THE CONCEPT OF CAPACITY IN THE THEORY OF FUNCTIONS WITH GENERALIZED DERIVATIVES

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INTRODUCTION

0.1. One of the fundamental ideas of classical potential theory is the concept of capacity. The capacity of a set is one of its very fine characteristics. However, the classical concept of capacity is in many cases insufficient for the study of classes of functions having generalized derivatives. It is the object of the present paper to remedy this shortcoming.

Our main tool is the concept of a Bessel potential. Bessel potentials are a very convenient means of studying questions in the theory of functions with generalized derivatives; this has been demonstrated in the work of a number of authors. (See for example, [2-5].) A Bessel kernel of order $l > 0$ in the space R^n is a function of the form

$$G_l(x) \equiv G_l^{(n)}(x) = \frac{2^{-\frac{n+l-2}{2}}\pi^{-\frac{n}{2}}}{\Gamma\left(\frac{l}{2}\right)} K_{\frac{n-l}{2}}(|x|)|x|^{\frac{l-n}{2}}.$$

where $K_\nu(r)$, $r > 0$ is a so-called Bessel function of third kind. The Fourier transform of the function $G_l(x)$ is given by the formula

$$\hat{G}_l(\xi) = (2\pi)^{-\frac{n}{2}}(1+|\xi|^2)^{-\frac{l}{2}}.$$

The function $G_l(x)$ for $l < n$ has a singularity of type $|x|^{l-n}$ at the origin and for $|x| \to \infty$ $G_l(x)$ is asymptotically equal to $C(1+|x|)^{\frac{l-n-1}{2}}e^{-|x|}$, where $C = \text{const}$.

The Bessel potential of order l in the space R^n of the measurable function f is the function

$$(G_l f)(x) = \int_{R^n} G_l(x-y) f(y)\,dy.$$

Let E be any set in the space R^n. We consider all nonnegative functions f which are summable on R^n in degree $p > 1$ and such that for all $x \in E$

$$(G_l f)(x) \geqslant 1.$$

The greatest lower bound of the quantity $(\|f\|_{L_p})$ taken over the set of all such functions f is called the (l, p) capacity of the set E.

The theory of the (l, p) capacity of sets is the subject of the present paper. We restrict ourselves here to the case in which $0 < l < n$, $p > 1$, and, in addition, the condition $lp \leq n$ is satisfied. The concept of (l, p) capacity can also be considered without these restrictions; however, in this case a number of special features arise which do not come up in our case.

The concept of (l, p) capacity is a special case of the general concept of the p-modulus of a family of measures which was introduced by B. Fuglede [1].

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The class of sets in R^n having zero (l, p) capacity coincides with the class of sets $\mathfrak{A}^{l,p}$ which was introduced and studied in detail in [3, 5]. The (l, p) capacity of a set $E \subset R^n$ is zero if and only if there exists a function f which is summable on R^n in degree p such that $(G_l f)(x) = \infty$ for all $x \in E$. The class $\mathfrak{A}^{l,p}$ is closely related to the theory of the function class L_l^p constructed in [3, 5]. A function u belongs to the class L_l^p if there exists a function v which is summable on R^n in degree p and such that $u(x) = (G_l v)(x)$ almost everywhere. The class L_l^p is an analogue of the class of functions with generalized derivatives. Indeed, if l is an integer, then $L_l^p(R^n)$ coincides with the well-known Sobolev class $W_l^p(R^n)$. The set of possible singularities of an arbitrary function of L_l^p is, as shown in [3], a set of class $\mathfrak{A}^{l,p}$, i.e., a set of zero (l, p) capacity in our terminology.

Different generalizations of the concept of capacity have also been studied by a number of other authors. We mention, in particular, the papers [6–8] in connection with boundary value problems in the theory of partial differential equations.

0.2. We now define certain basic concepts and introduce the terminology and notation used in the following considerations.

R^n henceforth denotes n-dimensional euclidean space. If $x = (x_1, x_2, \ldots, x_n)$, then we put

$$|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}.$$

Further, L_p, with $p \geq 1$, denotes the Banach space of functions which are defined and summable on R^n in degree p. The norm in L_p is defined as usual:

$$\|f\|_{L_p} = \left(\int_{R^n} |f(x)|^p \, dx \right)^{1/p}.$$

If $P(x)$ is some proposition, then $\{x \in A : P(x)\}$ denotes the set of all elements of the set A for which the proposition $P(x)$ is true.

If x is a point of R^n, $r > 0$, then $B(x, r)$ denotes the open sphere of radius r with center x, i.e.,

$$B(x, r) = \{y \in R^n : |y - x| < r\}.$$

Let A be any nonempty set of R^n and x a point of R^n. We then put $\rho(x, A) = \inf_{y \in A} |x - y|$.

Let x be a real number. We put $x^+ = \max(x, 0)$, $x^- = \max(x, 0)$. Clearly, $x^+ - x^- = x$, $x^+ + x^- = |x|$.

It is obvious that for any function $f \in L_p$

$$\|f^+\|_{L_p} \leqslant \|f\|_{L_p}, \|f^-\|_{L_p} \leqslant \|f\|_{L_p}.$$

The support of a function $f: R^n \to R$ is the smallest closed set $A \subset R^n$ such that $f(x) = 0$ for $x \notin A$. A function f is said to be finite if its support is compact.

We say that a function $f: R^n \to R$ belongs to the class C^∞ if it has continuous partial derivatives of every finite order.

§ 1. Definition of Capacity

1.1. Let $K(x, y)$ be a nonnegative function of the variables $x, y \in R^n$ which is defined and lower semicontinuous on $R^{2n} = R^n \times R^n$ (the value $K(x, y) = \infty$ is not excluded). We assume, moreover, that for any $x \in R^n$

$$\int_{R^n} K(x, y) \, dy < \infty. \tag{1.1}$$

Let f be a real, nonnegative, measurable function on R^n. We put

$$U_K f(x) = \int_{R^n} K(x, y) f(y) \, dy.$$

The quantity $U_K f(x)$ has a definite finite or infinite value for all $x \in R^n$.

Let $E \subset R^n$ be any set in R^n. We consider all nonnegative, measurable functions f belonging to the class $L_p(R^n)$ and such that for all $x \in E$

$$U_K f(x) \geqslant 1.$$

The greatest lower bound of the integral

$$\int_{R^n} [f(x)]^p dx,$$

taken over the set of all such functions is called the p capacity of the set E relative to the kernel K. We denote it by the symbol $\mathrm{Cap}_p(E; K)$.

If no such functions f exist, then we put $\mathrm{Cap}_p(E; K) = \infty$.

The concept of p capacity is a special case of the concept of the p modulus of a system of measures introduced by B. Fuglede [1]. Indeed, for $x \in R^n$ let

$$\mu_x(A) = \int_A K(x, y)\, dy.$$

This defines a family of measures μ_x in R^n. Let E be the subfamily consisting of measures μ_x where $x \in E$. Then $\mathrm{Cap}_p(E; K)$ coincides with the p modulus of the family of measures E in the sense of B. Fuglede [1].

1.2. We shall henceforth not consider the case of an arbitrary kernel K. We restrict ourselves to the case

$$K(x, y) = G_l(x - y),$$

where $0 < l < n$, and G_l is a function on R^n with Fourier transform

$$\hat{G}_l(\xi) = (1 + \xi^2)^{-\frac{l}{2}}.$$

The function G_l itself is given by

$$G_l(x) = C_{l,n} \frac{K_{\frac{n-l}{2}}(x)}{|x|^{\frac{n-l}{2}}}, \tag{1.2}$$

where $K_\nu(r)$ is a Bessel function of third kind and $C_{l,n}$ is a constant. We shall call the kernel $G_l(x-y)$ a Bessel kernel of order l.

We recall the following properties of the function G_l. Equation (1.2) implies that $G_l(x) = \beta_l(|x|)$, where the function $\beta_l(r)$ is defined for all $r > 0$ and such that $\beta_l(r) > 0$ for $r > 0$ and $\beta_l(r)$ is a monotone decreasing function. Moreover, $\beta_l(r) = o(e^{-r})$ for $r \to \infty$ and $\beta_l(r) = \frac{C}{r^{n-l}}[1 + o(1)]$ for $r \to 0$, where $C > 0$, $C = \mathrm{const}$. From this it follows that the function $G_l(x-y)$ satisfies all the conditions imposed on the function $K(x, y)$.

We henceforth denote the quantity $U_G f$ by the symbol $G_l f$ and call it the Bessel potential of order l of the function f.

For any $E \subset R^n$ we denote by $\mathfrak{M}_{l,p}(E)$ the set of all nonnegative functions $f \in L_p(R^n)$ such that $(G_l f)(x) \geq 1$ for all $x \in E$.

The p capacity relative to the kernel G_l we shall call the (l, p) capacity and denote it by the symbol $\mathrm{Cap}_{(l,p)} E$.

From the definition of (l, p) capacity we immediately obtain

LEMMA 1.1. Let E be any set in R^n and let $f \in L_p(R^n)$ be a nonnegative function such that $G_l f(x) > \alpha > 0$ for all $x \in E$. Then

$$\mathrm{Cap}_{(l,p)} E \leqslant \frac{1}{\alpha^p} \|f\|^p_{L_p}. \tag{1.2}$$

Proof. It is clear that the function $g = f/\alpha$ belongs to the class $\mathfrak{N}_{l,p}(E)$, and therefore

$$\mathrm{Cap}_{(l,p)}E \leqslant \|g\|^p_{L_p} = \frac{1}{\alpha^p}\|f\|^p_{L_p},$$

which was required to prove.

1.3. From the general properties of the modulus of a system of measures proved in [1] there follow the following propositions regarding the (l, p) capacity.

THEOREM 1.1. For any $E \subset R^n$ $\mathrm{Cap}_{(l,p)} E \geq 0$.

THEOREM 1.2. For any two sets E_1 and E_2 such that $E_1 \subset E_2$

$$\mathrm{Cap}_{(l,p)}E_1 \leqslant \mathrm{Cap}_{(l,p)}E_2.$$

THEOREM 1.3. Let $\{E_\nu\}$, $\nu = 1, 2, \ldots$, be any sequence of sets in R^n. Then

$$\mathrm{Cap}_{(l,p)}\bigcup_\nu E_\nu \leqslant \sum_\nu \mathrm{Cap}_{(l,p)}E_\nu.$$

THEOREM 1.4. Let $E \subset R^n$. Then in order that the (l, p) capacity of the set E be zero it is necessary and sufficient that there exist a nonnegative function $f \in L_p(R^n)$ such that $(G_l f)(x) = \infty$ for all $x \in E$.

THEOREM 1.5. If $\mathrm{Cap}_{(l,p)}E = 0$, then for any set $E' \subset E$ $\mathrm{Cap}_{(l,p)}E' = 0$.

THEOREM 1.6. The union of an at most countable set of sets of zero (l, p) capacity is a set of zero (l, p) capacity.

THEOREM 1.7. If $\mathrm{Cap}_{(l,p)} E = 0$, then for any $q < p$, $q > 1$

$$\mathrm{Cap}_{(l,q)}E = 0.$$

THEOREM 1.8. If $f \in L_p(R^n)$, $f \geq 0$, then the set of all $x \in R^n$ such that $G_l f(x) = \infty$ is a set of zero (l, p) capacity.

We say that a condition C is satisfied on a set $E \subset R^n$ almost everywhere with respect to the (l, p) capacity, or, more briefly, (l, p) almost everywhere on E, if the (l, p) capacity of the set of $x \in E$ for which condition C is not satisfied is zero.

THEOREM 1.9. Let

$$f_1 + f_2 + \ldots + f_\nu - \ldots \tag{1.3}$$

be any series in $L_p(R^n)$ such that $\sum_{\nu=1}^{\infty} \|f_\nu\|_{L_p} < \infty$. Then the series

$$(G_l f_1)(x) + (G_l f_2)(x) + \ldots + (G_l f_\nu)(x) + \ldots \tag{1.4}$$

converges (l, p) almost everywhere on R^n. Moreover, if f is the sum of the series (1.3), then the series (1.4) converges to $(G_l f)(x)$ (l, p) almost everywhere on R^n.

There is no general theorem corresponding to Theorem 1.9 in [1], and we shall therefore present its proof.

The hypotheses of the theorem imply that

$$\sum_{\nu=1}^{\infty} \|f_\nu^+\|_{L_p} < \infty, \sum_{\nu=1}^{\infty} \|f_\nu^-\|_{L_p} < \infty.$$

The functions $h = f_1^+ + f_2^+ + \ldots + f_\nu^+ + \ldots$ and $g = f_1^- + f_2^- + \ldots + f_\nu^- + \ldots$ belong to the class L_p. Since the integrands of the potentials $G_l f_\nu^+$ and $G_l f_\nu^-$ are nonnegative, we have for all $x \in R^n$

$$\sum_{\nu=1}^{\infty} G_l f_\nu^+(x) = G_l h(x), \tag{1.5}$$

$$\sum_{\nu=1}^{\infty} G_l f_\nu (x) = G_l g (x). \tag{1.6}$$

From this it is clear that the series (1.4) converges for any x such that $(G_l g)(x) < \infty$ and $(G_l h)(x) < \infty$, and hence the set of x for which the series (1.4) diverges is a set of zero (l, p) capacity. Moreover, if the series (1.5) and (1.6) converge at a point x, then the sum of the series (1.4) for this x is equal to the difference $(G_l h)(x) - (G_l g)(x) = (G_l f)(x)$. This completes the proof of Theorem 1.9.

THEOREM 1.10. Let f_ν, $\nu = 1, 2, \ldots$, be a sequence of functions in L_p such that $\|f_\nu - f\|_{L_p} \to 0$ for $\nu \to \infty$. Then there exists a sequence of indices $\{\nu_k\}$, $\nu_1 < \nu_2 < \ldots < \nu_k < \ldots$, such that

$$(G_l |f_{\nu_k} - f|)(x) \to 0$$

(l, p) almost everywhere on R^n as $k \to \infty$.

Theorem 1.10 is a special case of a general theorem of [1]. It can also be obtained as a corollary of Theorem 1.9.

THEOREM 1.11. For any set $E \subset R^n$ such that $\mathrm{Cap}_{(l, p)} E < \infty$, there exists a nonnegative function $f \in L_p$ such that $(G_l f)(x) \geq 1$ (l, p) almost everywhere on E and

$$\mathrm{Cap}_{(l, p)} E = \int_{R^n} [f(x)]^p dx.$$

Let E be any set in R^n. We denote by $\overline{\mathfrak{M}}^{l, p}(E)$ the closure of the set $\mathfrak{M}^{(l, p)}(E)$ in the space L_p.

LEMMA 1.2. In order that a function $f \geq 0$, $f \in L_p$ belong to the set $\overline{\mathfrak{M}}^{l, p}(E)$, it is necessary and sufficient that $(G_l f)(x) \geq 1$ (l, p) almost everywhere on E.

Proof. The necessity of the condition follows in an obvious way from Theorem 1.10. We prove sufficiency. Let $f \geq 0$, $f \in L_p$ be such that the (l, p) capacity of the set E' of all $x \in E$ for which $(G_l f)(x) < 1$ is equal to zero. By Theorem 1.4, there exists a function $v \geq 0$, $v \in L_p$ such that $G_l v(x) = \infty$ for all $x \in E$. We put $f_\nu = f + (1/\nu) v$, $\nu = 1, 2, \ldots$. Then $(G_l f_\nu)(x) \geq 1$ for all $x \in E$, and hence $f_\nu \in \mathfrak{M}^{(l, p)}(E)$ for all ν. For $\nu \to \infty$ $f_\nu \to f$ in L_p, whence it follows that $f \in \overline{\mathfrak{M}}^{l, p}(E)$. This completes the proof of the lemma.

THEOREM 1.12. The function $\mathrm{Cap}_{(l, p)}$ is invariant under motions of the space R^n, i.e., if the sets A and B can be obtained from one another by a motion of the space R^n, then $\mathrm{Cap}_{(l, p)} A = \mathrm{Cap}_{(l, p)} B$.

Proof. Let $B = \varphi A$, where φ is a motion of R^n. We take an arbitrary function $f \in \mathfrak{M}^{l, p}(A)$. Then for all $x \in A$

$$\int_{R^n} G_l(x - y) f(y) dy \geqslant 1.$$

This implies that for all $x \in B$

$$1 \leqslant \int_{R^n} G_l(\varphi x - y) f(y) dy = \int_{R^n} G_l(\varphi x - \varphi y) f(\varphi^{-1}(y)) dy = \int_{R_n} G_l(x - y) f(\varphi^{-1}(y)) dy.$$

This means that the function $g: y \to f(\varphi^{-1} y)$ belongs to $\mathfrak{M}^{l, p}(B)$. Hence,

$$\mathrm{Cap}_{(l, p)} B \leqslant \|g\|_{L_p}^p = \|f\|_{L_p}^p,$$

and since $f \in \mathfrak{M}^{l, p}(A)$ is arbitrary,

$$\mathrm{Cap}_{(l, p)} B \leqslant \mathrm{Cap}_{(l, p)} A.$$

The sets A and B are equivalent, whence it follows that

$$\mathrm{Cap}_{(l, p)} A \leqslant \mathrm{Cap}_{(l, p)} B.$$

and hence $\mathrm{Cap}_{(l, p)} A = \mathrm{Cap}_{(l, p)} B$, which was required to prove.

§ 2. Sets Measurable with Respect to the (l, p) Capacity

Let φ be a nonnegative set function defined on all subsets of the space R^n. The function φ is said to be a generalized capacity [9] if it satisfies the following axioms:

1) If $A \subset A'$, then $\varphi(A) \leq \varphi(A')$.

2) For any increasing sequence of sets $\{A_\nu\}$, $\nu = 1, 2, \ldots,$ in R^n

$$\varphi(\bigcup_\nu A_\nu) = \sup_\nu \varphi(A_\nu).$$

3) For any decreasing sequence of compact sets $\{K_\nu\}$, $\nu = 1, 2, \ldots,$ in R^n

$$\varphi(\bigcap_\nu K_\nu) = \inf_\nu \varphi(K_\nu).$$

A set $A \subset R^n$ is said to be measurable with respect to the generalized capacity φ if

$$\varphi(A) = \sup_{K \subset A} \varphi(K),$$

where K is compact.

THEOREM 2.1 [9]. Let φ be a generalized capacity in R^n. Then any analytic set in R^n is measurable with respect to φ.

LEMMA 2.1. If a function $v \in L_p$ is nonnegative, then its Bessel potential $G_l v$ is a function which is lower semicontinuous.

This lemma is an obvious corollary of Fatou's theorem on taking the limit under the sign of the Lebesgue integral.

LEMMA 2.2. Let $E \subset R^n$ be any set in R^n such that $\mathrm{Cap}_{(l,p)}E < \infty$. Then for any $\varepsilon > 0$ there exists an open set $U \supset E$ such that $\mathrm{Cap}_{(l,p)}U < \mathrm{Cap}_{(l,p)}E + \varepsilon$.

Proof. By definition, for any $\varepsilon > 0$ there exists a nonnegative function $v \in L_p$ such that $G_l v(x) \geq 1$ for all $x \in E$ and

$$\int_{R^n} [v(x)]^p \, dx < \mathrm{Cap}_{(l,p)} E + \frac{\varepsilon}{2}.$$

Let $\eta > 0$, $\eta < 1$ be arbitrary. Let U_η be the set of all $x \in R^n$ such that $(G_l v)(x) > 1 - \eta$. By Lemma 2.1, the set U_η is open. Moreover, $E \subset U_\eta$ and by Lemma 1.1,

$$\mathrm{Cap}_{(l,p)} U_\eta \leqslant \frac{1}{(1-\eta)^p} \int_{R^n} [v(x)]^p \, dx < \frac{\mathrm{Cap}_{(l,p)} E + \varepsilon/2}{(1-\eta)^p}.$$

We suppose that η has been chosen such that the right side of last inequality is less than $\mathrm{Cap}_{(l,p)}E + \varepsilon$. Then $U = U_\eta$ is the set required.

THEOREM 2.1. For any l and p such that $l > 0$, $p > 1$, $lp \leq n$, the (l, p) capacity is a generalized capacity.

Proof. By Theorem 1.1, the function $\mathrm{Cap}_{(l,p)}E$ is nonnegative. By Theorem 1.2, it satisfies axiom 1) for a generalized capacity.

We will show that it satisfies axiom 2). Let $\{A_\nu\}$, $\nu = 1, 2, \ldots,$ be an arbitrary increasing sequence of sets in R^n, $A = \bigcup_\nu A_\nu$. For each ν $\mathrm{Cap}_{(l,p)}A \geq \mathrm{Cap}_{(l,p)}A_\nu$, and hence

$$\mathrm{Cap}_{(l,p)} A \geqslant \sup_\nu \mathrm{Cap}_{(l,p)} A_\nu. \tag{2.1}$$

From (2.1) it follows that if $\sup_\nu \mathrm{Cap}_{(l,p)}A_\nu = \infty$, then the equality of axiom 2) is valid for the given sequence.

We now assume that

$$\gamma = \sup_\nu \mathrm{Cap}_{(l,p)} A_\nu < \infty. \tag{2.2}$$

Let $M_\nu = \mathfrak{M}^{(l,p)}(A_\nu)$. The sets M_ν are all closed and convex, and the sequence of sets M_ν is decreasing. We put

$$M = \bigcap_{\nu=1}^{\infty} M_\nu .$$

We show that M is nonempty. Indeed, let $d_\nu = (\mathrm{Cap}_{(l,p)} A_\nu)^{1/p}$ be the distance from the point 0 in the space L_p to the set M_ν. The sequence $\{d_\nu\}$ is increasing, and $d_\nu \to d_0 = \gamma^{1/p}$. By inequality (2.2), $d_0 < \infty$. We consider the sphere $B = \{u \in L_p : \|u\|_{L_p} \leq d_0\}$. For each ν, $M_\nu \cap B$ is a nonempty, closed convex set in L_p. Because of the weak compactness of the sphere B, the intersection

$$\bigcap_{\nu} (M_\nu \cap B) \subset M$$

is nonempty, and hence the set M is nonempty.

It is clear that M is closed and convex. Let u_0 be the point of M closest to the point 0. Then $\|u_0\| \leq d_0$. The function $u_0 \in M_\nu$ for all ν and hence for any ν

$$(G_l u_0)(x) \geq 1$$

(l, p) almost everywhere on A_ν. This implies that

$$(G_l u_0)(x) \geq 1$$

(l, p) almost everywhere on A, and hence $u_0 \in \mathfrak{M}^{(l,p)}(A)$; therefore

$$\gamma = d_0^p \geq \|u_0\|^p_{L_p} \geq \mathrm{Cap}_{(l,p)} A . \tag{2.3}$$

Comparing inequalities (2.2) and (2.3), we find that

$$\mathrm{Cap}_{(l,p)} A = \sup_{\nu} \mathrm{Cap}_{(l,p)} A_\nu$$

and this completes the verification of axiom 2).

We now shown that axiom 3) is satisfied. It is easily shown that the (l, p) capacity of any compact set in R^n is finite. Let $\{K_\nu\}$, $\nu = 1, 2, \ldots$, be an arbitrary decreasing sequence of compact sets

$$K = \bigcap_{\nu=1}^{\infty} K_\nu .$$

The set K is compact. We suppose that $\varepsilon > 0$ is given. By Lemma 2.2, there exists an open set $U \supset K$ such that $\mathrm{Cap}_{(l,p)} U < \mathrm{Cap}_{(l,p)} K + \varepsilon$. By the compactness of the set K_ν, there exists a ν_0 such that for $\nu > \nu_0$, $K_\nu \subset U$. For $\nu > \nu_0$ we have:

$$\mathrm{Cap}_{(l,p)} K \leq \mathrm{Cap}_{(l,p)} K_\nu \leq \mathrm{Cap}_{(l,p)} U < \mathrm{Cap}_{(l,p)} K + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\mathrm{Cap}_{(l,p)} K_\nu \to \mathrm{Cap}_{(l,p)} K$$

for $\nu \to \infty$. This completes the verification of axiom 3). The theorem has now been proved.

§ 3. The Dual Definition of the Concept of (l, p) Capacity

3.1. All measures henceforth considered are assumed to be defined on the σ-algebra of Borel sets in R^n.

We say that a measure μ is concentrated on a set $E \subset R^n$ if there exists a Borel set $E' \subset E$ such that for any Borel set A $\mu(A) = \mu(A \cap E')$.

We fix $l > 0$ and $p > 1$ such that $lp \leq n$. We put $q = p/(p-1)$.

For any measure μ in R^n we set

$$(G_l \mu)(x) = \int_{R^n} G_l(x-y)\, d\mu(y).$$

The function $G_l\mu$ is called the Bessel potential of order l of the measure μ. The function $G_l\mu$ is nonnegative and lower semicontinuous on R^n.

Let $E \subset R^n$. The dual (l, p) capacity of the set E is defined by

$$\sup_{\mu} \frac{[\mu(R^n)]^p}{\left\{\int\limits_{R^n} [(G_l\mu)(x)]^q\, dx\right\}^{p-1}}, \tag{3.1}$$

where the supremum is taken over the set of all measures μ concentrated on the set E. We denote the dual (l, p) capacity by the symbol $\overline{\mathrm{Cap}}_{(l,p)}E$.

The function of the measure μ standing to the right of the supremum in (3.1) is positive homogeneous of degree zero with respect to the measure μ, i.e., its value is unchanged if μ is multiplied by an arbitrary positive constant. This implies that the dual (l, p) capacity can also be defined as follows. We denote by $\mathfrak{A}_{l,p}(E)$ and $\mathfrak{B}_{(l,p)}(E)$ the sets consisting of all measures concentrated on E and satisfying the conditions in the case of $\mathfrak{A}_{l,p}(E)$: $\mu(R^n) \le 1$ and $G_l\mu \in L_q$, and in the case of $\mathfrak{B}_{(l,p)}(E)$:

$$\|G_l\mu\|_L \leqslant 1.$$

Then

$$\overline{\mathrm{Cap}}_{(l,p)}E = \sup_{\mu\in\mathfrak{A}_{l,p}(E)} \frac{1}{\|G_l\mu\|_{L_q}^{p-1}} = \sup_{\mu\in\mathfrak{B}_{l,p}(E)} [\mu(R^n)]^p \tag{3.2}$$

LEMMA 3.1. For any E_1 and E_2 in R^n such that

$$E_1 \subset E_2, \quad \overline{\mathrm{Cap}}_{(l,p)}E_1 \leqslant \overline{\mathrm{Cap}}_{(l,p)}E_2.$$

Proof. Indeed, if $E_1 \subset E_2$, then clearly $\mathfrak{A}_{(l,p)}(E_1) \subset \mathfrak{A}_{l,p}(E_2)$, whence by (3.2) the required inequality follows.

LEMMA 3.2. For any set $E \subset R^n$

$$\overline{\mathrm{Cap}}_{(l,p)}E \leqslant \mathrm{Cap}_{(l,p)}E.$$

Proof. We take an arbitrary measure $\mu \in \mathfrak{B}_{l,p}(E)$ and function $\mu \in \mathfrak{M}^{(l,p)}(E)$. We have the following inequalities:

$$\|u\|_{L_p} \geqslant \|u\|_{L_p}\|G_l\mu\|_{L_q}$$
$$\geqslant \int\limits_{R^n} u(y)(G_l\mu)(y)\,dy = \int\limits_{R^n} (G_lu)(x)\,d\mu(y) \geqslant \mu(R^n),$$

since $G_lu(x) \ge 1$ by hypothesis for all $x \in E$. Since $u \in \mathfrak{M}^{(l,p)}(E)$ and $\mu \in \mathfrak{B}_{l,p}(E)$ are arbitrary, this completes the proof of the lemma.

LEMMA 3.3. Let f be any function on R^n having compact support and belonging to the class C^∞. Then there exists a function $u \in C^\infty$ such that $u(x) = o(e^{-|x|})$ for $|x| \to \infty$ and $f = G_lu$.

Proof. Let m be a natural number such that $2m \ge l$. We put

$$u(x) = (2\pi)^n G_{2m-l}(I - \Delta)^m f,$$

where I is the identity operator and Δ is the Laplace operator. The function $v = (I-\Delta)^m f$ has compact support and belongs to the class C^∞. This implies that $u \in C^\infty$ and $u(x) = o(e^{-|x|})$ for $|x| \to \infty$. The equality

$$G_lu = f$$

is easy to check by considering the Fourier transforms of the function u and f. This completes the proof of the lemma.

THEOREM 3.1. For any compact set $A \subset R^n$ it is true that $\mathrm{Cap}_{(l,p)}A = \overline{\mathrm{Cap}}_{(l,p)}A$. Moreover, there exists a measure $\mu_0 \in \mathfrak{B}_{(l,p)}(A)$ such that $[\mu_0(R^n)]^p = \mathrm{Cap}_{(l,p)}A$. If the function $u_0 \in \mathfrak{M}^{(l,p)}(A)$ is such that $\mathrm{Cap}_{(l,p)}A = \|u_0\|_{L_p}$, then the measure μ_0 is concentrated on the set $E = \{x \in A: G_lu_0(x) \le 1\}$ and for all $x \in R^n$

$$[u_0(x)]^{p-1} = \|u_0\|_{L_p}^{p-1}(G_l\mu_0)(x).$$

Proof. We begin with the fact that by (3.2)

$$\overline{\mathrm{Cap}}_{(l,p)}(A)=\sup_{\mu\in\mathfrak{B}(l,p)(A)}[\mu(R^n)]^p.$$

If $\mathrm{Cap}_{(l,p)}A=0$, then Lemma 3.2 implies that $\overline{\mathrm{Cap}}(l,p)A=0$, i.e., $\mathrm{Cap}(l,p)A=\overline{\mathrm{Cap}}(l,p)A$.

We assume that $\mathrm{Cap}_{(l,p)}A>0$. Let u_0 be a function of $\overline{\mathfrak{M}}_{l,p}(A)$ such that $\mathrm{Cap}_{(l,p)}A=\|u_0\|$. We put

$$E=\{x\in A:\ (G_lu_0)\leqslant 1\}$$

Since by Lemma 2.1 G_lu_0 is lower semicontinuous, it follows that the set E is compact.

The set E is nonempty. Indeed, if it were empty, then for all $x\in A$ we would have $(G_lu_0)(x)>1$. Since G_lu_0 is lower semicontinuous and the set A is compact, the function G_lu_0 assumes its minimum on the set A at some point $x_0\in A$. For all $x\in A$ $(G_lu_0)(x)\geq (G_lu_0)(x_0)=1+\delta$, where $\delta>0$. The function

$$u_1=u_0/(1+\delta)\in\mathfrak{M}^{l,p}(A).\ \text{On the other hand}$$
$$\|u_1\|^p_{L_p}<\|u_0\|^p_{L_p},$$

which contradicts the definition of u_0. The contradiction thus obtained proves that at least for one $x\in A$ $(G_lu)(x)\leq 1$, i.e., E is nonempty.

We denote by S(E) the set of all functions $\eta\in L_p(R^n)$ such that the function $G_l\eta$ is continuous and for all $x\in E$ $(G_l\eta)(x)\geq 0$.

Further, we put

$$v_0=u_0^{p-1}/\|u_0\|^{p-1}_{L_p}.$$

It is then obvious that

$$v_0\in L_q(R^n)\text{ and }\|v_0\|_{L_q}=1\,.$$

We shall show that for any function $\eta\in S(E)$

$$\int_{R^n}v_0(x)\eta(x)\,dx\geqslant 0\,.$$

We first suppose that the function $\eta\in S(E)$ is such that $G_l\eta(x)>0$ for all $x\in E$. Then there exists an open set $U\supset E$ such that $G_l\eta(x)>0$ for all $x\in U$. We set $H=A\setminus U$. The set H is compact and for all $x\in H$ $(G_lu_0)(x)>1$. Since G_lu_0 is lower semicontinuous, this implies that for all $x\in H$ $(G_lu_0)(x)\geq 1+\delta$, where $\delta>0$.

We now show that there exists a $t_0>0$ such that $0<t<t_0$ $(G_lu_0)(x)+t(G_l\eta)(x)\geq 1$ (l,p) almost everywhere on A. Indeed, for $x\in A\cap U$ and all $t>0$

$$(G_lu_0)(x)+t(G_l\eta)(x)\geqslant (G_lu_0)(x)\geqslant 1$$

(l,p) almost everywhere on $A\cap U$. Let $x\in H=A\setminus U$. We set $M=\max|(G_l\eta)(x)|$. Then for $0<t<t_0=\delta/M$

$$(G_l(u_0+t\eta))(x)>1$$

for all $x\in H$. This implies that for $0<t<t_0$

$$(G_l(u_0+t\eta))(x)\geqslant 1$$

(l,p) almost everywhere on A.

Let $0<t<t_0$. Clearly

$$(G_l|u_0+t\eta|)(x)\geqslant (G_l(u_0+t\eta))(x)$$

for all x. This implies that for $0<t<t_0$ $|u_0+t\eta|\in\mathfrak{M}^{(l,p)}(A)$. Since the function u_0 gives the minimum value of the function $\|u\|^p_{L_p}$ on the set $\mathfrak{M}^{(l,p)}(A)$, this means that for any $t\in(0,\delta)$

$$\int_{R^n} |u_0(x)+t\eta(x)|^p dx \geqslant \int_{R^n} |u_0(x)|^p dx.$$

This implies that

$$0 \leqslant p \int_{R^n} |u_0(x)|^{p-1}\eta(x)\,dx = \lim_{\substack{t\to 0\\ t>0}} \int_{R^n} \frac{|u_0(x)+t\eta(x)|^p - |u_0(x)|^p}{t}\,dx$$

and hence

$$\int_{R^n} v_0(x)\eta(x)\,dx \geqslant 0. \tag{3.3}$$

This proves inequality (3.3) under the assumption that $(G_l\eta)(x) > 0$ for all $x \in E$. Let $\eta \in S(E)$ be such that $(G_l\eta)(x) \geq 0$ on E. Let $\varphi \in C^\infty$ be an arbitrary nonnegative function with compact support which is not identically zero. Then $(G_l\varphi)(x) > 0$ for all $x \in R^n$ and $G_l\varphi \in C^\infty$. The function $\eta + \tau\varphi \in S(E)$ for any $\tau > 0$ and $[G_l(\eta + \tau\varphi)](x) > 0$ for all $x \in E$. Thus, by what has been shown

$$\int_{R^n} v_0(x)[\eta(x)+\tau\varphi(x)]\,dx \geqslant 0$$

for any $\tau > 0$. Letting τ tend to zero, we obtain in the limit

$$\int_{R^n} v_0(x)\eta(x)\,dx \geqslant 0,$$

which was required to prove.

Let $\eta \in L_p$ be such that the function $(G_l\eta)(x)$ is continuous and for all $x \in E$ $(G_l\eta)(x) = 0$. Then

$$\int_{R^n} v_0(x)\eta(x)\,dx = 0. \tag{3.4}$$

Indeed, the functions $\eta(x)$ and $-\eta(x)$ belong to S(E), and hence

$$\int_{R^n} v_0(x)\eta(x)\,dx \geqslant 0, \quad \int_{R^n} v_0(x)[-\eta(x)]\,dx \geqslant 0,$$

which implies (3.4).

We denote by C(E) the set of all continuous real-valued functions defined on the set E. Let $C^\infty(E)$ be the set of all those functions of C(E), each of which is the restriction to E of some function $\varphi \in C^\infty(R^n)$ with compact support. For $f \in C(E)$ we put $\|f\|_{C(E)} = \max|f(x)|$. It is obvious that $C^\infty(E)$ is a dense linear subset of the Banach space C(E).

We now define a certain linear functional L on C(E). The functional L will first be defined on the set $C^\infty(E)$. Let $u \in C^\infty(E)$. Then there exists a function $\varphi \in C^\infty(R^n)$ such that $\varphi(x) = u(x)$ for all $x \in E$. By Lemma 3.3, there exists a function $\eta \in L_p(R^n)$ such that $\varphi(x) = (G_l\eta)(x)$. We put

$$L(u) = \int_{R^n} v_0(x)\eta(x)\,dx$$

The quantity L(u) does not depend on the choice of the extension φ of the function u. Indeed, let φ_1 and φ_2 be two functions with compact support in C^∞ such that $\varphi_1(x) = \varphi_2(x) = u(x)$ for all $x \in E$. Then $\varphi_1 = G_l\eta_1$, $\varphi_2 = G_l\eta_2$. We set $\eta = \eta_1 - \eta_2$. The function $G_l\eta$ is continuous, and $G_l\eta(x) = 0$ for all $x \in E$. By what has been proved, this means that

$$0 = \int_{R^n} v_0(x)\eta(x)\,dx = \int_{R^n} v_0(x)[\eta_1(x) - \eta_2(x)]\,dx,$$

whence

$$\int_{R^n} v_0(x)\eta_1(x)\,dx = \int_{R^n} v_0(x)\eta_2(x)\,dx .$$

This proves that the quantity L(u) does not depend on the choice of the extension φ of the function u.

It is clear that L is a linear functional on the linear subset $C^\infty(E)$ of the space C(E). By the properties of the functions of the class S(E) proved above, it follows that for any nonnegative function $u \in C^\infty(E)$ $L(u) \geq 0$. Thus, L is a nonnegative functional on $C^\infty(E)$. This implies that the functional L is bounded, and hence continuous on $C^\infty(E)$. We extend L by continuity to the entire space C(E). Such an extension exists uniquely and represents a linear functional on C(E).

By the well-known theorem on the representation of a nonnegative linear functional on the space of continuous functions on a compact metric space, there exists a measure μ_0 concentrated on the set E such that for all $u \in C(E)$

$$Lu = \int_E u(x)\,d\mu_0(x).$$

We extend the measure μ_0 to the entire space R^n by defining $\mu_0(B) = \mu_0(B \cap E)$ for any Borel set B.

Let $\eta \in C^\infty$ be any function with compact support, and let u be the restriction of the function $(G_l\eta)$ to the set E. $u \in C^\infty(E)$. We then have

$$\int_{R^n} v_0(x)\eta(x)\,dx = Lu = \int_{R^n} u(x)\,d\mu_0(x) = \int_{R^n} (G_l\eta)(x)\,\mu_0(dx) = \int_{R^n} (G_l\mu_0)(x)\,\eta(x)\,dx ,$$

by Fubini's theorem. We thus see that for any function $\eta \in C^\infty(R^n)$ with compact support in R^n

$$\int_{R^n} v_0(x)\eta(x)\,dx = \int_{R^n} (G_l\mu_0)(x)\,\eta(x)\,dx$$

Since η is arbitrary, this implies that for almost all $x \in R^n$

$$v_0(x) = (G_l\mu_0)(x). \tag{3.5}$$

The measure μ_0 is concentrated on the set A and

$$\|G_l\mu_0\|_{L_q} = \|v_0\|_{L_q} = 1,$$

whence it follows that $\mu_0 \in \mathfrak{B}_{(l,\,p)}(A)$.

The proof of the theorem is now completed as follows. By the definition of the function u_0,

$$\mathrm{Cap}_{(l,\,p)} A = \|u_0\|^p_{L_p(R^n)} = \left(\int_{R^n} v_0(x)\,u_0(x)\,dx\right)^p.$$

From this and equality (3.5) we conclude that

$$\mathrm{Cap}_{(l,\,p)} A = \left(\int_{R^n} u_0(x)(G_l\mu_0)(x)\,dx\right)^p = \left[\int_{R^n} (G_lu_0)(y)\,d\mu_0(y)\right]^p = \left[\int_E (G_lu_0)(y)\,d\mu_0(y)\right]^p.$$

By the definition of the set E $(G_lu_0)(y) \leq 1$ for all $y \in E$, whence it follows that

$$\int_{R_n} (G_lu_0)(y)\,\mu_0(dy) \leqslant \mu_0(E) = \mu_0(R^n).$$

As was shown above, the measure $\mu_0 \in \mathfrak{B}_{(l,\,p)}(A)$, and this means that $\mu_0(R^n) \leq [\overline{\mathrm{Cap}}_{(l,\,p)}A]^{1/p}$. We thus obtain the inequality

$$\mathrm{Cap}_{(l,\,p)} A \leqslant \overline{\mathrm{Cap}}_{(l,\,p)} A.$$

By Lemma 3.2, this implies that $\mathrm{Cap}_{(l,\,p)}A = \overline{\mathrm{Cap}}_{(l,\,p)}A$.

It is also easy to see that the measure μ_0 satisfies all the conditions required. This completes the proof of the theorem.

THEOREM 3.2. For any set $A \subset R^n$ which is measurable with respect to the (l, p) capacity, the (l, p) capacity and dual (l, p) capacity coincide.

Proof. Let A be any set which is measurable with respect to the (l, p) capacity. Then for any compact set $K \subset A$ we have:

$$\mathrm{Cap}_{(l,p)} A \geqslant \overline{\mathrm{Cap}}_{(l,p)} A \geqslant \overline{\mathrm{Cap}}_{(l,p)} K = \mathrm{Cap}_{(l,p)} K.$$

Since

$$\mathrm{Cap}_{(l,p)} A = \sup_{K \subset A} \mathrm{Cap}_{(l,p)} K,$$

this implies that

$$\overline{\mathrm{Cap}}_{(l,p)} A = \mathrm{Cap}_{(l,p)} A,$$

which was required to prove.

3.2. In the case $p = 2$ the (l, p) capacity coincides with the concept of capacity with respect to the Bessel potential which was introduced in the work of Aronszajn and Smith [2].

Let A be any set in R^n, and let α be a real number such that $0 < 2\alpha \leq n$. We denote by $\mathfrak{K}(A)$ the set of all measures μ concentrated on A such that $\mu(A) \geq 1$. For $\mu \in \mathfrak{K}(A)$ we set

$$I_{2\alpha}(\mu) = \int_{R^n \times R^n}\int G_{2\alpha}(x-y)\, d\mu(x)\, d\mu(y).$$

We put

$$C_{2\alpha}(A) = \sup_{\mu \in \mathfrak{K}(A)} \frac{1}{(I_{2\alpha}(\mu))^{1/2}}.$$

The quantity $C_{2\alpha}(A)$ is the (2α) capacity of the set A in the sense of Aronszajn and Smith [2].

THEOREM 3.3. For any set $A \subset R^n$

$$C_{2\alpha}(A) = \overline{\mathrm{Cap}}_{(\alpha,2)} A.$$

Proof. By (3.2)

$$\overline{\mathrm{Cap}}_{(\alpha,2)} A = \sup_{\mu \in \mathfrak{K}_{(\alpha,2)}(A)} \frac{1}{\| G_\alpha \mu \|^2_{L_2}} \tag{3.6}$$

We transform the expression in the denominator on the right-hand side:

$$\| G_\alpha \mu \|^2_{L_2} = \int_{R^n} [(G_\alpha \mu)(x)]^2\, dx = \int_{R^n}\Big(\int_{R^n} G_\alpha(x-y)\, d\mu(y)\Big)\Big(\int_{R^n} G_\alpha(z-x)\, d\mu(z)\Big) dx$$
$$= \int_{R^n}\int_{R^n}\Big[\int_{R^n} G_\alpha(x-y) G_\alpha(z-x)\, dx\Big] d\mu(y)\, d\mu(z),$$

by Fubini's theorem.

But since,

$$\int_{R^n} G_\alpha(x-y) G_\alpha(x-z)\, dx = G_{2\alpha}(z-y),$$

this means that,

$$\|G_\alpha\mu\|_{L_2}^2 = I_{2\alpha}(\mu)$$

which completes the proof of the theorem.

§ 4. Capacity and the Hausdorff Measure

4.1. We recall the concept of a Hausdorff h-measure. Let $h(r)$, $0 \le r < \infty$, be a nondecreasing function such that $h(0) = 0$ and $h(r) \to \infty$ for $r \to \infty$.

Let A be any set in R^n. Let $\varepsilon > 0$ be given, and suppose that $B_1, B_2, \ldots, B_\nu, \ldots$ is a sequence of openspheres such that $A \subset \bigcup_\nu B_\nu$, and the radii $r_1, r_2, \ldots, r_\nu, \ldots$ of the spheres does not exceed ε. The greatest lower bound of the sum

$$\sum_\nu h(r_\nu),$$

taken over the set of all sequences of spheres with the prescribed properties is denoted by $\mu_h(A, \varepsilon)$. The quantity $\mu_h(A, \varepsilon)$ is a nonincreasing function of ε. The limit $\lim_{\varepsilon\to 0} \mu_h(A, \varepsilon) = \mu_h(A)$ is called the Hausdorff h-measure of the set A. In the case $h(r) = r^\alpha$, $\alpha > 0$, $\mu_h(A)$ is called the α-dimensional Hausdorff measure and is denoted by the symbol $\mu_\alpha(A)$. The measure $\mu_1(A)$ is also called the linear Hausdorff measure.

If the functions $h_1(r)$ and $h_2(r)$ are such that $h_1(r) = h_2(r)$ for $0 \le r \le r_0$, $r_0 > 0$, then the corresponding Hausdorff measures μ_{h_1} and μ_{h_2} coincide. For this reason, in the definition of a Hausdorff h-measure it may be assumed that the function is defined initially only on some interval $[0, r_0]$, where $r_0 > 0$, and is extended beyond this interval in an arbitrary manner. The final result does not depend on how the function is extended.

In addition to the Hausdorff measure, we need one more property of a set. Let $h(r)$ be a monotone nondecreasing function defined for all $r \ge 0$ and such that $h(0) = 0$ and $h(r) \to \infty$ for $r \to \infty$. We consider all sequences of open spheres $\{B_\nu\}$, $\nu = 1, 2, \ldots,$ which cover a given set A. The greatest lower bound of the sum

$$\sum_{\nu=1}^{\infty} h(r_\nu),$$

where r_ν is the radius of the sphere B_ν, $\nu = 1, 2, \ldots,$ taken over all such sequences of spheres, is called the h-content of the set and is denoted by the symbol $\gamma_h(A)$. In the case $h(r) = r^\alpha$ in place of $\gamma_h(A)$ we write $\gamma_\alpha(A)$.

We note that if A and B are arbitrary sets in R^n, then the inclusion $A \subset B$ implies $\gamma_h(A) \le \gamma_h(B)$.

The h-content is a simpler property of a set; because of this, it will be more convenient to obtain subsequent estimates for it rather than for the Hausdorff measure. Actually, the h-content is equivalent in the Hausdorff h-measure in a certain sense, as is evident from the following lemma.

LEMMA 4.1. In order that the Hausdorff h-measure of a set $A \subset R^n$ be equal to zero, it is necessary and sufficient that its h-content be zero.

We leave the proof of this lemma to the reader.

LEMMA 4.2. Let μ be an arbitrary measure on R^n such that $\mu(R^n) < \infty$, and let $h(r)$, $0 \le r < \infty$, be a nondecreasing function such that $h(0) = 0$ and $h(r) \to \infty$ for $r \to \infty$. We denote by A_λ, where $\lambda > 0$, the set of all $x \in R^n$ such that for any $r \ge 0$ it is true that $\mu[B(x, r)] \le h(r)/\lambda$. Then the following estimate holds:

$$\gamma_h(R^n \setminus A_\lambda) \leqslant C_n \lambda \mu(R^n).$$

This is the well-known lemma of Cartan. For a proof in the two-dimensional case see, for example, [10]. For the case of arbitrary n a proof is given in [11].

We now establish a formula for the transformation of iterated integrals.

LEMMA 4.3. Let $F(r)$, $0 \le r < \infty$, be a nonnegative, decreasing function such that $F(r) \to 0$ for $r \to \infty$. Suppose that $F(r)$ has a continuous derivative $F'(r)$ for all $r > 0$. Then for any nonnegative, measurable function $u(x)$, $x \in R^n$, we have:

$$\int_{R_n} F(|x-y|)\, u(y)\, dy = -\int_0^\infty \Big(\int_{B(x,r)} u(y)\, dy \Big) F'(r)\, dr.$$

Proof. We put $\chi_r(x, y) = 1$ for $|x-y| < r$ and $\chi_r(x, y) = 0$ for $|x-y| \geq r$. Then

$$\int_{B(x,r)} u(y)\, dy = \int_{R^n} \chi_r(x, y)\, u(y)\, dy.$$

Hence

$$\int_0^\infty \Big(\int_{B(x,r)} u(y)\, dy \Big) F'(r)\, dr = \int_0^\infty \Big(\int_{R^n} \chi_r(x, y)\, u(y)\, dy \Big) F'(r)\, dr.$$

We apply Fubini's theorem to the integral on the right. As a result, we obtain

$$\int_0^\infty \Big(\int_{B(x,r)} u(y)\, dy \Big) F'(r)\, dr = \int_{R^n} \Big(\int_0^\infty \chi_r(x, y)\, F'(r)\, dr \Big) u(y)\, dy.$$

It is easy to see that for any x, y

$$\int_0^\infty \chi_r(x, y)\, F'(r)\, dr = -F(|x-y|).$$

This completes the proof of the lemma.

4.2. We have $G_l(x) = \beta_l(|x|)$. The properties of the function β_l which we need are given in Section 1.2.

THEOREM 4.1. Let $h(r)$, $0 \leq r < \infty$, be a nondecreasing function such that $h(0) = 0$ and $h(r) \to \infty$ for $r \to \infty$. We suppose further that

$$\int_0^\infty [h(r)]^{1/p} r^{n-\frac{n}{p}} |\beta_l'(r)|\, dr = h_0 < \infty.$$

Then for any set $E \subset R^n$ the following inequality holds:

$$\gamma_h(E) \leqslant \sigma_n^{p-1} C_n h_0^p \operatorname{Cap}_{(l,p)} E, \tag{4.1}$$

where σ_n is the volume of the unit sphere in R^n and C_n is the constant of Lemma 4.2.

Proof. Let $E \subset R^n$ be an arbitrary set. We assume that $\operatorname{Cap}_{(l,p)} E < \infty$, since otherwise inequality (4.1) is obvious.

We introduce the following notation. For any nonnegative, measurable function $f(x)$ on R^n we put

$$\theta(x, r, f) = \int_{B(x,r)} f(y)\, dy.$$

Now let $u \in L_p(R^n)$ be any nonnegative function such that $G_l u(x) \geq 1$ for all $x \in E$.

Using Lemma 4.3 to transform the integral, we obtain

$$(G_l u)(x) = \int_0^\infty \theta(x, r, u)\, |\beta_l'(r)|\, dr. \tag{4.2}$$

We estimate the quantity $\theta(x, r, u)$ by Hölder's inequality. This gives

$$\theta(x, r, u) \leqslant \sigma_n^{1-\frac{1}{p}} r^{n-\frac{n}{p}} [\theta(x, u^p, r)]^{1/p}.$$

Hence

$$(G_l u)(x) \leqslant \tau_n^{1-\frac{1}{p}} \int_0^\infty |\theta(x, r, u^p)^{1/p} r^{n-\frac{n}{p}} |\beta_l'(r)| \, dr.$$

We suppose that $\lambda > 0$ is arbitrary and denote by A_λ the set of all $x \in R^n$ for which

$$\theta(x, r, u^p) \leqslant h(r) / \lambda.$$

Applying Lemma 4.2 to the measure $\mu(E) = \int_E u^p dx$, we obtain

$$\gamma_h(R^n \setminus A_\lambda) \leqslant C_n \lambda \int_{R^n} [u(x)]^p dx,$$

For $x \in A_\lambda$ we have

$$(G_l u)(x) \leqslant \sigma_n^{1-\frac{1}{p}} \int_0^\infty [\theta(x, r, u^p)]^{1/p} r^{n-\frac{n}{p}} |\beta_l'(r)| \, dr \leqslant \frac{\sigma_n^{1-\frac{1}{p}}}{\lambda^{1/p}} \int_0^\infty [h(r)]^{\frac{1}{p}} r^{n-\frac{n}{p}} |\beta_l'(r)| dr = \frac{\sigma_n^{1-\frac{1}{p}} h_0}{\lambda^{1/p}}.$$

Now let λ be such that $\lambda^{1/p} > \sigma_n^{1-1/p} h_0$. Then for all $x \in A_\lambda$ we have $(G_l u)(x) < 1$. Since for all $x \in E$ $(G_l u)(x) \geq 1$, it follows that $E \subset R^n \setminus A_\lambda$. We thus obtain the estimate

$$\gamma_h(E) \leqslant \gamma_h(R^n \setminus A_\lambda) \leqslant C_n \lambda \int_{R^n} [u(x)]^p dx. \tag{4.3}$$

Since λ is here an arbitrary number greater than $\sigma_n^{p-1} h_0$, inequality (4.3) implies that

$$\gamma_h(E) \leqslant C_n \sigma_n^{p-1} h_0^p \int_{R^n} [u(x)]^p dx. \tag{4.4}$$

Since $u \in \mathfrak{M}^{(l,p)}(E)$ was arbitrary, inequality (4.4) implies that

$$\gamma_h(E) \leqslant C_n \sigma_n^{p-1} h_0^p \operatorname{Cap}_{(l,p)} E.$$

and this completes the proof of the theorem.

COROLLARY 1. If the (l, p) capacity of a set $E \subset R^n$, where $0 < l < n$, is equal to zero, then for any nondecreasing function $h(r)$, $0 \leq r < \infty$, such that $h(0) = 0$ and

$$\int_0^1 \frac{[h(r)]^{\frac{1}{p}}}{r^{\frac{n}{p} - l + 1}} dr < \infty, \tag{4.5}$$

the Hausdorff h-measure of the set A is equal to zero.

Proof. We redefine the function $h(r)$ on the interval $[1, \infty)$ by putting it equal to kr there, where $k = \text{const}$. The Hausdorff h-measure of the set A hereby remains unchanged. For $r \to 0$

$$|\beta_l'(r)| = \frac{C}{r^{n-l+1}} [1 + o(r)]$$

and $|\beta_l'(r)| = o(e^{-r})$ for $r \to \infty$. This implies that if inequality (4.5) is satisfied for the function h, then

$$h_0 = \int_0^\infty [h(r)]^{1/p} r^{n-\frac{n}{p}} |\beta_l'(r)| dr < \infty.$$

This means that if $\operatorname{Cap}_{(l,p)} E = 0$, then by Theorem 4.1 $\gamma_h(E) = 0$, and hence $\mu_h(E) = 0$ by Lemma 4.1.

COROLLARY 2. Let $E \subset R^n$ be such that $\operatorname{Cap}_{(l,p)} E = 0$, where $lp \leq n$. Then for any $\alpha > n - lp$ $\mu_\alpha(E) = 0$.

For the proof it suffices to take $h(r) = r^\alpha$ in Corollary 1.

4.3. We shall now establish some sufficient conditions in order that the (l, p) capacity be zero.

LEMMA 4.4. Let $\psi_{l,p}(r)$ be the (l, p) capacity of a sphere of radius r, where $lp \le n$. Then for $r \to 0$

$$\psi_{l,p}(r) = O(r^{n-lp})$$

if $n > lp$ and

$$\psi_{l,p}(r) = O\left[\left(\ln \frac{1}{r}\right)^{1-p}\right]$$

if $lp = n$.

Proof. Let $B_r = B(0, r)$ be a sphere of radius r with center at the origin. We shall assume that $r < 1$. We put $u(x) = 1/|x|^k$, where $k \ge n/p$ for $r \le |x| \le 1$, and $u(x) = 0$ for all other x. We have

$$\int_{R^n} [u(x)]^p dx = \omega_{n-1} \int_r^1 \frac{\rho^{n-1} d\rho}{\rho^{pk}} = \frac{\omega_{n-1}}{pk - n}\left(\frac{1}{r^{pk-n}} - 1\right)$$

if $k > n/p$ and

$$\int_{R^n} [u(x)]^p dx = \omega_{n-1} \ln \frac{1}{r}$$

if $k = n/p$.

We now estimate the potential $G_l u$ below on the sphere B_r. For $x \in B_r$ we have

$$(G_l u)(x) = \int_{r<|y|<1} G_l(x-y) \frac{dy}{|y|^k}. \tag{4.6}$$

It follows from the properties of the function G_l that there exists a constant $K_l > 0$ such that for $|x| \le 2$

$$G_l(x) \geqslant K_l / |x|^{n-l}. \tag{4.7}$$

Inequalities (4.6) and (4.7) imply that

$$(G_l u)(x) \geqslant K_l \int_{r<|y|<1} \frac{dy}{|x-y|^{n-l} |y|^k}$$

for all $x \in B_r$. For $x \in B_r$ and $y \notin B_r$ we obviously have

$$|x - y| \leqslant |x| + |y| \leqslant 2|y|,$$

since in this case $|x| \le |y|$. From this we find that for all $x \in B_r$

$$(G_l u)(x) \geqslant \frac{K_l}{2^{n-l}} \int_{r<|y|<1} \frac{dy}{|y|^{n-l+k}}.$$

The last integral is equal to $K'(1/r^{k-l} - 1)$ for $k \ne l$ and to $K' \ln(1/r)$ for $k = l$.

We have the following estimate for the (l, p) capacity of the sphere B_r:

$$\operatorname{Cap}_{(l,p)} B_r \leqslant \int_{R^n} [u(y)]^p dy / [\min_{x \in B_r} (G_l u)(x)]^p. \tag{4.8}$$

We assume first of all that $n > lp$. In this case we put $k > n/p > l$. Inequality (4.8) leads to the following estimate for $\operatorname{Cap}_{(l,p)} B_r$:

$$\operatorname{Cap}_{(l,p)} B_r \leqslant K'' \frac{r^{kp-lp}(r^{n-pk} - 1)}{(1 - r^{k-l})^p} = K'' \frac{r^{n-lp}(1 - r^{kp-n})}{(1 - r^{k-l})^p}.$$

This proves that $\mathrm{Cap}_{(l,p)}B_r = O(r^{n-lp})$ for $r \to 0$.

In the case $n = lp$ we put $k = n/p = l$; inequality (4.8) leads to the estimate

$$\mathrm{Cap}_{(l,p)}B_r \leqslant K^* \left(\ln \frac{1}{r}\right)^{1-p}.$$

This completes the proof of the lemma.

THEOREM 4.2. Let $h(r) = r^{n-lp}$ if $n > lp$ and $h(r) = [\ln(1/r)]^{1-p}$ for $0 < r \leq 1/2$ if $lp = n$. (It is assumed that then $p > 1$.) If the Hausdorff h-measure of the set $E \subset R^n$ is equal to zero, then its (l, p) capacity is equal to zero.

Proof. Let $B_1, B_2, \ldots, B_\nu, \ldots$ be any sequence of spheres covering the set E such that their radii $r_1, r_2, \ldots, r_\nu, \ldots$ do not exceed $\varepsilon \leq 1/2$. Then by Lemma 4.4

$$\mathrm{Cap}_{(l,p)}E \leqslant \sum_\nu \mathrm{Cap}_{(l,p)}B_\nu \leqslant K \sum_\nu h(r_\nu),$$

where K is a constant. Because the sequence of spheres $\{B_\nu\}$ was arbitrary, this implies that

$$\mathrm{Cap}_{(l,p)}E \leqslant K\mu_h(E, \varepsilon).$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$\mathrm{Cap}_{(l,p)}E \leqslant K\mu_h(E) = 0,$$

whence $\mathrm{Cap}_{(l,p)}E = 0$, which was required to prove.

§ 5. The Concept of (l, p) Capacity and Functions Having Generalized Derivatives

5.1. In this part of the paper we shall indicate some applications of the concept of (l, p) capacity to the study of functions having generalized derivatives. We first recall certain well-known facts from the theory of Bessel potentials.

The set of all real-valued functions u on R^n which admit the representation

$$u(x) = (G_l v)(x), \tag{5.1}$$

where $v \in L_p(R^n)$, $p > 1$, $l > 0$, we denote by $L_l^p(R^n)$. We introduce a norm in $L_l^{(p)}(R^n)$ as follows. If $u \in L_l^p(R^n)$ is represented as in Eq. (5.1), where $v \in L_p(R^n)$, then we put $\|u\|_{L^p_l(R^n)} = \|v\|_{L_p(R^n)}$. The representation of u by Eq. (5.1), if such a representation exists, is unique, and hence the norm is well defined. The space $L_l^p(R^n)$ is a Banach space.

The integral in (5.1) is interpreted as the Lebesgue integral $G_l v = G_l v^+ - G_l v^-$. By Theorem 1.8, the set of x for which $G_l v(x)$ is undefined or equal to $\pm\infty$, is a set of zero (l, p) capacity.

Let $l > 0$ be an integer. We denote by $W_p^l(R^n)$ the Sobolev class of real-valued functions defined on R^n and which have on R^n all generalized derivatives of order l summable on R^n in degree p. For $u \in W_p^l(R^n)$ we put

$$\|u\|_{W_p^l(R^n)} = \left[\int_{R^n} \sum_{0 \leqslant |\alpha| \leqslant l} |D^\alpha u(x)|^p dx\right]^{1/p}.$$

THEOREM 5.1 [3, 5]. For integral l the class $W_p^l(R^n)$ coincides with the class $L_l^p(R^n)$ in the following sense: for any function $u \in W_p^l(R^n)$ there exists a function $u^* \in L_l^p(R^n)$ such that $u(x) = u^*(x)$ almost everywhere on R^n and

$$\|u\|_{W_p^l(R^n)} \leqslant C\|u^*\|_{L_l^p(R^n)}, \quad \|u^*\|_{L_l^p(R^n)} \leqslant C\|u\|_{W_p^l(R^n)},$$

where the constant C does not depend on the choice of the function u.

THEOREM 5.2 [3, 5]. For $lp > n$ each of the functions of the class $L_l^p(R^n)$ is continuous and bounded. Moreover, there exists a constant C such that for any $u \in L_l^p(R^n)$

$$\sup_{x\in R^n} |u(x)| \leqslant C \|u\|_{L_l^p(R^n)}.$$

COROLLARY. If $l > 0$, $p > 1$, and $lp > n$, then the (l, p) capacity of any nonempty set of R^n is no less than the constant $\gamma = C^{-p}$, where C is the constant of Theorem 5.2.

Indeed, let E be an arbitrary nonempty set in R^n, and let $v \in L_p$. If $\|v\|^p_{L_p} < \gamma$, then $(G_l v)(x) < 1$ for all x, from which it follows that $v \in \mathfrak{M}_{(l,p)}(E)$ if and only if $\|v\|^p_{L_p} \geq \gamma$, and hence $\mathrm{Cap}_{(l,p)}E \geq \gamma$.

THEOREM 5.3 [3, 5]. Let $l > 0$, $p > 1$ be such that $lp \leq n$, and let $m < l$. Then for any q such that

$$1/q - 1/p + (l-m)/n \geqslant 0,$$

$L_m^q(R^n) \supset L_l^p(R^n)$. Moreover, there exists a constant $C = C(p, q, l, m, n)$ such that for any $u \in L_l^p(R^n)$

$$\|u\|_{L_m^q(R^n)} \leqslant C \|u\|_{L_l^p(R^n)}.$$

Theorems 1.9 and 1.10 imply the following results.

THEOREM 5.3. Let $u_1 + u_2 + \ldots + u_\nu + \ldots$ be a series of functions in $L_l^p(R^n)$ such that the numerical series

$$\|u_1\|_{L_l^p(R^n)} + \|u_2\|_{L_l^p(R^n)} + \ldots + \|u_\nu\|_{L_l^p(R^n)} + \ldots$$

converges. Then the series

$$u_1(x) + u_2(x) + \ldots + u_\nu(x) + \ldots$$

converges (l, p) almost everywhere on R^n.

THEOREM 5.4. Let $\{u_\nu\}$, $\nu = 1, 2, \ldots$, be an arbitrary sequence of functions of $L_l^p(R^n)$ such that $\|u_\nu - u\|_{L_l^p(R^n)} \to 0$ for $\nu \to \infty$. Then there exists a sequence of indices $\nu_1 < \nu_2 < \ldots < \nu_k < \ldots$ such that for $k \to \infty$

$$u_{\nu_k}(x) \to u(x)$$

(l, p) almost everywhere on R^n.

5.2. For functions of the class $L_l^p(R^n)$ the well-known properties of measurable functions contained in the theorems of Lusin and Egorov can be sharpened by using the concept of (l, p) capacity. It is assumed that $l > 0$, $p > 1$, $lp \leq n$.

THEOREM 5.5. Let $\sum_{\nu=1}^{\infty} u_\nu$ be any absolutely convergent series in the Banach space L_l^p, $lp \leq n$, and let u be the sum of this series. Then for any $\varepsilon > 0$ there exists an open set $U \subset R^n$ such that $\mathrm{Cap}_{(l,p)}U < \varepsilon$ and the series $\sum_{\nu=1}^{\infty} u_\nu$ converges to u uniformly on the set $R^n \setminus U$.

Proof. Let $\varepsilon > 0$ be arbitrary. For each ν we have $u_\nu = G_l v_\nu$, where $v_\nu \in L_p(R^n)$ and $\|u_\nu\|_{L_l^p(R^n)} = \|v_\nu\|_{L^p}$. We put

$$a_\nu = \|u_\nu\|_{L_l^p}, \quad \nu = 1, 2, \ldots.$$

We suppose first of all that the functions v_ν are nonnegative. Since the series $\sum_{\nu=1}^{\infty} \|v_\nu\|_{L_p}$ converges, there exists a sequence of natural numbers $\nu_1 < \nu_2 < \ldots < \nu_k < \ldots$ such that

$$\|v_{\nu_k-1} + v_{\nu_k+2} + \ldots + v_{\nu_{k+1}}\|_{L_p} < \frac{1}{2^{\frac{k+1}{p}+k}}.$$

We put

$$w_0 = v_1 + v_2 + \ldots + v_{\nu_0},\ a = \|w_0\|_{L_p},$$
$$w_k = v_{\nu_k+1} + v_{\nu_k+2} + \ldots + v_{\nu_{k+1}} \text{ for } k > 0.$$

Let $\varepsilon > 0$ be arbitrary, and let

$$W_0 = \left\{x \in R^n : (G_l w_0)(x) > \left(\frac{2}{\varepsilon}\right)^{1/p} a\right\},$$
$$W_k = \left\{x \in R^n : (G_l w_k)(x) > \frac{1}{2^k \varepsilon^{1/p}}\right\},\ k = 1, 2, \ldots .$$

The functions w_k, $k = 0, 1, 2, \ldots$, are nonnegative, and hence the functions $G_l w_k$ are lower semicontinuous on R^n. This implies that each of the sets W_k is open. The capacity of the set W_k for any k is no greater than $\varepsilon/2^{k+1}$.

We put

$$U = \bigcup_{k=0}^{\infty} W_k.$$

The set U is open, and the capacity of U does not exceed ε. We shall prove $\sum_{\nu=1}^{\infty} G_l v_\nu$ converges uniformly on the set $R^n \setminus U$. Indeed, from the construction of the set U it is clear that for all k

$$(G_l w_k)(x) \leqslant \frac{1}{2^k \varepsilon^{1/p}}$$

for $x \in R^n \setminus U$. We take any integers ν and q such that $\nu \geq \nu_k$, $k > 1$, $q > 0$ and consider the sum

$$V_{\nu,p}(x) = (G_l v_{\nu+1})(x) + (G_l v_{\nu+2})(x) + \ldots + (G_l v_{\nu+q})(x).$$

We suppose that $m > k$ is such that $\nu + q < \nu_m$. Then

$$0 \leqslant v_{\nu+1}(x) + v_{\nu+2}(x) + \ldots + v_{\nu+q}(x) \leqslant v_{\nu_k+1}(x) + v_{\nu_k+2}(x) + \ldots + v_{\nu_m}(x)$$
$$= w_k(x) + w_{k+1}(x) + \ldots + w_{m-1}(x).$$

This implies that for $x \in R^n \setminus U$

$$V_{\nu,q}(x) \leqslant (G_l w_k)(x) + (G_l w_{k+1})(x) + \ldots + (G_l w_{m-1})(x) \leqslant \frac{1}{\varepsilon^{1/p}}\left(\frac{1}{2^k} + \frac{1}{2^{k+1}} + \ldots + \frac{1}{2^{m-1}}\right) < \frac{1}{\varepsilon^{1/p} 2^{k-1}}.$$

Passing to the limit as $q \to \infty$, we find that for all $x \in R^n \setminus U$ and for $\nu > \nu_k$

$$V_{\nu,\infty}(x) = \sum_{m=k}^{\infty} G_l v_m(x) \leqslant \frac{1}{\varepsilon^{1/p} 2^{k-1}},$$

whence it clearly follows that the series $\sum_{\nu=1}^{\infty} G_l v_\nu(x)$ converges uniformly on $R^n \setminus U$.

We now consider the case in which the functions $u_\nu \in L_p(R^n)$ have arbitrary sign. For any $\nu = 1, 2, \ldots$ we obviously have $v_\nu = v_\nu^+ - v_\nu^-$. By what has been proved, there exist open sets U_1 and U_2, the (l, p) capacity of which is less than $\varepsilon/2$, such that the series

$$\sum_{\nu=1}^{\infty} (G_l v_\nu^+)(x) \text{ and } \sum_{\nu=1}^{\infty} (G_l v_\nu^-)(x)$$

converge uniformly on the sets $R^n \setminus U_1$ and $R^n \setminus U_2$ respectively. Let $U = U_1 \cup U_2$. Then $\mathrm{Cap}_{(l,p)} U < \varepsilon$ and the series $\sum_{\nu=1}^{\infty} G_l v_\nu(x)$ converges uniformly on the set $R^n \setminus U$.

The proof of the theorem is now complete.

THEOREM 5.6. Let u be any function of the class $L_l^p(R^n)$, where $lp \leq n$. Then for any $\varepsilon > 0$ there exists an open set $U \subset R^n$ such that $\mathrm{Cap}_{(l,p)}U < \varepsilon$ and the function u is continuous on the set $R^n \setminus U$.

Proof. We have $u = G_l v$, where $v \in L_p(R^n)$. Let w_ν, $\nu = 1, 2, \ldots$, be a sequence of functions with compact support in R^n which belong to the class C^∞ and are such that $\|v - w_\nu\|_{L_p} < 1/2^\nu$ for all ν. We put

$$v_1 = w_1, \quad v_\nu = w_\nu - w_{\nu-1} \quad \text{for } \nu > 1.$$

Then

$$v = v_1 + v_2 + \ldots + v_\nu + \ldots$$

and the series $\sum_{\nu=1}^{\infty} \|v_\nu\|_{L_p}$ converges. By Theorem 5.5, there exists an open set $U' \subset R^n$ such that $\mathrm{Cap}_{(l,p)}U' < \varepsilon$ and the series

$$\sum_{\nu=1}^{\infty} G_l v_\nu(x) \tag{5.2}$$

converges uniformly on the set $R^n \setminus U'$.

By Theorem 1.9, the sum of the series (5.2) is equal to $(G_l v)(x)$ (l, p) almost everywhere on R^n. Let S be the set of those x for which the series (5.2) either diverges or else converges to a sum different from $(G_l v)(x)$. Then $\mathrm{Cap}_{(l,p)}S = 0$. Let $V \supset S$ be an open set such that $\mathrm{Cap}_{(l,p)}V < \varepsilon - \mathrm{Cap}_{(l,p)}U'$. We put $U = V \cup U'$. Then

$$\mathrm{Cap}_{(l,p)}U \leqslant \mathrm{Cap}_{(l,p)}V + \mathrm{Cap}_{(l,p)}U'.$$

The series (5.2) converges uniformly on the set $R^n \setminus U$, and its sum is equal to $(G_l v)(x)$ for all $x \in R^n \setminus U$. Since each of the functions $G_l v_\nu$ is continuous, it follows that $G_l v$ is continuous on $R^n \setminus U$. This completes the proof of the theorem.

5.3. Let $f: U \to R$ be any locally summable function (U is an open set in R^n). We fix a point $x_0 \in U$. The number M is called the natural value of the function f at the point x_0 if

$$\lim_{h \to 0} \int_{|X|<1} |f(x_0 + hX) - M| \, dX = 0. \tag{5.3}$$

If $f(x_0)$ is defined and

$$\lim_{h \to 0} \int_{|X|<1} |f(x_0 + hX) - f(x_0)| \, dX = 0, \tag{5.4}$$

then we say that the function f is continuous at the point x_0 in the sense of convergence in L_1. By a well-known theorem of Lebesgue [3], the relation (5.4) is satisfied for almost all $x_0 \in U$, i.e., the value of the function f at the point x_0 is its natural value at this point for almost all x_0.

The set of those x for which the natural value of the function f at the point x does not exist is called the exceptional set. The exceptional set of any locally summable function is a set of measure zero. For functions of the class $L_l^p(R^n)$ this property of summable functions admits considerable sharpening.

THEOREM 5.7 [3]. For any function u of the class $L_l^p(R^n)$ the exceptional set is a set of zero (l, p) capacity.

5.4. Let E be a set in the space R^n. The set E is said to be p-exceptional with respect to k-dimensional surfaces (more briefly, exceptional in the sense [k, p]) if the p-modulus (see [1]) of the family of all Lipschitz surfaces passing through the points of the set E is equal to zero. It is shown in [1] that in order that E be exceptional in the sense [k, p], where $kp \leq n$, it is necessary and sufficient that there exist a function $f \geq 0$, $f \in L_p$, such that

$$U_f(x) = \int_{R^n} |x-y|^{k-n} f(y)\,dy = \infty$$

for all $x \in E$, while the function $U_f(x)$ is not identically equal to ∞.

THEOREM 5.8. The class of sets which are exceptional in the sense $[k, p]$ coincides with the class of sets of zero (k, p) capacity.

Proof. Let E be a set which is exceptional in the sense $[k, p]$. Then there exists a function $f \in L_p$ such that $U_f \not\equiv \infty$ and $U_f(x) = \infty$ for all $x \in E$. Since $U_f \not\equiv \infty$, it follows that

$$\int_{|y|>1} f(y)|y|^{k-n}\,dy < \infty$$

and this means that

$$\int_{|y-x|<1} |x-y|^{k-n} f(y)\,dy = \infty$$

for all $x \in E$. It is not difficult to see from this that $(G_l f)(x) = \infty$ for all $x \in E$, i.e., $\mathrm{Cap}_{(l,p)} E = 0$.

Conversely, suppose that $\mathrm{Cap}_{(l,p)} E = 0$. Let $E_m = \{x \in E: m-1 \leq |x| < m\}$, where $m = 1, 2, \ldots$. Then $E = \bigcup_{m=1}^{\infty} E_m$. Let $f_m \in L_p(R^n)$ be such that

$$(G_l f_m)(x) = \infty$$

for all $x \in E_m$. We may hereby assume that $f_m(x) = 0$ for $|x| < m-2$ and $|x| > m+1$ and that $\|f_m\|_{L_p} < 1/2^m$. We put $f = f_1 + f_2 + \ldots + f_m + \ldots$. It is then easy to check that $\int_{|y|>1} f(y)|y|^{k-m}dy < \infty$, and hence $U_f(x) \not\equiv \infty$. On the other hand, it is easy to verify that $U_f(x) = \infty$ for all $x \in E$. This completes the proof of the theorem.

§ 6. Variational Capacity

6.1. Henceforth l denotes an integer, $l > 0$, $p > 1$. Let A and B be closed sets in R^n. We say that the sets A and B form a regular pair if they have no common elements and one of them (say A) is bounded while the other (say B) is such that its complement is a bounded set. We denote by $C^\infty(A, B)$ the set of all functions $\varphi \in C^\infty$ such that $\varphi(x) = 1$ for $x \in A$ and $\varphi(x) = 0$ for all $x \in B$.

Let $\varphi \in C^\infty(A, B)$. We put

$$D_{l,p}(\varphi) = \int_{R^n} \Big\{ \sum_{|\alpha|=l} \frac{l!}{\alpha!} |D^\alpha \varphi|^2 \Big\}^{p/2} dx.$$

The integral $D_{l,p}$ is called the Dirichlet integral of type (l, p) for the function φ. The greatest lower bound of the quantity $D_{l,p}(\varphi)$ on the set $C^\infty(A, B)$ is called the variational (l, p) capacity of the pair of sets A, B. We shall denote it by the symbol $C.V._{(l,p)}(A, B)$.

We note some properties of the variational (l, p) capacity which follow directly from the definition.

LEMMA 6.1. Let (A_1, B_1) and (A_2, B_2) be two regular pairs of sets such that $A_1 \subset A_2$, $B_1 \subset B_2$. Then

$$C.V._{(l,p)}(A_1, B_1) \leqslant C.V._{(l,p)}(A_2, B_2).$$

The lemma is obvious from the inclusion $C^\infty(A_2, B_2) \subset C^\infty(A_1, B_1)$.

LEMMA 6.2. Let $\varphi \in C^\infty(R^n)$ be a function on R^n with compact support. Then for all $x \in R^n$ the following integral formula holds:

$$\varphi(x) = \gamma_{l,n} \int_{R^n} \sum_{|\alpha|=l} \frac{l!}{\alpha!} \frac{(x-y)^\alpha D^\alpha \varphi(y)}{|x-y|^n}\,dy,$$

where $\gamma_{l,n}$ is a constant.

Proof. Let e be any unit vector. Let $r_0 > 0$ be such that for $r > r_0$ $\varphi(x + re) = 0$. We take any $r > r_0$. Let $\Phi(r) = \varphi(x + re)$. Then $\varphi(x) = \Phi(0)$. We express $\Phi(0)$ by Taylor's formula with the remainder term in integral form involving the values of the derivatives of order no greater than l of the function Φ at the point r. We then obtain

$$\varphi(x) = \Phi(0) = \frac{1}{(l-1)!}\int_0^r (-\rho)^{l-1}\Phi^{(l)}(\rho)\,d\rho = \frac{(-1)^{l-1}}{(l-1)!}\int_0^\infty \sum_{|\alpha|=l}\frac{l!}{\alpha!} D^\alpha\varphi(x+re)e^\alpha\rho^{l-1}d\rho.$$

We integrate both sides of this equation with respect to the unit vector e over the sphere Ω_{n-1}, and after some obvious rearrangements we obtain the required formula. The constant $\gamma_{l,n}$ is hereby equal to $\omega_{n-1}/(l-1)!(-1)^{l-1}$.

THEOREM 6.1. Let (A, B) be a regular pair of sets in R^n. We suppose that A is bounded and let d be the diameter of $U = R^n \setminus B$. Then

$$\text{Cap}_{(l,p)}A \leqslant KC.V_{(l,p)}(A, B),$$

where K depends only on l, p, n, and d.

Proof. We take any function $\varphi \in C^\infty(A, B)$. By Lemma 6.2,

$$\varphi(x) = \gamma_{l,n}\int_{R^n}\sum_{|\alpha|=l}\frac{l!}{\alpha!}D^\alpha\varphi(y)(x-y)^\alpha\frac{dy}{|x-y|^n}.$$

It is easy to see that

$$\left|\sum_{|\alpha|=l}\frac{l!}{\alpha!}\frac{D^\alpha\varphi(y)(x-y)^\alpha}{|x-y|^n}\right| \leqslant \frac{1}{|x-y|^{n-l}}\left\{\sum_{|\alpha|=l}\frac{l!}{\alpha!}|D^\alpha\varphi(y)|^2\right\}^{1/2}.$$

The expression on the right side of the last inequality which multiplies $|x-y|^{l-n}$ we denote by v(y). We have

$$|\varphi(x)| \leqslant \gamma_{l,n}\int_{R^n}\frac{v(y)}{|x-y|^{n-l}}dy = \gamma_{l,n}\int_U\frac{v(y)}{|x-y|^{n-l}}dy,$$

since $v(y) = 0$ for $y \notin U$. From the properties of the function $G_l(x)$ noted in Section 1.2, there exists a constant L(d) such that for $|x| \leq d$ $|x|^{l-n} \leq L(d)G_l(x)$. Hence, for all $x \in A$

$$1 \leqslant |\varphi(x)| \leqslant \gamma_{l,n}L(d)\int_U G_l(x-y)v(y)\,dy = \gamma_{l,n}L(d)\int_{R^n}G_l(x-y)v(y)\,dy.$$

From this we obtain the estimate

$$\text{Cap}_{(l,p)}A \leqslant K\int_{R^n}[v(y)]^p dy = KD_{l,p}(\varphi),$$

where the constant $K = 1/[\gamma_{(l,n)}L(d)]^p$. Since φ was an arbitrary function in $C^\infty(A, B)$, this completes the proof of the theorem.

For any regular pair of closed sets A, B we put

$$\delta(A, B) = \inf_{x\in A, y\in B}|x-y| > 0.$$

LEMMA 6.3. For any regular pair of closed sets (A, B) there exists a function $\zeta \in C^\infty(A, B)$ such that for any α

$$|D^\alpha\zeta(x)| \leqslant K_\alpha/[\delta(A, B)]^\alpha$$

for all $x \in R^n$, where K_α does not depend on the sets A and B.

Proof. We put

$$\psi(x) = e^{\frac{1}{x^2-n}} \quad \text{for} \quad |x| < \sqrt{n},$$
$$\psi(x) = 0 \quad \text{for} \quad |x| > \sqrt{n}.$$

We set

$$\theta(x) = \sum_{\nu} \psi(x - \nu), \tag{6.1}$$

where ν runs through the set of all vectors with integer coordinates in R^n. It is easy to see that $\theta(x) > 0$ for all x. Moreover, for any point x there exists a neighborhood U in which only a finite number of terms on the right side of (6.1) are different from zero. This implies that $\theta \in C^\infty$.

We put $\eta(x) = \psi(x)/\theta(x)$. Then $\eta \in C^\infty(R^n)$, $\eta(x) = 0$ for $|x| \geq \sqrt{n}$, and for all $x \in R^n$

$$\sum_{\nu} \eta(x - \nu) = 1.$$

The functions $\eta(x-\nu)$ form a partition of unity in the space R^n. The support of the function $\eta(x-\nu)$ is hereby the sphere with center ν and radius $\sqrt{n}$.

Now let $h = (1/2\sqrt{n})\delta(A, B)$. We consider the system of functions $\eta[(x-\nu h)/h]$, where ν is a vector with integer coordinates. Let $\nu_1, \nu_2, \ldots, \nu_p$ be all the vectors ν for which the support of the function $\eta[(x - \nu h)/h]$ intersects the set A. We put

$$\zeta(x) = \sum_{i=1}^{p} \eta\left(\frac{x - \nu_i h}{h}\right). \tag{6.2}$$

It is clear that $\zeta \in C^\infty(R^n)$ and $\zeta(x) = 1$ for all $x \in A$. Further, by the choice of h each of the functions $\eta[(x - \nu_i h)/h]$ is equal to zero for $x \in B$. Hence $\zeta(x) = 0$ for $x \in B$. We note that for any point x there is a neighborhood U in which not more than $k_n < \infty$ of the functions $\eta[(x-\nu h)/h]$ are different from zero. It is easy to see that

$$\left|D^\alpha \eta\left(\frac{x - \nu h}{h}\right)\right| \leqslant \frac{M_\alpha}{h^{|\alpha|}},$$

where $M_\alpha = \max|D^\alpha \eta(x)|$. This implies that

$$|D^\alpha \zeta(x)| \leqslant k_n M_\alpha / h^{|\alpha|} = K_\alpha / [\delta(A, B)]^{|\alpha|}.$$

The proof of the lemma is now complete.

LEMMA 6.4. For any compact set $A \subset R^n$ (l, p)-capacity of A is equal to the greatest lower bound of the quantity $(\|u\|_{L_l^p})^p$ taken over the set of all functions $u \in C^\infty(R^n)$ belonging to L_l^p and such that $u(x) \geq 1$ on the set A.

Proof. Let $u = G_l v$, where $v \in \mathfrak{M}^{(l,p)}(R^n)$, $v \geq 0$. Then $u(x) \geq 1$ for all $x \in A$. Let $h > 0$ be arbitrary, and let

$$A_h = \{x \in R^n : \rho(x, A) \leqslant h\}.$$

We put $1 - \delta(h) = \inf_{x \in A_h} u(x)$. The function u(x) is lower semicontinuous. From this we easily conclude that $\delta(h) \to 0$ for $h \to 0$. We mollify the function u with the parameter h. We obtain the function

$$u_h(x) = M_h u(x) = (C_l M_h v)(x).$$

We note that for all $x \in A$

$$u_h(x) \geqslant 1 - \delta(h).$$

Let

$$\varphi_h(x) = \frac{u_h(x)}{1 - \delta(h)}, \quad \psi_h(x) = \frac{M_h v(x)}{1 - \delta(h)}.$$

Then $\psi_h \in \mathfrak{M}^{(l,p)}(A)$, $\psi_h \to v$ in $L_p(R^n)$, and the function $\varphi_h \in C^\infty$. We thus see that the set of those $v \in \mathfrak{M}^{(l,p)}(A)$ for which $G_l v \in C^\infty$ is dense in $\mathfrak{M}^{(l,p)}(A)$. This clearly implies the required result.

THEOREM 6.2. Let (A, B) be any regular pair of closed sets in R^n, where the set A is bounded. Then

$$C.\ V_{(l,\,p)}(A,\,B) \leqslant K\ \mathrm{Cap}_{(l,\,p)}A,$$

where K depends only on l, p, and $\delta(A, B)$.

Proof. Let $\varphi \in C^\infty(R^n) \cap L_l^p(R^n)$ be any function such that $\varphi(x) \geq 1$ for all $x \in A$. Now let ζ be the function whose existence was demonstrated in Lemma 6.3. We put

$$\psi(x) = \zeta(x)\varphi(x).$$

Then $\psi \in C^\infty(A, B)$. We have:

$$C.\ V_{(l,\,p)}(A,\,B) \leqslant D_{(l,\,p)}(\psi) = \int\limits_{R^n} \left\{ \sum_{|\alpha|=l} \frac{l!}{\alpha!} [D^\alpha(\zeta\varphi)]^2 \right\}^{p/2} dx. \tag{6.3}$$

It is easy to see that for $|\alpha| = l$

$$|D^\alpha[\zeta\varphi(x)]| \leqslant M \sum_{|\beta|\leq l} |D^\beta\varphi(x)|,$$

where $M = \max\limits_{x\in R^n,\ |\beta|\leq l} |D^\beta\zeta(x)|$. This implies the inequality

$$D_{l,p}(\psi) \leqslant K_1 \|\varphi\|^p_{W_p^l}, \tag{6.4}$$

where the constant K_1 depends only on $\max\limits_{x,|\beta|\leq l} |D^\beta\zeta(x)|$, and hence K_1 depends finally only on l, p, n, and $\delta(A, B)$. From the equivalence of the norms $\|\cdot\|_{W_p^l}$ and $\|\ \|_{L_l^p}$ for integral l, there exists a constant K_2 such that

$$\|\varphi\|^p_{W_p^l} \leqslant K_2 \|\varphi\|^p_{L_l^p}. \tag{6.5}$$

Comparing inequalities (6.3), (6.4), and (6.5), we obtain

$$C.\ V_{(l,\,p)}(A,\,B) \leqslant K \|\varphi\|^p_{L_l^p}.$$

Since $\varphi \in C^\infty(R^n) \cap L_l^p(R^n)$ was arbitrary, this implies the inequality

$$C.\ V_{(l,\,p)}(A,\,B) \leqslant K\ \mathrm{Cap}_{(l,\,p)}A.$$

The proof of the theorem is now complete.

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