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ONCE MORE ON THE FUNCTION $\sigma_A(M)$

A. Laurinčikas

Let, as usual,

$$\sigma_a(m) = \sum_{d|m} d^a,$$

and $\delta = \delta_T > 0,\ \delta_T \to 0$, as $T \to \infty$. In studying the remainder term of the mean square of the Riemann zeta-function $\zeta(s),\ s = \sigma + it$, near the critical line $\sigma = 1/2$

$$\int_0^T \left|\zeta(1/2 + \delta_T + it)\right|^2 dt, \qquad T \longrightarrow \infty, \tag{1}$$

it is useful to have a formula with an explicit remainder term for the sum

$$D_0(x, \delta_T) \stackrel{\text{def}}{=} \sum_{m \leqslant x} \sigma_{-\delta_T}(m).$$

Here x can be dependent on T. The mean values of the function $\sigma_a(m)$ have been studied by many authors. We note a rather complicated paper [1] where a bibliography on the identities for the sum

$$\sum_{m \leqslant x} \sigma_a(m)$$

also can be found. In the papers [2–5] formulas for the sums of coefficients are obtained for a wide class of Dirichlet series.

Assume $q > 0,\ 0 < \delta < 1/2,\ \Gamma(s)$ denotes the Euler gamma function and

$$D_{q-1}(x, \delta) = \frac{1}{\Gamma(q)} \sum_{m \leqslant x} (x - m)^{q-1} \sigma_{-\delta}(m).$$

In view of the inaccessibility of [1] and further research applications of the quantity (1), in the present note we shall give a simple proof of the identity for the sum $D_{q-1}(x, \delta)$ with $q > 1/2 + \delta$ that is based on the ideas of the paper [7] (see also [8]). Note that in [1] only integer values of q are considered. We shall also obtain an approximate formula for the sum $D_0(x, \delta)$. Moreover, we shall suppose that x is a non-integer positive number because in applications it is unimportant whether x is an integer or not.

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Let $J_\nu(z)$, $I_\nu(z)$, and $K_\nu(z)$ denote the Bessel functions, i.e.,

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{\Gamma(m+1)\Gamma(m+\nu+1)},$$
$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(z/2)^{2m}}{\Gamma(m+1)\Gamma(m+\nu+1)},$$
$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}.$$

Also, we put

$$^{+}J_\nu(z,\delta) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{\Gamma(m+1+\delta)\Gamma(m+\nu+1)},$$
$$^{+}I_\nu(z,\delta) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(z/2)^{2m}}{\Gamma(m+1+\delta)\Gamma(m+\nu+1)},$$
$$\lambda_\nu(z,\delta) = \frac{1}{2}\left(\frac{z}{2}\right)^{-\nu-\delta}\left(I_{\nu-\delta}(z) + J_{\nu-\delta}(z)\right) - \frac{1}{2}\left(\frac{z}{2}\right)^{-\nu}\left(^{+}I_\nu(z,\delta) - {}^{+}J_\nu(z,\delta)\right).$$

THEOREM 1. *For all non-integers $x > 0$ and $q > 1/2 + \delta$ the indentity*

$$D_{q-1}(x,\delta) = -\frac{1}{2}\zeta(\delta)\frac{x^{q-1}}{\Gamma(q)} + \frac{x^q\zeta(1+\delta)}{\Gamma(1+q)} + \frac{x^{q-\delta}\zeta(1-\delta)\Gamma(1-\delta)}{\Gamma(1+q-\delta)} + \frac{x^q(2\pi)^{1+\delta}}{\sin\frac{\pi\delta}{2}}\sum_{m=1}^{\infty}\sigma_\delta(m)\lambda_q\left(4\pi\sqrt{mx},\delta\right)$$

holds.

Let

$$\tilde{\lambda}_1(z,\delta) = J_{-1+\delta}(z) + J_{1-\delta}(z) - \frac{2}{\pi}\sin(\pi\delta)K_{1-\delta}(z),$$
$$\Delta_{-\delta}(x) = D_0(x,\delta) + \frac{1}{2}\zeta(\delta) - x\zeta(1+\delta) - \frac{x^{1-\delta}\zeta(1-\delta)}{1-\delta}.$$

From Theorem 1, when $q = 1$, we obtain the following assertion.

COROLLARY. *For all non-integers $x > 0$ the identity*

$$\Delta_{-\delta}(x) = \frac{x(2\pi)^{1+\delta}}{2\sin\frac{\pi\delta}{2}}\sum_{m=1}^{\infty}\sigma_\delta(m)\left(2\pi\sqrt{mx}\right)^{-1-\delta}\tilde{\lambda}_1\left(4\pi\sqrt{mx},\delta\right)$$

is valid.

In some cases it is useful to have an approximate formula for $D_0(x,\delta)$ with a finite sum of terms $\tilde{\lambda}_1(4\pi\sqrt{mx},\delta)$. Denote by B a function (not always the same) which is bounded by a constant.

THEOREM 2. *Let N be a natural number. Then for every $\varepsilon > 0$*

$$\Delta_{-\delta}(x) = \frac{x(2\pi)^{1+\delta}}{\sin\frac{\pi\delta}{2}}\sum_{m\leqslant N}\sigma_\delta(m)\left(2\pi\sqrt{mx}\right)^{-1-\delta}\tilde{\lambda}_1\left(4\pi\sqrt{mx},\delta\right) + Bx^\varepsilon + Bx^{\frac{1}{2}+\varepsilon}N^{-\frac{1}{2}} + Bx^{-\frac{\delta}{2}}N^{\frac{\delta}{2}+\varepsilon}.$$

COROLLARY. *For every* $\varepsilon > 0$ *the estimate*

$$\Delta_{-\delta}(x) = Bx^{\frac{1}{3}-\frac{\delta}{6}+\varepsilon}$$

is valid.

In the proof of Theorem 2 we shall use the following two propositions.

LEMMA 1. *Let* $c > 0$, $q > 0$, *and the Dirichlet series*

$$A(s) = \sum_{m=1}^{\infty} a_m m^{-s}$$

absolutely converge for $\sigma = c$. *Then for all non-integers* $x > 1$

$$\frac{1}{\Gamma(q+1)} \sum_{m \leqslant x} a_m (x-m)^q = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{A(s)\Gamma(s)x^{s+q}}{\Gamma(s+q+1)}\, ds.$$

The equation of Lemma 1 is a variant of the inversion formula for Dirichlet's series. Its proof can be found, for example, in [8, p. 487].

LEMMA 2. *Let* $|t| \to \infty$. *Then, uniformly in* σ *on any finite interval,*

$$\left|\Gamma(\sigma+it)\right| = e^{-\frac{1}{2}|t|}|t|^{\sigma-\frac{1}{2}}\sqrt{2\pi}\left(1+o(1)\right).$$

The assertion of Lemma 2 is a consequence of the well-known Stirling formula.
Let

$$f(s) = \frac{x^{s+q-1}}{\Gamma(s+\delta)\Gamma(s+q)\cos\frac{\pi s}{2}\cos\frac{\pi(s+\delta)}{2}}.$$

LEMMA 3. *Let* $\delta < c < 1/2$, $q > 3-\delta$, *and* $x > 0$. *Then*

$$J := \frac{1}{2\pi i} \int_{-c-i\infty}^{-c+i\infty} f(s)\, ds = \frac{2x^q}{\pi \sin\frac{\pi\delta}{2}} \lambda_q\left(2\sqrt{x}, \delta\right).$$

Proof. The function $f(s)$ has simple poles at the points

$$s = 2k+1 \quad \text{and} \quad s = 2k+1-\delta, \qquad k = 0, 1, 2, \dots .$$

If $z = s - (2k+1) \to 0$, then

$$\cos\frac{\pi s}{2} = \cos\left(\frac{(2k+1)\pi}{2} + \frac{\pi z}{2}\right) = (-1)^{k-1}\sin\frac{\pi z}{2} = (-1)^{k-1}\frac{\pi z}{2}\left(1+o(1)\right). \tag{2}$$

Similarly, if $w = s - (2k+1-\delta) \to 0$, then

$$\cos\frac{\pi(s+\delta)}{2} = (-1)^{k-1}\frac{\pi w}{2}\left(1+o(1)\right). \tag{3}$$

Let $L_1 = \{s\colon \ \sigma = -c, \ |t| \leqslant R\}$, $L_2 = \{s\colon \ s = -c + Re^{i\varphi}, \ |\varphi| \leqslant \pi/2\}$. In virtue of Lemma 2, the Stirling formula, and the well-known properties of the function cos s,

$$\lim_{R\to\infty} \int\limits_{L_2} f(s)\,ds = 0.$$

Consequently,

$$\int\limits_{-c-i\infty}^{-c+i\infty} f(s)\,ds = -\lim_{R\to\infty} \int\limits_{L_1\cup L_2} f(s)\,ds.$$

Since

$$\cos\frac{\pi(2k+1+\delta)}{2} = (-1)^{k-1}\frac{\sin\pi\delta}{2},$$
$$\cos\frac{\pi(2k+1-\delta)}{2} = (-1)^{k}\frac{\sin\pi\delta}{2},$$

we have, by the residue theorem and the Eqs. (2) and (3),

$$\begin{aligned} J = &-\frac{2}{\pi\sin\frac{\pi\delta}{2}}\sum_{k=0}^{\infty}\frac{x^{2k+q}}{\Gamma(2k+1+\delta)\Gamma(2k+q+1)} \\ &+\frac{2}{\pi\sin\frac{\pi\delta}{2}}\sum_{k=0}^{\infty}\frac{x^{2k+q-\delta}}{\Gamma(2k+1)\Gamma(2k+q+1-\delta)}. \end{aligned} \tag{4}$$

Since

$$\sum_{k=0}^{\infty}\frac{(\sqrt{x})^{4k-2\delta}}{\Gamma(2k+1)\Gamma(2k+q+1-\delta)} = \frac{1}{2}(\sqrt{x})^{-q-\delta}\big(J_{q-\delta}(2\sqrt{x}) + I_{q-\delta}(2\sqrt{x})\big),$$
$$\sum_{k=0}^{\infty}\frac{(\sqrt{x})^{4q}}{\Gamma(2k+1+\delta)\Gamma(2k+q+1)} = \frac{1}{2}(\sqrt{x})^{-q-\delta}\big({}^{+}J_{q}(2\sqrt{x}) + {}^{+}I_{q}(2\sqrt{x})\big),$$

Lemma 3 follows from (4).

LEMMA 4. *Let $q > 3-\delta$. Then the assertion of Theorem 1 is valid.*

Proof. It is well known that for $\sigma > \max(1, \operatorname{Re} a + 1)$

$$\zeta(s)\zeta(s-a) = \sum_{m=1}^{\infty}\frac{\sigma_a(m)}{m^s}. \tag{5}$$

Consequently, it follows from Lemma 1 that for $c > 1$

$$D_{q-1}(x,\delta) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{\zeta(s)\zeta(s+\delta)\Gamma(s)x^{s+q-1}\,ds}{\Gamma(s+q)}. \tag{6}$$

Let $\delta < b < 1/2$. Then, in virtue of Lemma 2, the estimate

$$\Gamma(s)\Gamma^{-1}(s+q) = B|t|^{-q} \tag{7}$$

is valid for sufficiently large $|t|$ in the strip $-b \leqslant \sigma \leqslant c$. Furthermore, the following estimates for the Riemann zeta-function are known to hold for large $|t|$:

$$\begin{aligned} \zeta(s) &= B|t|^{\frac{1}{2}+b}\log|t|, \qquad -b \leqslant \sigma \leqslant 0,\\ \zeta(s) &= B|t|^{\frac{1}{2}}\log|t|, \qquad 0 \leqslant \sigma \leqslant 1,\\ \zeta(s) &= B\log|t|, \qquad \sigma > 1,\\ \zeta(s+\delta) &= B|t|^{\frac{1}{2}+b-\delta}\log|t|, \qquad -b \leqslant \sigma. \end{aligned} \tag{8}$$

Since $q > 3-\delta$, in virtue of estimates (7) and (8), the integrand in the formula (6) is estimated as $B|t|^{-1-\varepsilon}$, $\varepsilon > 0$, in the strip $-b \leqslant \sigma \leqslant c$ for large $|t|$. This integrand has simple poles at $s=0$, $s=1$, and $s=1-\delta$. Taking into account the equality $\zeta(0) = -1/2$, we obtain, by the residue theorem,

$$\begin{aligned} D_{q-1}(x,\delta) = &-\frac{x^{q-1}\zeta(\delta)}{2\Gamma(q)} + \frac{x^q\zeta(1+\delta)}{\Gamma(1+\delta)} + \frac{x^{q-\delta}\zeta(1-\delta)\Gamma(1-\delta)}{\Gamma(1+q-\delta)}\\ &+\frac{1}{2\pi i}\int_{-b-i\infty}^{-b+i\infty}\zeta(s)\zeta(s+\delta)\frac{\Gamma(s)x^{s+q-1}}{\Gamma(s+q)}\,ds. \end{aligned} \tag{9}$$

By means of the functional equation for the Riemann zeta-function

$$\zeta(s) = \frac{(2\pi)^s}{2\Gamma(s)\cos\frac{\pi s}{2}}\zeta(1-s) = \chi(s)\zeta(1-s) \tag{10}$$

and the formula (5), we find that for $\sigma = -b$

$$\zeta(s)\zeta(s+\delta) = \frac{(4\pi^2)^s(2\pi)^\delta}{4\Gamma(s)\Gamma(s+\delta)\cos\frac{\pi s}{2}\cos\frac{\pi(s+\delta)}{2}}\sum_{m=1}^{\infty}\frac{\sigma_\delta(m)}{m^{1-s}}.$$

Whence, and from Eq. (9), using Lemma 3, we obtain the assertion of the lemma.

In order to prove Theorem 1 it remains to prove Lemma 4 for smaller values of q. For this aim we shall need the asymptotics of the quantity $\lambda_q(z,\delta)$.

LEMMA 5. *We have*

$$\begin{aligned} \lambda_q(z,\delta) = z^{-q-\delta-\frac{1}{2}}\sin\frac{\pi\delta}{2}\Bigg(&A_1(q,\delta)\cos\left(z-\frac{\pi(q-\delta)}{2}-\frac{\pi}{4}\right)\\ &+A_2(q,\delta)\cos\left(z+\frac{\pi(q-\delta)}{2}-\frac{\pi}{4}\right)+A_3(q,\delta)\sin\left(z-\frac{\pi(q-\delta)}{2}-\frac{\pi}{4}\right)\Bigg)\\ &+Bz^{-q-\delta-\frac{3}{2}}\sin\frac{\pi\delta}{2}+Bz^{-4}\sin\frac{\pi\delta}{2} \qquad \text{for} \quad z\to\infty. \end{aligned}$$

Here the quantities $A_j(q,\delta)$, $j=1,2,3$, are bounded for all z and q, and the constant bounding the factor B is independent of q on any finite part of the q-plane.

Proof. Let n be a sufficiently large natural number and

$$I(z) \stackrel{\text{def}}{=} -\frac{1}{2\pi i}\int_{-n+\frac{1}{2}-i\infty}^{-n+\frac{1}{2}+i\infty}\frac{(z/2)^{2s}\,ds}{\Gamma(s+1+\delta)\Gamma(s+q+1)\sin\pi s\sin\pi(s+\delta)}.$$

From Lemma 2 and the properties of the function sin s it follows that the integral $I(z)$ is the sum of the residues of the integrand at the points

$$s = k, \qquad k = 0,1,2,\dots,$$
$$s = k-\delta, \qquad k = 0,1,2,\dots,$$
$$s = -k, \qquad \begin{cases} k = 1,2,\dots,n-1 & \text{if } q \text{ is non-integer,} \\ k = 1,2,\dots,q & \text{if } q \text{ is integer.} \end{cases}$$

Consequently,

$$I_q(z) = \frac{1}{\pi \sin \pi\delta}\left(\left(\frac{z}{2}\right)^{-q} + I_q(z,\delta) - \left(\frac{z}{2}\right)^{-q-\delta} I_{q-\delta}(z) + \sum_{k=1}^{n-1} \frac{(z/2)^{-2k}}{\Gamma(-k+1+\delta)\Gamma(-k+q+1)}\right). \tag{11}$$

On the other hand, the definition of the integral $I(z)$ implies the estimate

$$I(z) = Bz^{-2n}, \qquad z \to \infty.$$

Whence, and from Eq. (11), using the supplementary formula for the Γ-function, we obtain

$$\left(\frac{z}{2}\right)^{-q} + I_q(z,\delta) - \left(\frac{z}{2}\right)^{-q-\delta} I_{q-\delta}(z) = \frac{\sin \pi\delta}{\pi} \sum_{k=1}^{n-1} \frac{(z/2)^{-2k}(-1)^k \Gamma(k-\delta)}{\Gamma(-k+q+1)} + Bz^{-2n} \sin \pi\delta. \tag{12}$$

When q is a natural number we can obtain an exact formula for the left-hand side of (12). We have

$$\sum_{k=q}^{\infty} \frac{(z/2)^{2k-2q}}{\Gamma(k+1)\Gamma(k-q+1+\delta)} - \left(\frac{z}{2}\right)^{-q-\delta} I_{q-\delta}(z)$$
$$= \left(\frac{z}{2}\right)^{-q-\delta} I_{-q+\delta}(z) - \left(\frac{z}{2}\right)^{-q-\delta} I_{q-\delta}(z) - \left(\frac{z}{2}\right)^{-2q} \sum_{k=0}^{q-1} \frac{(z/2)^{2k}}{\Gamma(k+1)\Gamma(k-q+1-\delta)}.$$

In particular, when $q = 1$, the identity

$$\begin{aligned} \left(\frac{z}{2}\right)^{-1} + I_1(z,\delta) - \left(\frac{z}{2}\right)^{-1-\delta} I_{1-\delta}(z) &= \left(\frac{z}{2}\right)^{-1-\delta} \left(I_{-1+\delta}(z) - I_{1-\delta}(z)\right) - \left(\frac{z}{2}\right)^{-2} \frac{1}{\Gamma(\delta)} \\ &= \frac{2}{\pi} \sin \pi\delta K_{1-\delta}(z) \left(\frac{z}{2}\right)^{-1-\delta} - \left(\frac{z}{2}\right)^{-2} \frac{1}{\Gamma(\delta)} \end{aligned} \tag{13}$$

follows.

Now we shall consider the integral

$$J(z) \stackrel{\text{def}}{=} -\frac{1}{2\pi i} \int_{-n+\frac{1}{2}-i\infty}^{-n+\frac{1}{2}+i\infty} \frac{(z/2)^{2s-2}\, ds}{\Gamma(s+\delta)\Gamma(q+s) \sin \pi(q+s) \sin \pi s \sin \pi(s+\delta)}.$$

First, let the numbers q and $q-\delta$ be non-integers. Then the integral $J(z)$ is equal to the sum of the residues of the integrand at its simple poles

$$s = k, \qquad k = 1,2,\dots,$$
$$s = k-\delta, \qquad k = 1,2,\dots,$$
$$s = k-q, \qquad k = 1,2,\dots,$$
$$s = -k, \qquad k = 0,1,\dots,\ n-1.$$

Reasoning as in the case of the integral $I(z)$, we find that

$$-\frac{1}{\pi\sin\pi q\sin\pi\delta}\left(\frac{z}{2}\right)^{-q}+J_q(z,\delta)+\frac{1}{\pi\sin\pi\delta\sin\pi(q-\delta)}\left(\frac{z}{2}\right)^{-q-\delta}J_{q-\delta}(z)$$
$$-\frac{1}{\pi\sin\pi q\sin\pi(q-\delta)}\left(\frac{z}{2}\right)^{-q-\delta}J_{-q+\delta}(z)$$
$$+\frac{1}{\pi\sin\pi\delta\sin\pi q}\sum_{k=0}^{n-1}\frac{(z/2)^{2k-2}(-1)^k}{\Gamma(-k+\delta)\Gamma(-k+q)}=Bz^{-2n-2}.$$

From this we find that

$$\left(\frac{z}{2}\right)^{-q}+J_q(z,\delta)-\left(\frac{z}{2}\right)^{-q-\delta}J_{q-\delta}(z)$$
$$=\left(\frac{\sin\pi q}{\sin\pi(q-\delta)}-1\right)\left(\frac{z}{2}\right)^{-q-\delta}J_{q-\delta}(z)-\frac{\sin\pi\delta}{\sin\pi(q-\delta)}\left(\frac{z}{2}\right)^{-q-\delta}J_{-q+\delta}(z)$$
$$+\frac{\sin\pi\delta}{\pi}\sum_{k=1}^{n}\frac{(z/2)^{-2k}\Gamma(k-\delta)}{\Gamma(-k+q+1)}+\frac{B\sin\pi\delta|\sin\pi q|}{z^{2n+2}}. \tag{14}$$

It is easy to see that Eq. (14) also remains true when q is an integer.

Let m be an integer. Below we shall need the Bessel function $Y_m(z)$ that is defined as

$$Y_m(z)=\frac{1}{\pi}\left(\frac{\partial J_\nu(z)}{\partial\nu}-(-1)^m\frac{\partial J_{-\nu}(z)}{\partial\nu}\right)\bigg|_{\nu=m}.$$

If $q-\delta$ is an integer, then taking into account continuity and using L'Hospital's rule, we deduce from Eq. (14) that

$$\left(\frac{z}{2}\right)^{-q}+J(z,\delta)-\left(\frac{z}{2}\right)^{-q-\delta}J_{q-\delta}(z)$$
$$=\left(\frac{z}{2}\right)^{-q-\delta}(\cos\pi\delta-1)J_{q-\delta}(z)+\left(\frac{z}{2}\right)^{-q-\delta}Y_{q-\delta}(z)$$
$$+\frac{\sin\pi\delta}{\pi}\sum_{k=1}^{n}\frac{(z/2)^{-2k}\Gamma(k-\delta)}{\Gamma(-k+q+1)}+\frac{B\sin\pi\delta|\sin\pi q|}{z^{2n+2}}. \tag{15}$$

It is well known (see [9]) that as $z\to\infty$

$$J_\nu(z)=\frac{C_1(\nu)}{\sqrt{z}}\cos\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)+Bz^{-3/2},$$
$$J_{-\nu}(z)=\frac{C_2(\nu)}{\sqrt{z}}\cos\left(z+\frac{\pi\nu}{2}-\frac{\pi}{4}\right)+Bz^{-3/2},$$
$$Y_\nu(z)=\frac{C_3(\nu)}{\sqrt{z}}\sin\left(z-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)+Bz^{-3/2}.$$

Here $C_1(\nu)$, $C_2(\nu)$, and $C_3(\nu)$ are bounded for all values of z and ν, and a constant, bounding the factor B, is independent of ν on any finite region of the ν-plane.

LEMMA 6. *Let $q>3/2+\delta$. Then the assertion of Theorem 1 is valid.*

Proof. If $q>3/2+\delta$, it follows from Lemma 5 that the series in the formula for $D_{q-1}(x,\delta)$ converges absolutely and uniformly with respect to x on any closed interval which does not contain 0. Moreover, for a fixed x, the convergence is uniform with respect to q on any finite region of the half-plane $\operatorname{Re}q\geqslant 3/2+\delta+\varepsilon$, $\varepsilon>0$. Consequently, both sides of the equation of Theorem 1 are analytic functions of q in such a half-plane. Thus, the assertion of the lemma follows from Lemma 4 by analytic continuation.

Lemma 7. *Let $0 < \sigma_0 \leqslant \sigma < 2$. Then for all $y > 1$*

$$\zeta(\sigma) = \sum_{m \leqslant y} \frac{1}{m^\sigma} + \frac{y^{1-\sigma}}{\sigma - 1} + By^{-\sigma}.$$

A constant bounding the factor B depends only on σ_0.

The proof of the lemma can be found, for example, in [10].

Lemma 8. *Let $x > 1$. Then*

$$\sum_{m \leqslant x} \sigma_{-\delta}(m) = \frac{x^{1-\delta}\zeta(1-\delta)}{1-\delta} + x\zeta(1+\delta) + Bx^{\frac{1}{2}-\frac{\delta}{2}}.$$

Proof. It is easy to see that

$$\begin{aligned}\sum_{m \leqslant x} \sigma_{-\delta}(m) &= \sum_{mn \leqslant x} \sum m^{-\delta} = \sum_{n \leqslant \sqrt{x}} \sum_{m \leqslant \sqrt{x}} m^{-\delta} \\ &+ \sum_{m \leqslant \sqrt{x}} m^{-\delta} \sum_{\sqrt{x} < n \leqslant x/m} 1 + \sum_{n \leqslant \sqrt{x}} \sum_{\sqrt{x} < m \leqslant x/m} m^{-\delta} \stackrel{\text{def}}{=} S_1 + S_2 + S_3.\end{aligned} \tag{16}$$

Let $[u]$ denote the fractional part of u. Then we have

$$S_1 = [\sqrt{x}] \sum_{m \leqslant \sqrt{x}} m^{-\delta},$$

$$S_2 = \sum_{m \leqslant \sqrt{x}} m^{-\delta}\left(\left[\frac{x}{m}\right] - [\sqrt{x}]\right),$$

and consequently, in virtue of Lemma 7,

$$\begin{aligned}S_1 + S_2 &= \sum_{m \leqslant \sqrt{x}} m^{-\delta}\left[\frac{x}{m}\right] = x \sum_{m \leqslant \sqrt{x}} \frac{1}{m^{1+\delta}} + B \sum_{m \leqslant \sqrt{x}} m^{-\delta} \\ &= x\left(\zeta(1+\delta) - \frac{(\sqrt{x})^{-\delta}}{\delta} + Bx^{-\frac{1}{2}-\frac{\delta}{2}}\right) + Bx^{\frac{1}{2}-\frac{\delta}{2}}.\end{aligned} \tag{17}$$

Since

$$\sum_{m \leqslant x} m^{-\delta} = \frac{x^{1-\delta}}{1-\delta} + A + Bx^{-\delta},$$

where A is some constant, we obtain, using Lemma 7,

$$\begin{aligned}S_3 &= \sum_{n \leqslant \sqrt{x}} \left(\left(\frac{x}{n}\right)^{1-\delta} \frac{1}{1-\delta} - (\sqrt{x})^{1-\delta} \frac{1}{1-\delta} + Bx^{-\delta}n^\delta + Bx^{-\frac{\delta}{2}}\right) \\ &= \frac{x^{1-\delta}}{1-\delta}\zeta(1-\delta) + \frac{x^{1-\frac{\delta}{2}}}{\delta} + Bx^{\frac{1}{2}-\frac{\delta}{2}}.\end{aligned}$$

Whence, and from (16) and (17), the assertion of the lemma follows easily.

Lemma 9. *We have*

$$\lambda'_\nu(z,\delta) = -\nu\left(\frac{z}{2}\right)^{-1}\lambda_\nu(z,\delta) + \left(\frac{z}{2}\right)^{-1}\lambda_{\nu-1}(z,\delta).$$

Proof. It is known [9] that

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{2}\left(\frac{z}{2}\right)^{-1} J_\nu(z), \tag{18}$$

$$I'_\nu(z) = I_{\nu-1}(z) - \frac{\nu}{2}\left(\frac{z}{2}\right)^{-1} J_\nu(z). \tag{19}$$

In a similar manner we find also that

$$^+J'_\nu(z,\delta) = {}^+J_{\nu-1}(z,\delta) - \frac{\nu}{2}\left(\frac{z}{2}\right)^{-1} {}^+J_\nu(z,\delta), \tag{20}$$

$$^+I'_\nu(z,\delta) = {}^+I_{\nu-1}(z,\delta) - \frac{\nu}{2}\left(\frac{z}{2}\right)^{-1} {}^+I_\nu(z,\delta). \tag{21}$$

From Eqs. (18)–(21) and from the definition of the quantity $\lambda_\nu(z,\delta)$ we deduce the equality of the lemma.

LEMMA 10. *We have*

$$\lambda''_\nu(z,\delta) = \left(\frac{\nu}{2}+\nu^2\right)\left(\frac{z}{2}\right)^{-2}\lambda_\nu(z,\delta) + \left(\frac{1}{2}-2\nu\right)\left(\frac{z}{2}\right)^{-2}\lambda_{\nu-1}(z,\delta) + \left(\frac{z}{2}\right)^{-2}\lambda_{\nu-2}(z,\delta).$$

Lemma 10 follows from Lemma 9.

Proof of Theorem 1. Let

$$\begin{aligned} r_0(x) &= \Delta_{-\delta}(x), \\ r_1(x) &= \int_0^x r_0(t)\,dt = \sum_{m\leqslant x}(x-m)\sigma_{-\delta}(m) \\ &\quad - \frac{x^2}{2}\zeta(1+\delta) - \frac{x^{2-\delta}\zeta(1-\delta)}{(2-\delta)(1-\delta)} + \frac{x}{2}\zeta(\delta). \end{aligned}$$

If M and N, $M < N$ are natural numbers and $f(x)$ is a function having a continuous second derivative in $[M,N]$, then

$$\begin{aligned} \sum_{m=M+1}^{N} f(m)\sigma_{-\delta}(m) &= \int_M^N f(t)\,dD_0(t,\delta) \\ &= \int_M^N f(t)\,dr_0(t) + \int_M^N f(t)\big(\zeta(1+\delta)+t^{-\delta}\zeta(1-\delta)\big)\,dt \\ &= f(t)r_0(t)\big|_M^N - f'(t)r_1(t)\big|_M^N + \int_M^N f''(t)r_1(t)\,dt \\ &\quad + \int_M^N f(t)\big(\zeta(1+\delta)+t^{-\delta}\zeta(1-\delta)\big)\,dt. \end{aligned} \tag{22}$$

Since $0<\delta<1/2$, the estimate

$$r_0(x) = Bx^{\frac{1}{2}-\frac{\delta}{2}} \tag{23}$$

follows from Lemma 8 for large x. Applying Lemma 6 with $q=2$ and Lemma 5, we find that for sufficiently large x

$$r_1(x) = Bx^{\frac{3}{4}-\frac{\delta}{2}}. \tag{24}$$

Now let $f(t)=t^{\delta}\lambda_q(4\pi\sqrt{xt},\delta)$. Then in virtue of the estimates (23) and (24), and Lemmas 9 and 5 we have, for large M and $x\in[x_0,X_0]$, where x_0 and X_0 are fixed positive numbers,

$$\left(f(t)r_0(t)-f'(t)r_1(t)\right)\Big|_M^N=BM^{-\frac{q}{2}+\frac{1}{4}}\sin\frac{\pi\delta}{2}, \tag{25}$$

$$\int_M^N f(t)\left(\zeta(1+\delta)+t^{-\delta}\zeta(1-\delta)\right)dt=BM^{-\frac{q}{2}+\frac{\delta}{2}+\frac{1}{4}}. \tag{26}$$

In view of Lemma 6,

$$r_1(t)=\frac{t^2(2\pi)^{1+\delta}}{\sin\frac{\pi\delta}{2}}\sum_{m=1}^{\infty}\sigma_{\delta}(m)\lambda_2(4\pi\sqrt{mt},\delta).$$

By Lemma 5 the series above converges absolutely and uniformly with respect to t on the interval $[M,N]$. Thus,

$$\int_M^N r_1(t)f''(t)\,dt=\frac{(2\pi)^{1+\delta}}{\sin\frac{\pi\delta}{2}}\sum_{m=1}^{\infty}\sigma_{\delta}(m)\int_M^N t^2\lambda_2(4\pi\sqrt{mt},\delta)\,dt. \tag{27}$$

By Lemmas 9, 10, and 5, we find that

$$\begin{aligned}f''(t)&=x^{-\frac{q}{2}-\frac{\delta}{2}+\frac{3}{4}}t^{-\frac{q}{2}+\frac{\delta}{2}-\frac{5}{4}}\sin\frac{\pi\delta}{2}\left(C_1(t,x,q,\delta)\times\cos\left(4\pi\sqrt{tx}-\frac{\pi(q-2-\delta)}{2}-\frac{\pi}{4}\right)\right.\\&+C_2(t,x,q,\delta)\cos\left(4\pi\sqrt{tx}+\frac{\pi(q-2-\delta)}{2}-\frac{\pi}{4}\right)\\&\left.+C_3(t,x,q,\delta)\sin\left(4\pi\sqrt{tx}-\frac{\pi(q-2-\delta)}{2}-\frac{\pi}{4}\right)\right)+Bx^{-\frac{q}{2}-\frac{\delta}{2}+\frac{1}{4}}t^{-\frac{q}{2}+\frac{\delta}{2}-\frac{7}{4}}\sin\frac{\pi\delta}{2}+Bx^{-2}t^{\delta-4}\sin\frac{\pi\delta}{2}.\end{aligned}$$

Here the quantities $C_j(t,x,q,\delta)$, $j=1,2,3$, are bounded for all t, x, and q, and a constant bounding the factor B is independent of q on any finite region of the q-plane. Since x is non-integer and $0<\delta<1/2$, whence and from (22), (25)–(27), by Lemma 5, we find that if $q>1/2+\delta$, then the series

$$\sum_{m=1}^{\infty}\sigma_{\delta}(m)\lambda_q(4\pi\sqrt{mx},\delta)$$

converges uniformly with respect to $x\in[x_0,X_0]$. Moreover, for fixed x this convergence is uniform with respect to q on any finite part of the half-plane $\operatorname{Re}q\geqslant 1/2+\delta+\varepsilon$. Thus, by analytic continuation and Lemma 6, the theorem follows.

Proof of Corollary. If $q=1$, then we find, by (14),

$$\begin{aligned}&\left(\frac{z}{2}\right)^{-1}+J_1(z,\delta)-\left(\frac{z}{2}\right)^{-1-\delta}J_{1-\delta}(z)\\&\quad=-\left(\frac{z}{2}\right)^{-1-\delta}J_{1-\delta}(z)-\left(\frac{z}{2}\right)^{-1-\delta}J_{-1+\delta}(z)-\left(\frac{z}{2}\right)^{-2}\frac{1}{\Gamma(\delta)}.\end{aligned}$$

Whence, and from the formula (13), we deduce that

$$\begin{aligned}\lambda_1(z,\delta)&=\frac{1}{2}\left(\frac{z}{2}\right)^{-1-\delta}\left(J_{1-\delta}(z)+J_{-1+\delta}(z)-\frac{2}{\pi}\sin\pi\delta K_{1-\delta}(z)\right)\\&=\frac{1}{2}\left(\frac{z}{2}\right)^{-1-\delta}\tilde{\lambda}_1(z,\delta).\end{aligned}$$

Thus, it remains to use Theorem 1.

To prove Theorem 2 we shall need the following known lemma.

LEMMA 11. *Let $F(x)$ be a real differentiable function such that either $F'(x) \geqslant m > 0$ or $F'(x) \leqslant -m < 0$ for $x \in [a,b]$, and let $G(x)$ be a monotonic function for $x \in [a,b]$ such that $|G(x)| \leqslant G$. Then*

$$\left| \int_a^b G(x) e^{iF(x)}\, dx \right| \leqslant \frac{4G}{m}.$$

Proof of the lemma can be found, for example, in [8].

Proof of Theorem 2. We put $\sigma_x = 1 + 1/\log x$. By Eq. (5) we find that for $\sigma \to 1+0$

$$\zeta(\sigma)\zeta(\sigma+\delta) = \sum_{m=1}^{\infty} \frac{\sigma_{-\delta}(m)}{m^\sigma} = \frac{B}{(\sigma-1)(\sigma+\delta-1)} = \frac{B}{(\sigma-1)^2}.$$

Whence and from the estimate

$$\sigma_{-\delta}(m) = Bm^\varepsilon,$$

which is valid for every fixed $\varepsilon > 0$, repeating the proof of the Perron formula (see [11, pp. 427–428]), we find that for $U > 0$

$$\begin{aligned} \sum_{m \leqslant x} \sigma_{-\delta}(m) = \frac{1}{2\pi i} \int_{\sigma_x - iU}^{\sigma_x + iU} \zeta(s)\zeta(s+\delta)\frac{x^s}{s}\, ds \\ + \frac{Bx^{\sigma_x}}{U(\sigma_x-1)^2} + \frac{Bx^{1+\varepsilon}}{U} + Bx^\varepsilon. \end{aligned} \tag{28}$$

Let $c > 0$. We replace the contour in the integral of Eq. (28) by the new contour joining the points $\sigma_x - iU$, $-c - iU$, $c + iU$, $\sigma_x + iU$. The union of the former and the latter contours embraces the poles of the integrand at $s = 0$, $s = 1$, and $s = 1 - \delta$. Consequently, by the residue theorem, the formula (28), and the functional equation (10), taking $g(s) = \chi(s)\chi(s+\delta)(x^s/s)$, we have

$$\begin{aligned} \Delta_{-\delta}(x) &= Bx^\varepsilon + \frac{Bx^{1+\varepsilon}}{U} + BU^{2c-\delta}x^{-c} + \frac{1}{2\pi i}\int_{-c-iU}^{-c+iU} \zeta(s)\zeta(s+\delta)\frac{x^s}{s}\, ds \\ &= \frac{1}{2\pi i}\sum_{m=1}^{\infty} \sigma_\delta(m) \int_{-c-iU}^{-c+iU} \frac{g(s)\, ds}{m^{1-s}} + Bx^\varepsilon + \frac{Bx^{1+\varepsilon}}{U} + BU^{2c-\delta}x^{-c}. \end{aligned} \tag{29}$$

Let N be a natural number, $N = Bx^A$, $A > 0$, and $U^2/4\pi^2 x = N + 1/2$. Now we shall estimate the sum

$$Z_N \stackrel{\text{def}}{=} \sum_{m > N} \sigma_\delta(m) \int_{-c-iU}^{-c+iU} \frac{g(s)\, ds}{m^{1-s}}.$$

It is well known that for the function $\chi(s)$ the following asymptotic formula holds:

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{s-\frac{1}{2}} e^{i\left(t+\frac{\pi}{4}\right)}\left(1 + \frac{B}{t}\right), \qquad t \geqslant t_0 > 0.$$

Consequently,

$$\begin{aligned} \chi(s)\chi(s+\delta) = \exp\Big\{ &(2\sigma+\delta-1)\log 2\pi - (2\sigma+\delta-1)\log t + \frac{\pi i}{2} \\ &+ 2it\log 2\pi - 2it\log t + 2it \Big\}\left(1 + \frac{B}{t}\right). \end{aligned}$$

Thus,

$$\chi(s)\chi(s+\delta)x^{it}m^{it}s^{-1} = \exp\left\{(2\sigma+\delta-1)\log 2\pi + iF(t)\right\}t^{-2\sigma-\delta}\left(1+\frac{B}{t}\right),$$

where

$$F(t) = 2t\log 2\pi - 2t\log t + 2t + t\log x + t\log m.$$

We have

$$F'(t) = \log\frac{4\pi^2 xm}{t^2}.$$

Hence, applying Lemma 11, we find

$$\begin{aligned} Z_N &= Bx^{-c}\sum_{m>N}\frac{\sigma_\delta(m)}{m^{1+\varepsilon}}\left|\int_1^U t^{2c-\delta}e^{iF(t)}\,dt\right| \\ &\quad + Bx^{-c}N^{-c+\delta+\varepsilon} + Bx^{-c}U^{2c-\delta}N^{-c+\delta+\varepsilon} \\ &= Bx^{-c}U^{2c-\delta}\sum_{m>N}\frac{1}{m^{1+c-\delta-\varepsilon}\log\left(m/\left(N+\frac{1}{2}\right)\right)} \\ &\quad + Bx^{-c}N^{-c+\delta+\varepsilon} + Bx^{-\delta/2}N^{\delta/2+\varepsilon} + Bx^{-c}U^{2c-\delta}N^{-c+\delta+\varepsilon} \\ &= Bx^{-c}N^{-c+\delta+\varepsilon} + Bx^{-\delta/2}N^{\delta/2+\varepsilon}. \end{aligned} \tag{30}$$

Now from (29) and (30) it follows that

$$\begin{aligned} \Delta_{-\delta}(x) &= \frac{1}{2\pi i}\sum_{m\leqslant N}\frac{\sigma_\delta(m)}{m}\int_{-c-iU}^{-c+iU} m^s g(s)\,ds + Bx^{\varepsilon} + Bx^{1/2+\varepsilon}N^{-1/2} \\ &\quad + BN^{c-\delta/2}x^{-\delta/2} + Bx^{-c}N^{-c+\delta+\varepsilon} + Bx^{-\delta/2}N^{\delta/2+\varepsilon}. \end{aligned} \tag{31}$$

It is clear that

$$\int_{-c-iU}^{-c+iU} m^s g(s)\,ds = \int_{-i\infty}^{i\infty} m^s g(s)\,ds - \left(\int_{iU}^{i\infty} + \int_{-i\infty}^{-iU} + \int_{-iU}^{-c-iU} + \int_{-c+iU}^{iU}\right) m^s g(s)\,ds. \tag{32}$$

Using Lemma 11 again, we find that the two first integrals in brackets of Eq. (32) are estimated as

$$BU^{-\delta}\left(\log\frac{N+\frac{1}{2}}{m}\right)^{-1}.$$

Thus, the contribution of these two integrals to the right-hand side of Eq. (31) does not exceed the quantity

$$BU^{-\delta}\sum_{m\leqslant N}\frac{\sigma_\delta(m)}{m\log\left((N+1)/m\right)} = Bx^{-\delta/2}N^{\delta/2+\varepsilon}. \tag{33}$$

The contribution of the two other integrals in brackets of Eq. (32) is

$$B\sum_{m\leqslant N}\frac{\sigma_\delta(m)}{m}\int_{-c}^{0}\frac{(mx)^\sigma}{U^{2\sigma+\delta}}\,d\sigma = Bx^{-\delta/2}N^{c-\delta/2} + Bx^{-\delta/2}N^{\delta/2+\varepsilon}. \tag{34}$$

Let $b = 1 - 3\delta/2$. Then

$$\int_{-i\infty}^{i\infty} m^s g(s)\, ds = \int_{b-i\infty}^{b+i\infty} m^s g(s)\, ds. \tag{35}$$

Indeed, the intengrand is regular in the strip $0 \leqslant \sigma \leqslant b$. Moreover,

$$\lim_{U\to\infty} \left(-\int_{iU}^{b+iU} + \int_{-iU}^{b-iU} \right) m^s g(s)\, ds = 0.$$

Thus, it follows from (31)–(35) that

$$\Delta_{-\delta}(x) = \frac{1}{2\pi i} \sum_{m\leqslant N} \frac{\sigma_\delta(m)}{m} \int_{b-i\infty}^{b+i\infty} m^s g(s)\, ds + Bx^\varepsilon + Bx^{1/2+\varepsilon} N^{-1/2} \\ + BN^{c-\delta/2} x^{-\delta/2} + Bx^{-c} N^{-c+\delta+\varepsilon} + Bx^{-\delta/2} N^{\delta/2+\varepsilon}. \tag{36}$$

Repeating the proof of Lemma 3, we find

$$\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} m^s g(s)\, ds = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{(2\pi)^\delta (4\pi^2 mx)^s\, ds}{4\Gamma(s+1)\Gamma(s+\delta)\cos\frac{\pi s}{2}\cos\frac{\pi}{2}(s+\delta)} \\ = \frac{xm(2\pi)^{1+\delta}}{2\sin\frac{\pi\delta}{2}} \left(2\pi\sqrt{mx}\right)^{-1-\delta} \tilde{\lambda}_1\left(4\pi\sqrt{mx}, \delta\right).$$

Whence and from (36), choosing an appropriate number c, we obtain the assertion of the theorem.

Proof of the corollary follows from Theorem 2, taking $N = [x^{1/3+\delta/3}]$.

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