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ALGORITHMIC DIMENSION OF NILPOTENT GROUPS

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The concept of algorithmic dimension was introduced in [1] for algebraic systems $\mathfrak{A}$. In the present article, we investigate the connection between the algorithmic dimension ($\dim_A \mathfrak{A}$) and the algebraic properties of the system $\mathfrak{A}$.

It is well known [2, 3] that constructible Abelian groups of infinite rank (Prüfer) are of infinite algorithmic rank. Thus, for torsion-free constructible Abelian groups, we have the following correlation between algorithmic and algebraic properties [3]: a group G is self-stable if and only if it has finite rank. The results that follow (Theorems 1-3) will show that for constructibel torsion-free nilpotent groups, this correlation already takes on a more complicated character. In Secs. 2 and 3 we will construct examples to illustrate this; in Sec. 3 we give some sufficient conditions for nilpotent groups to be self-stable.

§1. All the necessary notation and definitions used in this article have been introduced in [4-6]. Let us recall some of them.

A countable algebraic system $\mathfrak{A}$ is said to be constructible if there exists an enumeration $\nu\colon \mathbf{N} \to |\mathfrak{A}|$ of the underlying set of the system, relative to which the fundamental predicates and functions of the system become recursive. In this situation, the enumeration is called a constructivization and the pair (G, ν) is said to be a constructive algebraic system. Two constructivizations ν and μ of an algebraic system $\mathfrak{A}$ are said to be auto-equivalent if there exists a recursive isomorphism φ of the constructive system $(\mathfrak{A}, \nu)$ onto $(\mathfrak{A}, \mu)$; i.e., φ is an automorphism of the system $\mathfrak{A}$, such that

$$(\forall n \in \mathbf{N})\,(\varphi\nu(n) = \mu f(n)),$$

where f is a general recursive function. The maximal number of nonequivalent constructivizations of a system $\mathfrak{A}$ is called its algorithmic dimension ($\dim_A \mathfrak{A}$).

Suppose that a group G is given by generators $X = \{x_i/_{i\in\mathbf{N}}\}$ and defining relations $\{g_i/_{i\in\mathbf{N}}\}$. We will say that G is locally finitely presented if for every finite subset $\{x_{i_0}, x_{i_1}, \ldots, x_{i_l}\}$ of X the subgroup G' of G generated by the elements of this subset can be presented in the form

$$G' = \langle x_{i_0}, x_{i_1}, \ldots, x_{i_l};\, g_{j_0}(x_{i_0}, x_{i_1}, \ldots, x_{i_l}), \ldots, g_{j_t}(x_{i_0}, x_{i_1}, \ldots, x_{i_l})\rangle. \quad (1.1)$$

The functions $t = \delta(i_0, i_1, \ldots, i_l)$, $j_r = \sigma_r(i_0, i_1, \ldots, i_l)$ $(r = 1, 2, \ldots, t)$ are called defining functions of the group G.

Proposition 1.1. If the group $G = \langle\{x_i/_{i\in\mathbf{N}}\};\, \{g_j/_{j\in\mathbf{N}}\}\rangle$ is locally finitely presented, with every finitely generated subgroup of G residually finite, and the defining functions of G are general recursive, then the word problem is solvable in G.

Proof. We will indicate an algorithm to decide the condition $G \models f(x_{i_0}, x_{i_1}, \ldots, x_{i_k}) = e$, where $f(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$ is an arbitrary word in the generators $\{x_i/_{i\in\mathbf{N}}\}$. When we apply the hypotheses to the sequence $\langle i_0, i_1, \ldots, i_k\rangle$ we obtain a subgroup G' of G of the form (1.1). Observe first of all that

$$G \models f(x_{i_0}, x_{i_1}, \ldots, x_{i_k}) = e \Leftrightarrow G' \models f(x_{i_0}, x_{i_1}, \ldots, x_{i_k}) = e.$$

We now proceed to define two processes.

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I. By hypothesis, the relations

$$g_{j_0}, g_{j_1}, \ldots, g_{j_t} \tag{1.2}$$

among the generators $x_{i_0}, x_{i_1}, \ldots, x_{i_k}$ can be found effectively. If $G' \models f(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$, then, by enumerating the consequences of the system of defining relations (1.2), we get to the word $f(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$ after a finite number of steps.

II. There can be at most countably many finite groups with generators $x_{i_0}, x_{i_1}, \ldots, x_{i_k}$. Let G_i $(i \in \mathbf{N})$ be some Gödel enumeration of the groups of this set.

If the relation $G' \models f(x_{i_0}, x_{i_1}, \ldots, x_{i_k}) \neq e$ applies, then, in light of G' being residually finite, there exists a homomorphism φ of the group G' onto some group G_s in the set $\{G_i/_{i \in \mathbf{N}}\}$, such that $G_s \models f(x'_{i_0}, x'_{i_1}, \ldots, x'_{i_k}) \neq e$, where $x'_{i_j} = \varphi(x_{i_j})$ $(j = 0, 1, \ldots, k)$.

The essence of the second process is in our finding a homomorphism φ of the group G' onto the group G, in the class $\{G_i/_{i \in \mathbf{N}}\}$, such that $G_s \models g_{i_r}(x'_{i_0}, x'_{i_1}, \ldots, x'_{i_k}) = e$, $G_s \models f(x'_{i_0}, x'_{i_1}, \ldots, x'_{i_k}) \neq e$ $(r = 0, 1, \ldots, t)$, where $x'_{i_j} = \varphi(x_{i_j})$ $(j = 0, 1, \ldots, k)$.

By implementing the processes I and II in parallel, we can determine after a finite number steps whether or not the relation $G \models f(x_{i_0}, x_{i_1}, \ldots, x_{i_k}) = e$ applies.

We will need the following result [6, Theorem 32.21].

THEOREM 1.1. Every finitely generated torsion-free nilpotent group is residually a finite p-group for every prime p.

We will say that a group $\hat{G}$ is X-generated $(X \subset \hat{G})$ over G if $\hat{G} = \langle G, X \rangle$ and $X \cap G = \varnothing$. If X is finite, we say that $\hat{G}$ is finitely generated over G; if X is infinite, then $\hat{G}$ is infinitely generated over G.

§2. Let $F = F(x, y, y_1, y_2, \ldots, y_n, \ldots)$ $(n \in \mathbf{N})$ be a free group in the variety of two-step nilpotent groups, freely generated by $x, y, y_1, y_2, \ldots, y_n, \ldots$. We define H to be the minimal normal subgroup in F generated by the relations

$$y_i^{p_i}[x, y]^{-1} = e \quad (i = 1, 2, \ldots, n, \ldots), \tag{2.1}$$

$$[y_i, y_j] = e \quad (i, j = 1, 2, \ldots, n, \ldots), \tag{2.2}$$

$$[x, y_i] = e \quad (i = 1, 2, \ldots, n, \ldots), \tag{2.3}$$

$$[y, y_i] = e \quad (i = 1, 2, \ldots, n, \ldots), \tag{2.4}$$

where p_i is the i-th prime. Put

$$G = F/H. \tag{*}$$

LEMMA 2.1. Every element of the group G is uniquely representable in the form

$$g = x^{\varepsilon_1} y^{\varepsilon_2} y_1^{l_1} y_2^{l_2} \ldots y_n^{l_n} [x, y]^n,$$

where $\varepsilon_1, \varepsilon_2, l_i \in \mathbf{N}$, $0 \leqslant l_i < p_i$ $(i = 1, 2, \ldots, n)$.

Proof. According to [6, Theorem 31.52],

$$g = x^{\varepsilon_1} y^{\varepsilon_2} y_1^{l_1} y_2^{l_2} \ldots y_n^{l_n} c, \quad c \in [G, G]. \tag{2.5}$$

It follows from the relations (2.1)-(2.4) that [G, G] is an infinite cyclic group generated by [x, y]; i.e., in (2.5), $c = [x, y]^l$ for some $l \in \mathbf{N}$. As a result of (2.1), we may assume in (2.5) that $0 \leqslant l_i < p_i$ $(i = 1, 2, \ldots, n)$.

Suppose now that

$$x^{\varepsilon_1} y^{\varepsilon_2} y_1^{l_1} y_2^{l_2} \ldots y_n^{l_n} [x, y]^l = e. \tag{2.6}$$

Consider the homomorphism φ from G into the group

$$G' = \mathbf{Z} \oplus \mathbf{Z} \oplus \sum_{i=1}^{n} \mathbf{Z}(p_i),$$

that extends the homomorphism

$$\hat{\varphi}(x) = (1, 0, 0, \ldots, 0),$$

$$\widehat{\varphi}(y)=(0,\ 1,\ 0,\ \ldots,\ 0),$$
$$\widehat{\varphi}(y_i)=(0,\ 0,\ \ldots,\ 1(\mathrm{mod}\ p_i),\ \ldots,\ 0)\quad (i=1,\ 2,\ \ldots,\ n).$$

[The homomorphism φ, extending $\widehat{\varphi}$, exists since $\varphi(x)$, $\varphi(y)$, $\varphi(y_i)$ obviously satisfy relations (2.1)-(2.4), and generate G'.]

By applying the map φ to both sides of (2.6), we obtain

$$(\varepsilon_1,\ \varepsilon_2,\ l_1(\mathrm{mod}\ p_1),\ l_2(\mathrm{mod}\ p_2),\ \ldots,\ l_n(\mathrm{mod}\ p_n))=0. \tag{2.7}$$

In light of the structure of the group G', we deduce from (2.7) that $\varepsilon_1=0$, $\varepsilon_2=0$, $l_i\equiv 0(\mathrm{mod}\ p_i)$ (i = 1, 2,...,n). Since $0\leqslant l_i<p_i$, we have $\ell_i = 0$ (i = 1, 2,...,n). By (2.6), we then arrive at the equation $[x, y]^{\ell} = e$, which can only be valid if $\ell = 0$, the group [G, G] being torsion free, as pointed out earlier.

COROLLARY 2.1. The group G is torsion free.

LEMMA 2.2. Let $t\in G$; then $t=y_i^{\delta_i}[x,y]^{\sigma_i}$, where $\delta_i+\sigma_i p_i=1$, if and only if $t^{p_i} = [x, y]$.

Proof. Let $t_i=y_i^{\delta_i}[x,y]^{\sigma_i}$ and $\delta_i+\sigma_i p_i=1$. Then $t_i^{p_i}=y_i^{p_i\delta_i}[x,y]^{\sigma_i p_i}$. By formula (2.1), $t_i^{p_i} = [x,y]^{\delta_i+\sigma_i p_i}=[x,y]$. Suppose now that $t\in G$ and $t^{p_i} = [x, y]$. By Lemma 2.1, $t=x^{\varepsilon_1}y^{\varepsilon_2}y_1^{l_1}y_2^{l_2}\ldots y_n^{l_n}[x, y]^{\ell}$. Then

$$t^{p_i}=x^{\varepsilon_1 p_i}y^{\varepsilon_2 p_i}y_1^{l_1 p_i}\ldots y_n^{l_n p_i}[x,y]^{lp_i-\varepsilon_1\varepsilon_2 p_i}=[x,y].$$

In view of the equality $y_i^{l_i p_i}=[x,y]^{l_i}$, it follows from this that

$$x^{\varepsilon_1 p_i}y^{\varepsilon_2 p_i}\prod_{j\neq i}^{n} y_j^{l_j p_i}[x,y]^{l_i+p_i(l-\varepsilon_1\varepsilon_2)-1}=e,$$

whence, by Lemma 2.1, $\varepsilon_1 p_1=0$, $\varepsilon_2 p_2=0$, $l_j p_i\equiv 0(\mathrm{mod}\ p_j)$ $(j\neq i)$, $l_i+p_i(l-\varepsilon_1\varepsilon_2)-1=0$, i.e., $\varepsilon_1=0$, $\varepsilon_2=0$, $l_j\equiv 0(\mathrm{mod}\ p_j)$, $l_i+p_i(l-\varepsilon_1\varepsilon_2)=1$. In light of the inequalities $0\leqslant l_i<p_i$, we conclude that $l_j=0$. Therefore, $t=y_i^{\delta_i}[x,y]^{\delta_i+p_i\sigma_i}$, where $\delta_i=l_i$, $\sigma_i=l-\varepsilon_1\varepsilon_2$.

LEMMA 2.3. The map $\widehat{\varphi}(x)=x$, $\widehat{\varphi}(y)=y$, $\widehat{\varphi}(y_i)=y^{\delta_i}[x,y]^{\sigma_i}$, where $\delta_i+\sigma_i p_i=1$ $(i=1,\ 2,\ \ldots,\ n)$, extends to an automorphism φ of the group G.

Proof. We will show that the elements x, y, $t_i=y_i^{\delta_i}[x,y]^{\sigma_i}$ (i = 1, 2,...,n) generate the group G. To this end, we express y_i in terms of the elements x, y, t_i (i = 1, 2,...,n,...). Observe that the condition $\delta_i + \sigma_i p_i = 1$ implies the equality $(\delta_i, p_i) = 1$; i.e., there exist u_i, $v_i\in\mathbf{Z}$ such that $u_i\delta_i + v_i p_i = 1$. So $t_i^{u_i}=y_i y_i^{-v_i p_i}[x,y]^{\delta_i u_i}$. Since, by (2.1), $y_i^{p_i}=[x,y]$, it follows that $t_i^{u_i}=y_i[x,y]^{\delta_i u_i-v_i}$. From this we deduce that $y_i=t_i^{u_i}[x,y]^{v_i-\delta_i u_i}$ (i = 1, 2, ..., n).

Also observe that the elements x, y, t_i obviously satisfy the relations (2.1)-(2.4); i.e., the map $\widehat{\varphi}$ extends to a homomorphism φ. We will now show that $\ker\varphi=\{e\}$.

Let $g=x^{\varepsilon_1}y^{\varepsilon_2}y_1^{l_1}y_2^{l_2}\ldots y_n^{l_n}[x,y]^{l}$. Then

$$\varphi(g)=x^{\varepsilon_1}y^{\varepsilon_2}y_1^{l_1\delta_1}[x,y]^{l_1\sigma_1}y_2^{l_2\delta_2}[x,y]^{l_2\sigma_2}\ldots y_n^{l_n\delta_n}[x,y]^{l_n\sigma_n}[x,y]^{l}=x^{\varepsilon_1}y^{\varepsilon_2}y_1^{l_1\delta_1}y_2^{l_2\delta_2}\ldots y_n^{l_n\delta_n}[x,y]^{l+\sum_i l_i\sigma_i}=e,$$

with

$$0\leqslant l_i<p_i,\ (\delta_i,\ p_i)=1\quad (i=1,\ 2,\ \ldots,\ n). \tag{2.8}$$

By Lemma 2.1, $\varepsilon_1=\varepsilon_2=0$, $l_i\delta_i\equiv 0(\mathrm{mod}\ p_i)$, $l+\sum_i l_i\sigma_i=0$. In light of (2.8), we get that $\ell_i = 0$ (i = 1, 2,...,n). It then follows from $l+\sum_i l_i\sigma_i=0$ that $l=0$.

LEMMA 2.4. The group G is constructible.

Proof. According to the definition of the system of relations (2.1)-(2.4), G is locally finitely presented. Moreover, by Theorem 1.1, every finitely generated subgroup of G is residually finite. By Proposition 1.1, the word problem is solvable in G, i.e., G is constructible.

LEMMA 2.5. The group G is self-stable.

Proof. Let ν be some fixed constructivization and $\nu(n_i)=y_i$ $(i\in\mathbf{N})$, $\nu(s_0)=e$, $\nu(s_1)=x$, $\nu(s_2)=y$. Let μ be any constructivization of the group G, and $\mu(t_0)=e$, $\mu(t_1)=x$, $\mu(t_2)=y$. For every i, we choose m_i in accordance with the condition $(\mu(m_i))^{p_i}=[\mu(t_1),\mu(t_2)]$ $(i\in\mathbb{N})$. By virtue of Lemmas 2.2, 2.3, the map $\widehat{\varphi}(x)=x$, $\widehat{\varphi}(y)=y$, $\widehat{\varphi}(y_i)=\mu(m_i)$ is extendable to an automorphism φ of the group G. We define a general recursive function f, satisfying $\varphi\nu(n)=\mu f(n)$, as follows:

$$f(s_i)=t_i\ (i=0,1,2),\quad f(n_i)=m_i.$$

Furthermore, if $\nu(n)=\prod_{i=0}^{k}(x^{\varepsilon_i}y^{\delta_i})\prod_{j=1}^{r_i}y_j^{l_{ji}}$, then we put

$$f(n)=\mu^{-1}\left(\prod_{i=0}^{k}(\mu(t_1))^{\varepsilon_1}(\mu(t_2))^{\varepsilon_2}\prod_{j=1}^{r_i}(\mu(m_j))^{l_{ji}}\right).$$

THEOREM 2.1. There exists a constructible two-step torsion-free nilpotent group G such that G/[G, G] is of infinite rank, [G, G] is of finite rank and $\dim_A G = 1$.

Proof. Let G be the group defined by (*). According to Lemma 2.1, [G, G] is a group of rank 1. By Corollary 2.1, G is torsion free. It is easy to see that the elements $y_i[G,G]$ $(i\in\mathbb{N})$ of the quotient G/[G, G] are linearly independent (Lemma 2.1); i.e., G/[G, G] is of infinite rank. By Lemma 2.4, the group of G is constructible; according to Lemma 2.5 it is self-stable.

Note that G/[G, G] is a torsion-free Abelian group; i.e., $\dim_A(G/[G,G])=\infty$, and G is finitely generated over G/[G, G]; so, we have

THEOREM 2.2. There exist groups G and $\hat{G}$ such that

a) G and $\hat{G}$ are torsion free,

b) G is Abelian and $\dim_A G=\infty$,

c) $\hat{G}$ is two-step nilpotent and finitely generated over G,

d) $\dim_A \widehat{G}=1$.

§3. Let $F=F(\{x_i/_{i\in\mathbb{N}}\},\{y_j/_{j\in\mathbb{N}}\})$ be a free group of the generators x_i, y_j $(i, j\in\mathbb{N})$ in the variety of two-step nilpotent groups, let H be the minimal normal subgroup of F determined by the relations

$$[x_n,y_m]=e\quad(n\neq m;\ n,m\in\mathbb{N}),\tag{3.1}$$

$$[x_{2n},x_{2n+1}]=[y_{2n},y_{2n+1}]=a\quad(n\in\mathbb{N}),\tag{3.2}$$

$$[x_{2n+1},x_{2n+2}]=[y_{2n+1},y_{2n+2}]=b\quad(n\in\mathbb{N}).\tag{3.3}$$

Put G = F/H.

LEMMA 3.1. Each element g of the group G can be uniquely represented in the form $g=\prod_{i=0}^{k_1}x_i^{\varepsilon_i}\prod_{i=0}^{k_2}y_i^{\delta_i}c$, where $c\in[G,G]$.

Proof. The proof is similar to that of Lemma 2.1.

LEMMA 3.2. The commutator subgroup [G, G] of G is torsion free.

Proof. It follows from (3.1)-(3.3) that the group [G, G] is freely generated by the elements $[x_n,y_n]$ $(n\in\mathbb{N})$, a, b, $[x_i,y_j]$ $(j>i+1)$, $[y_l,y_k]$ $(k>l+1)$.

COROLLARY 3.1. The group [G, G] is Abelian of infinite rank.

COROLLARY 3.2. The group G is torsion free.

LEMMA 3.3. Let $t_{ij}\in G$ $(i\in\mathbb{N}, j\in\{1,2\})$. Then

(a) $t_{n1}=x_{n+1}c_{n1}$, $c_{n1}\in[G,G]$ if and only if

$$\begin{cases}[t_{n1},y_n]=e,\\ [x_n,t_{n1}]=\begin{cases}a, & \text{if } n \text{ is even},\\ b, & \text{if } n \text{ is odd};\end{cases}\end{cases}\tag{3.4}$$

(b) $t_{n2}=y_{n+1}c_{n2}$, $c_{n2}\in[G,G]$ if and only if

$$\begin{cases}[t_{n2},x_n]=e,\\ [y_n,t_{n2}]=\begin{cases}a, & \text{if } n \text{ is even},\\ b, & \text{if } n \text{ is odd}.\end{cases}\end{cases}\tag{3.5}$$

Proof. Because of the symmetry, we will only carry out the proof of (a).

It is obvious that if $t = x_{n+1}c$ and $c \in [G, G]$, then t satisfies (3.4). We will now show that every solution of (3.4) is of the form $x_{n+1}c$, with $c \in [G, G]$. Suppose that $t \in G$ and t satisfies (3.4). By Lemma 3.1,

$$t = \prod_{i=0}^{k_1} x_i^{\varepsilon_i} \prod_{i=0}^{k_2} y_i^{\delta_i} c \quad (c \in [G, G]).$$

It follows from this that

$$[t, y_n] = \prod_{i=0}^{k_1} [x_i, y_n]^{\varepsilon_i} \prod_{i=0}^{k_2} [y_i, y_n]^{\delta_i} = e. \tag{3.6}$$

By (3.6), taking into account the relations (3.1)-(3.3) and Corollary 3.1, $\varepsilon_n = 0$, $\delta_i = 0$ $(i \neq n,\ i \in \{0, 1, \ldots, k_2\})$. Thus,

$$t = \prod_{i,\ n}^{k_1} x_i^{\varepsilon_i} y_n^{\delta_n} c.$$

By substituting this expression for t into the second equation of the system (3.4), we may write

$$[x_n, t] = \prod_{i \neq n}^{k_1} [x_n, x_i]^{\varepsilon_i} [x_n, y_n]^{\delta_n} = \begin{cases} [x_{2k}, x_{2k+1}], & \text{if} \quad n = 2k, \\ [x_{2k+1}, x_{2k+2}], & \text{if} \quad n = 2k+1. \end{cases} \tag{3.7}$$

By (3.7), taking into account the relations (3.1)-(3.3) and Corollary 3.1, we get $\varepsilon_i = 0$ $(i \neq n+1)$, $\delta_n = 0$, $\varepsilon_{n+1} = 1$. Thus, $t = x_{n+1}c$.

LEMMA 3.4. The map $\widehat{\varphi}(x_0) = x_0$, $\widehat{\varphi}(y_0) = y_0$, $\widehat{\varphi}(x_i) = x_i c_{i1}$, $\widehat{\varphi}(y_i) = y_i c_{i2}$ $(i = 1, 2, \ldots)$ extends to an automorphism φ of the group G.

Proof. The proof is analogous to the proof of Lemma 2.3.

LEMMA 3.5. The group G is constructible.

Proof. The proof is analogous to that of Lemma 2.4.

LEMMA 3.6. The group G is self-stable.

Proof. Let ν be some fixed constructivization of the group G, with $\nu(n_i) = x_i$, $\nu(m_i) = y_i$ $(i \in \mathbf{N})$, $\nu(s_0) = e$, $\nu(s_1) = a$, $\nu(s_2) = b$, $\nu(r_1) = x_0$, $\nu(r_2) = y_0$.

Let μ be an arbitrary constructivization of the group G with $\mu(s_0') = e$, $\mu(s_1') = a$, $\mu(s_2') = b$, $\mu(r_1') = x_0$, $\mu(r_2') = y_0$. We will now show how one can effectively find the μ-number of the elements $x_k c_{k1}$ and $y_k c_{k2}$ $(c_{k1}, c_{k2} \in [G, G],\ k = 1, 2, \ldots)$. Suppose that $\mu(n_i') = x_i c_{i1}$, $\mu(m_i') = y_i c_{i2}$. Consider the systems

$$\begin{cases} [t, \mu(m_i')] = \mu(s_0'), \\ [\mu(n_i'), t] = \begin{cases} \mu(s_1'), & \text{if } i \text{ is even}, \\ \mu(s_2'), & \text{if } i \text{ is odd}; \end{cases} \end{cases} \tag{3.4'}$$

$$\begin{cases} [t, \mu(n_i')] = \mu(s_0'), \\ [\mu(m_i'), t] = \begin{cases} \mu(s_1'), & \text{if } i \text{ is even}, \\ \mu(s_2'), & \text{if } i \text{ is odd}. \end{cases} \end{cases} \tag{3.5'}$$

By Lemma 3,3, the solutions to these systems are of the form $\mu(n_{i+1}') = x_{i+1} c_{i+11}$, $\mu(m_{i+1}') = y_{i+1} c_{i+12}$ $(c_{i+11}, c_{i+12} \in [G, G])$. By Lemma 3.4 the map $\widehat{\varphi}(x_0) = x_0$, $\widehat{\varphi}(y_0) = y_0$, $\widehat{\varphi}(x_i) = x_i c_{i1}$, $\widehat{\varphi}(y_i) = y_i c_{i2}$ $(i = 1, 2, \ldots)$ extends to an automorphism of the group G. We define a general recursive function f(x), satisfying the condition $\varphi\nu(n) = \mu f(n)$, by the rule $f(s_i) = s_i'$ $(i = 0, 1, 2)$, $f(r_i) = r_i'$ $(i = 1, 2)$, $f(n_i) = n_i'$, $f(m_i) = m_i'$ $(i = 1, 2, \ldots)$. Furthermore, if $\nu(n) = \prod_{j=1}^{s} \left(\prod_{i=0}^{k_j} x_i^{\varepsilon_{ij}} \prod_{i=0}^{l_j} y^{\delta_{ij}} \right)$, then we put

$$f(n) = \mu^{-1} \left(\prod_{j=1}^{s} \left(\prod_{i=0}^{k_j} (\mu(n_i'))^{\varepsilon_{ij}} \prod_{i=0}^{l_j} (\mu(m_i'))^{\delta_{ij}} \right) \right).$$

THEOREM 3.3. There exists a torsion-free two-step nilpotent group G such that G/[G, G] and [G, G] are of infinite rank and $\dim_A G = 1$.

Observe that the groups G/[G, G] and [G, G] are torsion free Abelian; i.e., $\dim_A (G/[G, G])=\infty$, $\dim_A [G, G]=\infty$, and the group G is infinitely generated over G/[G, G]. So, we have the following theorem.

THEOREM 3.3'. There exist groups G and $\hat{G}$ such that

a) G and $\hat{G}$ are torsion free;

b) G is Abelian and $\dim_A G = \infty$,

c) $\hat{G}$ is a two-step nilpotent group, infinitely generated over G,

d) $\dim_A \hat{G} = 1$.

§4. We will find some necessary conditions for constructible nilpotent groups to be self-stable.

Definition 4.1. Let G be nilpotent group, and $G=G_1>G_2>\ldots>G_n>G_{n+1}=\{e\}$ its lower central series. We will say that G is strongly torsion free if G, G_i/G_{i+1} (i = 1, 2,...,n) are torsion-free groups.

LEMMA 4.1. Let G be a group satisfying the conditions of Definition 4.1. If the quotient G_1/G_2 is of finite rank, then so is every quotient G_i/G_{i+1} (i = 2, 3,...,n).

Proof. The proof will be carried out by induction on the nilpotency degree n.

(a) n = 2. In this case, the lower central series of G is of the form G > [G, G] > {e}. Let

$$g_1[G, G],\ g_2[G, G],\ \ldots,\ g_s[G, G] \tag{4.1}$$

be a basis for G/[G, G]. We will show that the elements $[g_i, g_j]$ ($i < j \in \{1, 2,\ldots,s\}$) form a basis for the group [G, G]. The group [G, G] is generated by the elements $[x, y]$, $x, y \in G$.

Since the elements displayed in (4.1) form a basis for G/[G, G], we have

$$x^{\xi}[G, G]=g_1^{\sigma_1}g_2^{\sigma_2}\ldots g_s^{\sigma_s}[G, G], \tag{4.2}$$

$$y^{\eta}[G, G]=g_1^{\delta_1}g_2^{\delta_2}\ldots g_s^{\delta_s}[G, G]. \tag{4.3}$$

whence

$$[x^{\xi}, y^{\eta}]=[x, y]^{\xi\eta}=\prod_{i,j=1}^{s}[g_i, g_j]^{\sigma_i\delta_j}. \tag{4.4}$$

Let $g\in[G, G]$. Then

$$g=[x_1, y_1]^{r_1}[x_2, y_2]^{r_2}\ldots[x_t, y_t]^{r_t}. \tag{4.5}$$

For each commutator $[x_l, y_l]$ $(1\leqslant l\leqslant t)$ we find, as in (4.4), that

$$[x_l, y_l]^{\xi_l\eta_l}=\prod_{i_l,j_l=1}^{s}[g_{i_l}, g_{j_l}]^{\sigma_{i_l}\delta_{i_l}}, \tag{4.4$_\ell$}$$

Let m be the least common multiple of the number $\xi_l\eta_l$ $(1\leqslant l\leqslant t)$. When we raise both sides of (4.5) to the power m and perform some obvious manipulations, we get

$$g^m=\prod_{i,j=1}^{s}[g_i, g_j]^{\lambda_{ij}}.$$

This equality shows that the set of elements $[g_i, g_j]$ ($i < j \in \{1, 2,\ldots,s\}$) constitute a basis for [G, G].

(b) Suppose that for all groups of nilpotency degree less than n the lemma has been proved, and let G be a group of nilpotency degree n, satisfying the conditions of the lemma. Then G/G_n is a nilpotent group of degree no greater than n − 1. The lower central series of G/G_n looks as follows:

$$G/G_n=G_1/G_n>G_2/G_n>\ldots>G_{n-1}/G_n>G_n/G_n=\{e\}.$$

Each quotient in this series is torsion free, since

$$(G_i/G_n)/(G_{i+1}/G_n)\cong G_i/G_{i+1}\ \ (i=1,\ 2,\ \ldots,\ n-1). \tag{4.6}$$

The first of these factors is isomorphic to G_1/G_2, which, by hypothesis, is an Abelian group of finite rank. Therefore, all the conditions of the lemma are met by the group G/G_n, and it follows from the induction hypothesis that all its quotients are of finite rank. In light of the isomorphisms (4.6), we conclude that the quotients

$$G_2/G_3,\ G_3/G_4,\ \ldots,\ G_{n-1}/G_n$$

are of finite rank. It only remains to note that the group G_n also has this property, which can be proved in a similar way to part (a).

LEMMA 4.2. If a group G satisfies the conditions of Definition 4.1, and $g_{i1}G_{i+1}, g_{i2}G_{i+1}, \ldots, g_{is_i}G_{i+1}$ is a basis for the group G_i/G_{i+1} ($i = 1, 2, \ldots, n$), then for each element $g \in G$ we can find a unique sequence

$$\langle l_1, r_{11}, r_{12}, \ldots, r_{1s_1}; l_2, r_{21}, r_{22}, \ldots, r_{2s_2}; l_n, r_{n1}, r_{n2}, \ldots, r_{ns_n}\rangle, \tag{4.7}$$

with

$$\begin{cases} g^{l_1} = g_{11}^{r_{11}} g_{12}^{r_{12}} \ldots g_{1s_1}^{r_{1s_1}} c_2, & c_2 \in G_2, \\ c_2^{l_2} = g_{21}^{r_{21}} g_{22}^{r_{22}} \ldots g_{2s_2}^{r_{2s_2}} c_3, & c_3 \in G_3, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ c_{n-1}^{l_{n-1}} = g_{n-11}^{r_{n-11}} g_{n-12}^{r_{n-12}} \ldots g_{n-1s_{n-1}}^{r_{n-1s_{n-1}}} c_n, & c_n \in G_n, \\ c_n^{l_n} = g_{n1}^{r_{n1}} g_{n2}^{r_{n2}} \ldots g_{ns_n}^{r_{ns_n}}. & \end{cases} \tag{4.8}$$

Proof. According to Lemma 4.2, the system of equations (4.8) exists and is uniquely defined; the sequence (4.7) is derived from this system.

THEOREM 4.1. If G is a constructible nilpotent group, with G_1/G_2 of finite rank, and G strongly torsion free, then G is self-stable.

Proof. Let ν be some fixed constructivization of the group G, with $\nu(n_{ij}) = g_{ij}$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, s_i$), where $g_{i1}G_{i+1}, g_{i2}G_{i+1}, \ldots, g_{is_i}G_{i+1}$ is a basis for the group G_i/G_{i+1} ($i = 1, 2, \ldots, n$).

Let μ be any other constructivization of the group G, with $\mu(m_{ij}) = g_{ij}$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, s_i$). We define as follows a general recursive function $f(x)$, satisfying the condition $\varphi\nu(n) = \mu f(n)$, where φ is an automorphism of the group G. We put $f(n_{ij}) = m_{ij}$ ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, s_i$). Let $n \in \mathbf{N}$. According to Lemma 4.2, there is a unique sequence of type (4.7) associated with the element $\nu(n)$. Note that, in light of the existence and uniqueness of the system of equations (4.8), this sequence can be found effectively in terms of the number n of the element $\nu(n)$ and the number n_{ij} of g_{ij} ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, s_i$). Having defined the sequence of type (4.7) corresponding to $\nu(n)$, we consider the system

$$\begin{cases} (\mu(x))^{l_1} = (\mu(m_{11}))^{r_{11}} (\mu(m_{12}))^{r_{12}} \ldots (\mu(m_{1s_1}))^{r_{1s_1}} \mu(x_2), \\ (\mu(x_2))^{l_2} = (\mu(m_{21}))^{r_{21}} (\mu(m_{22}))^{r_{22}} \ldots (\mu(m_{2s_2}))^{r_{2s_2}} \mu(x_3), \\ \quad\ldots \qquad\qquad\qquad \ldots \\ (\mu(x_{n-1}))^{l_{n-1}} = (\mu(m_{n-11}))^{r_{n-11}} (\mu(m_{n-12}))^{r_{n-12}} \ldots \\ \ldots (\mu(m_{n-1s_{n-1}}))^{r_{n-1s_{n-1}}} \mu(x_n), \\ (\mu(x_n))^{l_n} = (\mu(m_{n1}))^{r_{n1}} (\mu(m_{n2}))^{r_{n2}} \ldots (\mu(m_{ns}))^{r_{ns_n}}. \end{cases} \tag{4.8'}$$

By Lemma 4.2, the elements $\mu(x_n), \mu(x_{n-1}), \ldots, \mu(x_2), \mu(x)$ exist, and, since G is a torsion-free nilpotent group, they are uniquely determined by (4.8'). From (4.8') we can effectively find the μ-number m of an element x. We then put $f(n) = m$.

COROLLARY 4.1. If G is a constructible full nilpotent group of degree 2, with G and [G, G] torsion free and G/[G, G] of finite rank, then G is self-stable.

Proof. All the conditions of Theorem 4.1 will be fulfilled if we can establish that G/[G, G] is torsion free. Let $g \in G$, $g^n \in [G, G]$ $(n > 1)$, $g^n = [x_1, y_1]^{\varepsilon_1}[x_2, y_2]^{\varepsilon_2} \ldots [x_k, y_k]^{\varepsilon_k}$. If z_i satisfies the equation $z_i^n = x_i$ ($i = 1, 2, \ldots, k$), then

$$g^n = [z_1^n, y_1]^{\varepsilon_1}[z_2^n, y_2]^{\varepsilon_2} \ldots [z_k^n, y_k]^{\varepsilon_k}. \tag{4.9}$$

Since G is a torsion-free two-step nilpotent group, we deduce from (4.9) that

$$g^n = ([z_1, y_1]^{\varepsilon_1}[z_2, y_2]^{\varepsilon_2} \ldots [z_k, y_k]^{\varepsilon_k})^n,$$

i.e., $g=[z_1, y_1]^{\varepsilon_1}[z_2, y_2]^{\varepsilon_2}\dots[z_k, y_k]^{\varepsilon_k}$. This shows at the same time that if $g^n\in[G, G]$ then $g\in[G, G]$. Thus, G/[G, G] is torsion free.

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PLESIOCOMPACT HOMOGENEOUS SPACES

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One can consider this paper as a continuation of [1, 2]. Some results on the study of compact homogeneous spaces with the help of the concept of decomposition of Lie groups (considered in detail in [1]) are given in [2, 3]. However the methods used turned out to be applicable also (after some refinement) to some classes of not necessarily compact homogeneous spaces. One such class, the plesiocompact homogeneous spaces, is studied in this paper. In particular, all compact homogeneous spaces and also all homogeneous spaces having a finite invariant measure are plesiocompact.

In Sec. 1 we introduce and study the concept of plesiocompact homogeneous space and also the concept of plesiouniform subgroup of a Lie group which is connected with it. Some properties of subgroups equivalent with plesiouniformity and also properties of plesiocompact homogeneous spaces are considered, approximating them with compact homogeneous spaces.

In Sec. 2 we consider regular transitive actions of Lie groups (an action of a Lie group G on a homogeneous space M = G/H of it is called regular, if $N_G(H_0)S = G$, where S is a Levy subgroup of the Lie group G).

In Sec. 3 we consider some modifications of transitive actions of Lie groups, i.e., transitions to actions whose properties are in some sense simpler than the properties of the original actions. The basic result of Sec. 3 is Theorem 3.3.

In the present paper we give only the foundations of the general theory of plesiocompact homogeneous spaces. By analogy with the case of compact homogeneous spaces (for which, cf., e.g., [2, 3]) subsequent study of plesiocompact homogeneous spaces should contain, in particular, a construction of natural and structural fibrations, the isolation and study of separate classes of plesiocompact homogeneous spaces.

We shall denote Lie groups by upper case Latin letters, their Lie algebras by the corresponding lower case German ones. If G is a Lie group, then G_0 is its connected component of the identity, and $\pi_0(G) = (G/G_0)$ is its group of connected components. We denote by Z(G) the center of the Lie group G, by $\mathscr{L}(G)$ (if G is connected) its linearizer (i.e., the smallest normal subgroup for which the quotient group is linear), by $N_G(H)$ the normalizer of the subgroup H in the group G. The Levy decomposition of a connected Lie group G is usually written as follows: G = SR, where R is the radical and S is the Levy subgroup (i.e., a maximal semisimple Lie subgroup of G). By $M' \to M \to M''$ we denote a smooth locally trivial fibration of the smooth manifold M over the base M" with fiber M'.

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