Multivar. Anal., 4, 325-337 (1977).
13. V. A. Statulevicius, "On limit theorems for random processes," in: Second Japan—USSR Symp. Probab. Theory, Kyoto (1972), pp. 138-152.
14. V. A. Statulevicius, "On limit theorems for dependent random variables," in: Second Intern. Vilnius Conf. Probab. Theory and Math. Stat. Abstracts of Commun., Vol. 3 (1977), pp. 212-215.
15. V. A. Statulevicius, "On limit theorems for dependent random variables," in: Third Intern. Vilnius Conf. Probab. Theory and Math. Stat. Abstracts of Commun., Vol. 3 (1981), pp. 320-323.
16. V. A. Statulevicius, "Limit theorems for random processes," in: Proc. Second Symp. of the Southern Africa Math. Sci. Assoc., Univ. Zimbabwe (1983), pp. 84-106.
17. V. A. Statulevicius, "On some methods of the proof of limit theorems for the sums of dependent random variables," in: Fourth Japan—USSR Symp. Probab. Theory Abstracts of Commun., Kyoto (1986), pp. 51-52.
18. V. A. Statulyavichus, "Improvements of limit theorems for weakly dependent random variables," in: Proc. Sixth All-Union Conference on Probab. Theory and Math. Stat. [in Russian], Vilnius (1962), pp. 113-119.
19. D. A. Yakimavichyus, "Estimates of centered moments of stochastic processes with mixing," in: 28th Conference of the Lithuanian Mathematical Society. Abstracts of Reports [in Russian], Vilnius (1987), pp. 386-387.
20. D. A. Yakimavichyus, "Estimates of mixed semiinvariants of stochastic processes with mixing," in: 28th Conference of the Lithuanian Mathematical Society, Abstracts of Reports [in Russian], Vilnius (1987), pp. 384-385.
21. V. A. Statulyavichus and D. A. Yakimavichyus, "Estimates of k-th-order semiinvariants of a sum of mixing random variables," in: 28th Conference of the Lithuanian Mathematical Society. Abstracts of Reports [in Russian], Vilnius (1987), p. 322.
22. I. G. Zhurbenko, "Estimate of mixed semiinvariants for a class of stochastic processes," Teor. Veroyatn. Primen., 15, No. 3, 541-543 (1970).
23. I. G. Zhurbenko, "Strong estimates of mixed semiinvariants," Sib. Mat. Zh., 13, No. 2, 293-308 (1972).

24. I. G. Zhurbenko and N. M. Zuev, "Estimates of higher spectral densities of stationary processes, satisfying Cramer's condition with mixing à la Rosenblatt," Liet. Mat. Rinkinys, 15, No. 1, 111-124 (1975).
25. A. V. Bulinskii, "Estimates of mixed semiinvariants and hiigher covariances of bounded random variables," Teor. Veroyatn. Primen., 19, No, 4, 869-873 (1974).
26. N. M. Zuev, "Estimates of mixed semiinvariants of stochastic processes," Mat. Zametki, 13, No. 4, 581-586 (1973).
27. N. M. Zuev, "Estimate of mixed semiinvariants of stochastic processes satisfying a mixing condition of almost Markov type," Liet. Mat. Rinkinys, 21, No. 2, 81-87 (1981).
28. M. Iosifescu, "Recent advances in mixing sequences of random variables," in: Third Intern. Summer School on Probab. Theory and Math. Statist. Varna, 1978, Sofia (1980), pp. 111-138.
29. P. M. Lappo, "Estimates of mixed semiinvariants of stochastic processes with mixing and their applications," Candidate's Dissertation, Minsk (1985).
30. M. Abramovits and I. Stigan (eds.), Handbook on Special Functions [in Russian], Nauka, Moscow (1979).
31. G. Andrews, Theory of Partitions [Russian translation], Nauka, Moscow (1982).
32. V. N. Sachkov, Introduction to Combinatorial Methods of Discrete Mathematics [in Russian], Nauka, Moscow (1982).
33. S. V. Nagaev, "Limit theorems for homogeneous Markov chains," Teor. Veroyatn. Primen., 2, No. 4, 389-416 (1957).
34. S. V. Nagaev, "Improvement of limit theorems for homogeneous Markov chains," Teor. Veroyatn. Primen., 6, No. 1, 67-86 (1961).

ADMISSIBILITY OF THE NONPARAMETRIC ANALOG OF THE PITMAN ESTIMATE IN THE CASE OF ONE UNKNOWN PARAMETER

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In this paper we prove the admissibility of the estimate (1) of an element $g_\theta\varphi$ of a Hilbert space H, where φ is a known element, g_t, $t \in R$, is a one-parameter group of affine isometries of the space H, and θ is an unknown number. Special cases of (1) are the Pitman estimates (1') (cf. Example 1) and (1") of the function $\varphi(x-\theta), \varphi\in L^2(-\infty, \infty)$ under an unknown translation θ of the argument (cf. Example 2). The method applied was developed in the proof of Theorem 3 of [8].

Let H be a separable real or complex Hilbert space, g_t, $t \in R = (-\infty, \infty)$ be a continuous group of isometric affine transformations of the space H; in other words: 1) $g_{t_1}g_{t_2}=g_{t_1+t_2}$, $t_1, t_2\in\mathbf{R}$; 2) for any $\varphi \in H$ the map $t\mapsto g_t\varphi$ is continuous; 3) $g_t\varphi=U_t\varphi+\psi(t)$, $t\in\mathbf{R}$, $\varphi\in H$, where U_t are orthogonal (unitary in the complex case) operators, $\psi(t) \in H$; it is clear that g_t are isometric, i.e., $\|g_t\varphi-g_t\psi\|=\|\varphi-\psi\|$, $t\in\mathbf{R}$, $\varphi, \psi\in H$. Such groups can be of two types.

<u>Type 1:</u> "rotation" about a point $\psi_0 \in H$, i.e., $g_t\varphi=U_t(\varphi-\psi_0)+\psi_0$, where $U_t=U_1^t$, $t\in\mathbf{R}$, is a group of orthogonal (unitary) operators.

<u>Type 2:</u> $g_t\varphi=U_t\varphi+t\psi_0$, where $U_t=U_1^t$, $t\in\mathbf{R}$, is a group of orthogonal (unitary) operators, ψ_0 is a characteristic element of them (if U_1 is the identity transformation E, then $g_t\varphi=\varphi+t\psi_0$ is translation by the element $t\psi_0$ if $U_1 \neq E$, then $g_t\varphi$ is a helical motion along the axis $t\psi_0$, $-\infty<t<\infty$).

For transformations of type 1, $\|g_t\varphi-g_u\varphi\|\leqslant C=\|\varphi\|+2\|\psi_0\|$, $t\in\mathbf{R}$, $\varphi\in H$. For transformations of type 2, $\|g_t\varphi-g_u\varphi\|\asymp|t-u|$, $\varphi\in H$.

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