INCORRECT PROBLEMS IN TOPOLOGICAL SPACES

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INTRODUCTION

1. Incorrectly formulated (strongly unstable) problems can be usually reduced to the determination of x from the equation

$$Ax = y, \tag{1}$$

where x and y are elements of the metric spaces X and Y, whereas $A:X \to Y$ is a mapping of X into Y, with no continuous dependence of x on y. In most cases, X and Y are endowed with a linear structure and a number of other assumptions (such as completeness and the property of being Hilbert spaces) are made; the operator A is assumed to be linear and continuous, etc.

In recent years it was shown that in many cases it is possible to replace continuity by closure [1-4]: a number of propositions, formulated at first for linear equations, were found to be valid also for nonlinear equations [5, 6]. An ever-greater role is being played by weak topology, which is nonmetrizable in nonseparable spaces.

These results show that it would be useful to consider Eq. (1) under the most general assumptions, by minimizing the requirements towards the spaces X and Y, and the operator A. This is precisely what we are trying to do in this paper. Let us also note that incorrectly formulated problems in topological spaces were considered by Gorbunov [7, 8] under a different aspect.

2. Our problem can be formulated as follows. Let X and Y be topological Hausdorff spaces, and $A:X \to Y$ a mapping with a closed graph. For a $y_0 \in Y$ (the exact value of the right-hand side) there exists a unique pre-image $x_0 \in X$. Let $\{V_\delta\}$ be a filter of neighborhoods of the point y_0. To each neighborhood V_δ it is required to assign a point $x_\delta \in X$ such that the generalized sequence $\{x_\delta\}$ converges to x_0.

In § 1 we analyze the concept of a "correct" problem. Already the first rigorous approach to the solving of incorrectly formulated problems (Tikhonov [9]) was based on a topological lemma, according to which a continuous one-to-one mapping of a bicompact space into a separable space is a homeomorphism. In § 2 we extend this lemma to mappings with a closed graph (Theorems 3 and 5). A particular case of such an extension is examined in [1]. In § 3 we present a generalized abstract analog of the method of quasi-solutions [10]. This generalization is new also for metric spaces.

The results of § 4 can be regarded as an abstract analog of variational methods of solution of incorrectly formulated problems [11, 12, 6].

In the following we shall assume throughout that X and Y are Hausdorff spaces and that the image AX of the space X is a set, dense in Y.

§ 1. Correct and Incorrect (Unstable) Problems

3. Let $A:X \to Y$ be a mapping of the Hausdorff space X into a Hausdorff space Y. We shall consider Eq. (1) at the point $y = y_0$. Hadamard's well-known correctness conditions, applied to Eq. (1) in local interpretation, can be formulated as follows. For a given point y_0 the solution x_0: 1) exists, 2) is unique, and 3) depends continuously on y.

Condition 3 will be called the stability of the solution. Let us note that Conditions 1 and 2 are not related to the topologies in X and Y; only Condition 3 is of topological character.

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Let us sharpen the concept of correctness in relation to Eq. (1).

Let $\{V_\delta\}$ be a filter of neighborhoods of the point y_0. From the denseness of the AX in the space Y it follows that any neighborhood V_δ (irrespective of the existence of a solution of Eq. (1) at the point y_0) has a nonempty complete pre-image $A^{-1}V_\delta$ in X. Here δ are subscripts, belonging to a partially ordered set Δ.

Definition 1. Let $\{V_\delta\}$ be a filter of neighborhoods of y_0. We shall say that the problem of solving Eq. (1) for $y = y_0$ has been correctly formulated if:

1) the intersection $\bigcap_\delta A^{-1}V_\delta$ of complete pre-images contains only one point x_0;

2) a filter in X, generated by the totality of complete pre-image $A^{-1}V_\delta$, converges in x_0.

It follows from Condition 1 that the totality of sets $A^{-1}V_\delta$ is a basis of the filter $\{A^{-1}V_\delta\}$. Condition 1 of Definition 1 signifies the existence and uniqueness of the solution of Eq. (1) for $y = y_0$, whereas Condition 2 amounts to the requirement that the solution be continuously dependent on the initial data.

In the definition of correctness it could be assumed that Y is an abstract set, by supposing that together with a topology in Y we are given a filter $\{V_\delta\}$ with the following property: the intersection of all sets V_δ consists only of one point y_0. Yet we shall assume Y to be a topological space.

Let us note that our definition of correctness is weaker than the ordinary definition even in the case of topological spaces (see [8]), differing in two respects:

1) It has local character.

2) We do not require that y_0 be an interior point of the domain of values of the mapping A. Therefore, we can have, as closely as desired to y_0, points y which do not have pre-images in X. For such y, Eq. (2) has no solutions.

4. In solving incorrectly formulated problems, the existence and uniqueness of the solution are assumed known in most cases (in the absence of uniqueness we usually seek a "normal solution" in the sense of Tikhonov). Of basic significance in such problems is usually the construction of an approximate solution x_δ on the basis of an approximate right-hand side y_δ. According to Tikhonov [10], and approximate solution is an element $x_\delta \in X$, constructed by certain rules from the approximate right-hand side y_δ, such that $x_\delta \to x_0$ for $y_\delta \to y_0$. It is not required that x_δ satisfy condition (1) for $y = y_\delta$.

Developing this idea, we shall extend the concept of approximate solution of the right-hand side. More precisely, an approximate assignment of y_0 is understood in the sense of assigning a neighborhood V_δ of the point y_0. This can be done on the basis of the following consideration. If Y is a metric space and a point of this space is assigned with an accuracy $\delta > 0$, this will signify that we do not distinguish between points which are less than a distance δ apart. This means that instead of a "point," we always have a set of small diameter. In the following we shall assume that Condition 1 of Definition 1 always holds, i.e., for $y = y_0$ Eq. (1) has a unique solution $x_0 \in X$. Our problem is as follows.

Problem Suppose that under a mapping $A: X \to Y$ the point $y_0 \in Y$ has in X a unique pre-image x_0. To each element V_δ of a filter of neighborhoods of the point y_0 it is required to assign a point $x_\delta \in X$ such that $x_\delta \to x_0$. Any x_δ, possessing this property, will be called an approximate solution of Eq. (1), corresponding to a neighborhood V_δ of the point y_0.

5. The problem, formulated in Subsection 4, is easy to solve if it is correct, i.e., in addition to Condition 1 of Definition 1 we satisfy also Condition 2 of stability. As x_δ we can take any point belonging to a complete pre-image $A^{-1}V_\delta$ of a neighborhood V_δ. The above condition of completeness of the domain of values R(A) of the mapping A into Y ensures the existence of such x_δ. Such a method requires the determination of x from the condition $Ax \in V_\delta$, which in the case of metric spaces reduces to the solving of inequalities.

With our definition of correctness, a problem will be incorrect at a point y_0 if it satisfies none of the conditions of correctness. We confine ourselves here to a study of incorrect problems that do not satisfy Condition 2 of Definition 1. For these problems we require the fulfillment of the conditions of existence and uniqueness of the solution of Eq. (1), but we do not require stability. This means that the set of complete pre-image $A^{-1}V_\delta$ of neighborhoods of y_0 has as its intersection a single point x_0, and hence it is a basis of the filter $\{A^{-1}V_\delta\}$, though the convergence of this filter to the point x_0 is not required. Such problems will be called unstable problems.

For solving unstable problems (in the sense of Subsection 2) it is not possible to use directly the method of Subsection 4. One possibility of solution is to restrict the mapping A to a bicompact set M of the space X, which under additional assumptions makes it possible to use the method of Subsection 4. We shall show that this can be done on the assumption that A is a mapping with a closed graph.

§ 2. Mappings with a Closed Graph

6. As is well known, a set

$$G = \{x, Ax : x \in X\} \quad (1a)$$

in the topological product X × Y is called a graph of the mapping A. In considering Eq. (1), we shall assume henceforth that A is a mapping with a closed graph. For the sake of generality, it will be assumed in Definition 2 that the domain of definition D(A) of the mapping A may not coincide with X. For a mapping with a closed graph it is possible to give two equivalent definitions.

Definition 2. A mapping $A:X \to Y$ is said to be a mapping with a closed graph if:

A) the graph G of the mapping A is a closed set in X×Y, or

B) from the fact that the filter $\{E_\alpha\}$ on D(A) converges in X to $\bar{x} \in X$, and a filter, generated by the sets $AE_\alpha \subset Y$, converges to $\bar{y} \in Y$, it follows that $\bar{x} \in D(A)$, $\bar{y} = A\bar{x}$.

In (B) we can use generalized sequences instead of filters.

THEOREM 1. Parts A and B of Definition 2 are equivalent.

The equivalence of A and B can be proved in the same way as the equivalence of the two definitions of a closed operator in the theory of normed linear spaces; it is only necessary to replace denumerable sequences by filters (see also p. 27 in [3]).

It is well known that the graph of a continuous mapping is closed ([14], p. 84); therefore closed mappings are a generalization of continuous mappings.

7. Everywhere in the following, A will denote a mapping with a closed graph whose domain of definition D(A) coincides with X.

THEOREM 2. In a mapping with a closed graph $A:X \to Y$ the image of a bicompact set is closed.

Proof. Let M be a bicompact set in X, let N = AM be its image in Y, and $\bar{y}$ a limit point of N. Then there exists on N a filter $\{F_\alpha\}$, convergent to $\bar{y}$. The totality of complete pre-images $A^{-1}F_\alpha$ forms a basis of the filter $\{A^{-1}F_\alpha\}$ on M.

Let $\{U_\alpha\}$ be an ultrafilter on M that contains $\{A^{-1}F_\alpha\}$. Owing to the bicompactness of M, the ultrafilter $\{U_\alpha\}$ will converge to a point $\bar{x} \in M$. The image $\{AU_\alpha\}$ of the ultrafilter $\{U_\alpha\}$ is a filter that majorizes the filter $\{F_\alpha\}$; therefore $AU_\alpha \to \bar{y}$. Since the graph of the mapping A is closed, we then have by definition $2\bar{y} = A\bar{x}$, and since $\bar{x} \in M$, it follows that $\bar{y} \in N$.

A direct consequence of this theorem is the following:

THEOREM 3. Let $A: M \to Y$ be a one-to-one mapping of the bicompact space M into a Hausdorff space Y that has a closed graph. Then the inverse mapping will be continuous on the image N = AM.

Theorem 3 can be also obtained from Exercises 5 on p. 129 of [14]. This assertion is made under stronger assumptions also in [1] (Theorem 2 on p. 834), where it is used for the solving of incorrectly formulated problems. By requiring (instead of closure) the continuity of A, we obtain a well-known topological lemma (p. 148 in [15]) that lies at the basis of the theory of incorrectly formulated problems since the appearance of Tikhonov's paper [9].

THEOREM 4. If the pre-image M of a bicompact set $N \subset Y$ is not empty under a mapping with a closed graph $A:X \to Y$, then M will be a closed set.

The scheme of the proof of this theorem is the same as in Theorem 2.

8. In this subsection we shall consider a stronger version of Theorem 3, needed by us.

LEMMA. Let M be a bicompact space, and $\{E_\delta\}$ a filter on M that has a unique tangency point $\bar{x}$. Then $E_\delta \to \bar{x}$.

Remark. If the bicompactness of M is not required by us, the assertion will be false. Example: a filter on the set of real numbers whose basis consists of the sets $F_n = \{0\} \cup (n, +\infty)$, $n = 1, 2, \ldots$ ([16], p. 14).

Proof. By Δ we shall denote a directed set of indices δ. We shall assume that $\bar{x}$ is the unique point of tangency of the filter basis, but the filter does not converge to it. This means that there exists a neighborhood $U(\bar{x})$ of the point $\bar{x}$ such that none of the sets E_δ is completely contained in $U(\bar{x})$. Then the set $P_\delta = E_\delta \setminus U(\bar{x})$ is not empty for any $\delta \in \Delta$. The set P_δ consists of the basis of a filter. Indeed, let $\alpha \in \Delta$ and $\beta \in \Delta$. Then $E_\gamma = E_\alpha \cap E_\beta$ belongs to the filter $\{E_\delta\}$ and $P_\gamma = E_\gamma \setminus U(x_0)$ belongs to the intersection $P_\alpha \cap P_\beta$. The filter $\{P_\delta\}$, defined on a bicompact space M, has a tangency point $\{x_1\}$. The filter $\{E_\delta\}$ is weaker than the filter $\{P_\delta\}$; therefore x_1 is also a tangency point of the filter $\{E_\delta\}$, and in view of its uniqueness it coincides with $\bar{x}$. But this is impossible, since all the sets P_δ lie outside a neighborhood $U(\bar{x})$ of the point $\bar{x}$.

9. THEOREM 5. Let $A: M \to Y$ be a mapping, with a closed graph, of the bicompact space M into a Hausdorff space Y, and y_0 a point of Y that has in M a unique pre-image x_0. If $\{V_\delta\}$ is a filter of neighborhoods of the point y_0, the complete pre-images $E_\delta = A^{-1}V_\delta$ will form the basis of a filter that converges to the point x_0.

Proof. It is easy to check that the set E_δ forms the basis of a filter on M. Owing to the bicompactness of M, the filter $\{E_\delta\}$ has on M a tangency point. It suffices to prove that every tangency point $\bar{x}$ of the filter $\{E_\delta\}$ coincides with x_0; by virtue of the lemma it hence follows that $E_\delta \to x_0$.

Thus let $\bar{x}$ be a tangency point of the filter $\{E_\delta\}$. There exists a filter $\{Q_\gamma\}$ that majorizes $\{E_\delta\}$ and converges to $\bar{x}$. As a basis of such a filter we can take the intersection of the sets E_δ with the neighborhoods of the point $\bar{x}$. The filter of the images $\{AQ_\gamma\}$ majorizes the filter $\{V_\delta\}$; hence $AQ_\delta \to y_0$.

Thus $Q_\gamma \to \bar{x}$, $AQ_\gamma \to y_0$, and since A is a mapping with a closed graph, it follows that $A\bar{x} = y_0$. But y_0 has in M a unique pre-image x_0; therefore $\bar{x} = x_0$.

COROLLARY. On a bicompact space, Condition 2 in the definition of correctness (Definition 1 of Subsection 3) is a consequence of Condition 1.

§3. Extension of the Method of Quasisolutions

10. In this section we consider an extension of the method of quasisolutions [10] to the problem, examined in Subsection 2. We shall assume that for a given $y = y_0$ the equation

$$Ax = y,$$

where A is a mapping with a closed graph, has a unique solution x_0 that belongs to a given compact set $M \subset X$. For any neighborhood V_δ, belonging to the basis of a filter of neighborhoods $\{V_\delta\}$, it is required to construct an $x_\delta \in X$ such that $x_\delta \to x_0$.

Let us denote by E_δ a complete pre-image of the set V_δ in M:

$$E_\delta = M \cap A^{-1}V_\delta.$$

The sets E_δ are not empty; each of them contains the point x_0, and they form the basis of a filter $\{E_\delta\}$, which according to Theorem 5 converges to the point x_0. In each set E_δ let us select a point x_δ. By virtue of the corollary of Theorem 5 we then have $x_\delta \to x_0$, and the generalized sequence $\{x_\delta\}$ is that sought.

11. Let us consider the case that X and Y are metric spaces. As the indices δ we usually take positive numbers $(0 < \delta \le \delta_0)$, and for each δ we assign a y_δ such that $\rho(y_0, y_\delta) < \delta$ (the approximate value of y_0). The elements of the basis of a filter of neighborhoods of the point y_0 are taken in the form of spheres

$$V_\delta = \{y : \rho(y, y_\delta) \le \delta\}, \qquad (2)$$

each of which contains y_0 and y_δ. Usually, the bicompact set M is likewise assigned with the aid of a system of inequalities. Then the selection of x_δ reduces to the solving of a system of inequalities that ensure the fulfilment of the relations

$$x \in M, \ Ax \in V_\delta. \qquad (3)$$

In [10] it is proposed to take x_δ in the form of an x in M that minimizes $\rho(Ax, y_\delta)$. This x_δ satisfies inequalities (2), yet the inequalities (2) offer greater possibilities for the selection of x_δ. On the other hand the accuracy of quasisolutions in the sense of order of magnitude does not exceed the accuracy of the elements, satisfying (2).

For $\eta > 0$, let

$$\omega(\eta; x_0) = \sup \rho_x(x, x_0) \quad \text{for} \quad x \in M,\ \rho_y(Ax, Ax_0) \leqslant \eta.$$

From Theorem 5 it follows that

$$\lim_{\eta \to 0} \omega(\eta, x_0) = 0.$$

If V_δ is defined by (2) and x_δ is any element, satisfying (3), it follows from the fact that Ax_δ and y_0 belong to V_δ that

$$\rho(Ax_\delta, y_0) = \rho(Ax_\delta, y_\delta) + \rho(y_\delta, y_0) < 2\delta,$$

hence

$$\rho(x_\delta, x_0) \leqslant \omega(2\delta; x_0). \tag{4}$$

If y_δ has a pre-image $\bar{x}_\delta$ in M, this pre-image will be a quasisolution, since $\rho(A\bar{x}_\delta, y_\delta) = 0$. On the other hand, as can be seen from the definition of the quantity $\omega(\eta; x_0)$, it may happen that

$$\rho(\bar{x}_\delta, x_0) = \omega(\delta; x_0), \tag{5}$$

i.e., we cannot guarantee that the quasisolution deviates from the exact solution by less than $\omega(\delta; x_0)$. The only advantage in using a quasisolution $\bar{x}_\delta$ instead of x_δ, satisfying (3), consists in the coefficient 2 of δ, which does not affect the order of the deviation for $\delta \to 0$.

§ 4. Variational Method

12. To topological spaces it is possible to extend a type of variational method [11, 12, 6] that was rigorously formulated for the first time by A. N. Tikhonov. In Hilbert spaces and (as is shown in [17]) in Efimov-Stechkin spaces [18] this method is based on weak compactness. We shall describe an abstract analog of this method.

Definition 3. In a space X let us define a nonnegative numerical functional $\Omega(x)$ with the following properties: 1) For any positive c the set

$$M_c = \{x : \Omega(x) \leqslant c\}$$

is nonempty and bicompact;

2) for any $c \geq 0$ there exists a point $x \in X$ such that $\Omega(x) = c$.

Under these conditions, $X = \bigcup_{0} M_c$.

Such a functional is said to be stabilizing.

Example. Let X be a Hilbert space with a weak topology, $\Omega(x) = \|x\|$.

In applications it is often assumed that $S = \bigcup_{c>0} M_c$ is the intrinsic part of the space X, but Condition 2 does not reduce the generality, since the space X can be always taken in the form of the set S.

Owing to Property 2 of Definition 3, there exists for any nonempty set $E \subset X$ a point x_E in X at which $\Omega(x)$ reaches its exact infimum on E: $\Omega(x_E) = \inf_{x \in E} \Omega(x)$, yet this point may not belong to E. We shall say that a function $\Omega(x)$ has the property of minimality if for sets E whose closure $\bar{E}$ is bicompact we always have $x_E \in \bar{E}$.

13. In this and the following subsections we shall assume that in Eq. (1) the quantity A is a mapping with a closed graph, and for $y = y_0$ this equation has in X a unique solution x_0. We shall show that in the presence of a stabilizing functional the assumption of existence of a solution and the condition $\Omega(x_0) > 0$ are sufficient for solving the problem, posed in Subsection 2. In the present subsection we additionally assume

that the minimizing functional has the property of minimality. In Subsection 14 we do not make any additional assumptions.

The construction of a generalized sequence $\{x_\delta\}$, convergent to x_0, is based on the following reasoning.

Let $\Omega(x_0) = c_0$ and $c_0 > 0$.

The set

$$M_0 = \{x: \Omega(x) \leq c_0\} \tag{6}$$

is nonempty and bicompact. As in Subsection 10, there exists here a filter $\{E_\delta\}$, convergent to x_0, such that $E_\delta = M_0 \cap A^{-1}V_\delta$, but the selection of the elements x_δ from each E_δ is more difficult here in view of the fact that the set M_0 is not given, since the element x_0 is unknown. This difficulty can be overcome by taking x_δ in the form of an element that minimizes $\Omega(x)$ on the closure $\bar{E}_\delta$ of the set E_δ. Then $\Omega(x_\delta) \leq c_0$ and $x_\delta \in M_0$.

This idea is formulated in the following theorem.

THEOREM 6. Let us assume that in Eq. (1), $A: X \to Y$ is a mapping with a closed graph, the point y_0 has in X a unique pre-image x_0, $\Omega(x)$ is a stabilizing functional, and there exists a point at which $\Omega(x)$ reaches its exact infimum under the condition $Ax \in V_\delta$, where V_δ is an element of the basis of a filter of neighborhoods of the point y_0. If $\Omega(x)$ has the property of minimality, then $x_\delta \to x_0$.

Proof. Let M_0 be defined by (6), and let $E_\delta = M_0 \cap A^{-1}V_\delta$ be complete pre-image (in M_0) of elements of the basis of a filter of neighborhoods of the point y_0. By Theorem 5 the sets E_δ form the basis of a filter, convergent to x_0. Since M_0 is regular in a relative topology (as a compact space), it follows that a filter $\{\bar{E}_\delta\}$, generated by the closures of the sets E_δ, will also converge to x_0. Each $\bar{E}_\delta$ is a closed subset of a bicompact set M_0, being therefore bicompact, too. By the property of minimality we have $x_\delta \in \bar{E}_\delta$ and therefore $x_\delta \to x_0$. The minimality condition for $\Omega(x)$ can be relaxed, but with our method it cannot be completely dropped. This can be seen from the following example.

Let $X = Y = [0, 2]$ be a segment of the number interval with a natural topology, and A an identity mapping such that Eq. (1) has the form

$$x = y.$$

Let us write $y_0 = 2$; hence $x_0 = 2$. Here the problem is even correct.

Let us define the stabilizing functional as follows: $\Omega(x) = 1 - x$ for $0 \leq x \leq 1$; $\Omega(x) = x/2 - (1/2)^n$ for $2 - (1/2)^{n-1} < x \leq 2 - (1/2)^n$, $n = 1, 2, \ldots$; $\Omega(2) = 1$.

$\Omega(x)$ is a function, continuous from the left. It is easy to see that the conditions of Definition 2 are satisfied. For neighborhoods $V_\delta(y_0) = [2 - \delta, 2]$ and $\delta = (1/2)^{n-1}$, $n = 2, 3$, we have $\inf \Omega(x) = 1 - (1/2)^{n-1}$. But this infimum is reached at the point $x_\delta = (1/2)^{n-1}$, and for $\delta \to 0$ the convergence $x_\delta \to x_0$ does not occur.

14. It is possible to give another method of construction of a sequence $\{x_\delta\}$, convergent to x_0, that is based only on the properties of the stabilizing functional $\Omega(x)$, contained in Definition 2, without requiring the fulfillment of the minimality property.

Let Δ_1 be a set of indices, defining the basis of a filter of neighborhoods $\{V_\delta\}$ of the point y_0. We shall assume that on Δ_1 a positive bounded numerical function $\eta(\delta)$ $(0 < \eta(\delta) \leq \eta_0)$ is defined, with the following properties:

a) for $\delta_2 \geq \delta_1$ we always have $\eta(\delta_2) \leq (\delta_1)$ (monotonicity);

b) $\lim \eta(\delta) = 0$.

THEOREM 7. Let us assume that in Eq. (1), $A: X \to Y$ is a mapping with a closed graph, the point y_0 has in X a unique pre-image x_0, and $\Omega(x)$ is a stabilizing functional. For any $\delta \in \Delta_1$ let the element x_δ be constructed such that $Ax_\delta \in V_\delta$ and it satisfies the inequality $\gamma \leq \Omega(x_\delta) < \gamma_\delta + \eta_\delta$, where $\gamma_\delta = \inf \Omega(x)$ for $Ax \in V_\delta$. Hence $x_\delta \to x_0$

Proof. From the definition of an exact infimum follows that for any $\delta \in \Delta_1$ there exist x_δ, satisfying (7).

Let us introduce the set

$$M_1 = \{x: \Omega(x) \leq c_0 + \eta_0\}, \ c_0 = \Omega(x_0).$$

This set is bicompact and $x_0 \in M_1$. By Theorem 5 the sets $E_\delta = M_1 \cap A^{-1}V_\delta$ form a basis of a filter on M_1, convergent to x_0.

From the inequalities

$$\gamma_\delta \leqslant \Omega(x_\delta) \text{ and } \gamma_\delta + \eta(\delta) \leqslant c_0 + \eta_0$$

and inequality (7) follows that $x_\delta \in M_1$. Since $Ax \in V_\delta$, we conclude that $x_\delta \in E_\delta$. But in this case $x_\delta \to x_0$, which completes the proof of the theorem. With the use of this theorem, the determination of x_δ for a given neighborhood V_δ of the point y_0 reduces to the determination of $\gamma_\delta = \inf \Omega(x)$ for $Ax \in V_\delta$, and of x_δ from the relations

$$\gamma_\delta \leqslant \Omega(x) \leqslant \gamma_\delta + \eta_\delta, \quad Ax \in V_\delta;$$

as in § 3, in the case of metric spaces this normally reduces to the solving of systems of simultaneous inequalities.

CONCLUSIONS

15. According to the methods of § 3 and § 4, we are considering the solution on a bicompact set M, taking as approximate solutions the elements of this set, i.e., we restrict X to a bicompact space M.

By virtue of the corollary of Theorem 5, this re-establishes the stability of the problem. Thus by using bicompact sets and the related additional information about the solution, it is possible to go over from an incorrect problem to a problem which is correct in the sense of Tikhonov (see [19], p. 4).

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