A CLASS OF INVERSE PROBLEMS FOR ELLIPTIC EQUATIONS

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UDC 517.95

In this paper we consider problems of determining the right-hand side and the coefficients of a second-order elliptic equation, not depending on one of the spatial variables, where complete Cauchy data are specified on the domain boundary. Various versions have been studied of similar formulations for parabolic equations in [1-3] and for elliptic equations in [4] (problem concerning the density in an inverse problem of potential theory), where only theorems of uniqueness were obtained in an overdetermined formulation. The reader, wishing to acquaint himself with the modern state of the theory of inverse problems, should consult [5, 6]. Preliminary results relating to this problem, as well as an existence theorem for an unknown coefficient, may be found in [7, 8]. Part of our results in this paper were given previously in [9].

In Sec. 1 estimates are established for a Schauder-type solution; to do this, the well-known method of "freezing of coefficients" [10-12] is modified in an appropriate way. In Sec. 2, under other conditions, uniqueness theorems are proved by Novikov's method [4, 13] and an application is given to the problem of recovery of a coefficient and the right-hand side. Finally, in Sec. 3, based on the results in Secs. 1 and 2, conditions are obtained for the existence of a solution in Hölder classes.

We denote by x a point $(x_1, \ldots, x_n)$ of an n-dimensional euclidean space $\mathbf{R}^n$ and by x' the projection $(x_1, \ldots, x_{n-1}, 0)$ of this point on the hyperplane $x_n = 0$; let $|u|^{k+\lambda}(\Omega)$ be the Hölder norm of function u on the open subset $\Omega \subset \mathbf{R}^n$ [11]. Here, and in what follows, λ will denote a fixed number from (0, 1). We consider a linear differential operator

$$A = -\sum_{j,k=1}^{n} a^{jk}(x)\,\partial^2/\partial x_j\partial x_k + \sum_{j=1}^{n} a^j(x)\,\partial/\partial x_j + a(x)$$

with coefficients a^{jk}, a^j, a of class $C^\lambda(\mathbf{R}^n)$, satisfying the uniform ellipticity condition

$$\varepsilon_0|\xi|^2 \leqslant \sum_{j,k=1}^{n} a^{jk}(x)\,\xi_j\xi_k \tag{1}$$

for all $\xi \in \mathbf{R}^n$ and $x \in \Omega$. By A' we will mean operator A minus the terms containing derivatives with respect to x_n.

1. ESTIMATES OF SCHAUDER TYPE

Let Ω' be a domain in $\mathbf{R}^{n-1}$ with boundary $\partial\Omega'$ of class $C^{2+\lambda}$, and let $\Omega = \Omega' \times (-H, 0)$, where H is some positive number. We fix weight functions $\rho_1, \rho_2 \in C^\lambda(\mathbf{R}^n)$ such that

$$0 < \varepsilon_0 \leqslant \det\begin{pmatrix}\rho_1(x',0) & \rho_2(x',0)\\ \rho_1(x',-H) & \rho_2(x',-H)\end{pmatrix} \text{ on } \Omega'. \tag{2}$$

In Theorem 1 we assume that the coefficients a^{jk} of operator A do not depend on x_n and that the coefficients a^{jn} for $j < n$ satisfy the following condition at corner points of the boundary Ω:

$$0 = a^{jn}(x, \tau) \text{ on } \partial\Omega' \text{ for } \tau = 0, -H. \tag{3}$$

We consider the problem of finding a triple of functions (u, q_1, q_2) satisfying the following conditions:

$$Au = \rho_1 q_1 + \rho_2 q_2 + f,\ \partial q_j/\partial x_n = 0 \text{ on } \Omega, \tag{4}$$

$$u = g \text{ on } \partial\Omega,\ \partial u/\partial x_n = h \text{ on } \Gamma_0 \cup \Gamma_H, \tag{5}$$

Samarkand. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 31, No. 4, pp. 149-159, July-August, 1990. Original article submitted December 11, 1985.

where $\Gamma_\tau = \Omega' \times \{-\tau\}$. In what follows we denote by C constants depending on Ω, the C^λ-norm of coefficients of operator A and of the functions ρ_1, ρ_2, as well as on ε_0. Dependence of C on other parameters will be noted separately.

THEOREM 1. There exists a constant C such that

$$|u|^{2+\lambda}(\Omega)+|q_1|^\lambda(\Omega)+|q_2|^\lambda(\Omega)\leqslant$$
$$\leqslant C(|f|^\lambda(\Omega)+|g|^{2+\lambda}(\partial\Omega)+|h|^{1+\lambda}(\Gamma_0\cup\Gamma_H)+|u|^0(\Omega)) \tag{6}$$

for arbitrary functions $u \in C^{2+\lambda}(\overline{\Omega})$ and q_1, $q_2 \in C^\lambda(\overline{\Omega})$, satisfying conditions (4) and (5).

The proof is based on a series of lemmas in which the following notation is used: Ω_1 denotes the layer $\{x \in \mathbf{R}^n;\ -1 < x_n < 0\}$, $B_\varepsilon(\tau)$ is the ball $\{x:\ |x - y| < \varepsilon\}$ in $\mathbf{R}^n$, where $y = (0, \ldots, 0, -\tau)$, $B_\varepsilon^-(0)$ is the lower half-ball $B_\varepsilon(0) \cap \Omega_1$, $\sigma_\varepsilon(0)$ is the $(n-1)$-dimensional ball $\partial\Omega_1 \cap B_\varepsilon(0)$, $B_\varepsilon^+(1)$ is the upper half-ball $B_\varepsilon(1) \cap \Omega_1$, and $\sigma_\varepsilon(1)$ is the $(n-1)$-dimensional ball $\partial\Omega_1 \cap B_\varepsilon(1)$. Here and elsewhere we assume that $\varepsilon < 1/4$.

LEMMA 1. There exists a constant C such that the following estimate is valid for a solution of problem (4), (5) with Ω_1 in place of Ω, $A = -\Delta$, and with ρ_1, ρ_2 constant on $B_{2\varepsilon}(0)$ and on $B_{2\varepsilon}(1)$, respectively:

$$|u|^{2+\lambda}(B_\varepsilon^-(0)\cup B_\varepsilon^+(1)) + \varepsilon^2|q_1|^\lambda(\sigma_\varepsilon(0)) + \varepsilon^2|q_2|^\lambda(\sigma_3(0))\leqslant$$
$$\leqslant C\left(\varepsilon^2|f|^\lambda(B_{2\varepsilon}^-(0)\cup B_{2\varepsilon}^+(1)) + |g|^{2+\lambda}(\sigma_{2\varepsilon}(0)\cup\sigma_{2\varepsilon}(1)) + \varepsilon|h|^{1+\lambda}(\sigma_{2\varepsilon}(0)\cup\sigma_{2\varepsilon}(1) + |u|^1(B_{2\varepsilon}^-(0)\cup B_{2\varepsilon}^+(1))\right)$$

for arbitrary functions $u \in C^{2+\lambda}(\overline{B}_{2\varepsilon}^-(0) \cup \overline{B}_{2\varepsilon}^+(1))$ and q_1, $q_2 \in C^\lambda(\sigma_{2\varepsilon}(0))$.

Proof. We denote the solution of the following problem by u_1:

$$-\Delta u_1 = f \text{ on } B_{2\varepsilon}^-(0)\cup B_{2\varepsilon}^+(1),\quad u_1 = g \text{ on } \sigma_{2\varepsilon}(0)\cup\sigma_{2\varepsilon}(1),$$

the norm of which in $C^{2+\lambda}(B_{2\varepsilon}^-(0) \cup B_{2\varepsilon}^+(1))$ is estimated in terms of norms f and g, indicated in relation (6). To construct u_1 on $B_{2\varepsilon}^-$ it is sufficient to go over to new variables $y = \varepsilon^{-1}x$, to include domain $B_{2\varepsilon}^-$ in a wider bounded domain D_3 with boundary of class $C^{2+\lambda}$ such that $\sigma_{2\varepsilon} = \partial B_{2\varepsilon}^- \cap \partial D_3$, to extend the data f and g in the new variables onto $\overline{D}_3$ with norm estimates extended on the basis of Whitney's theorems (see, for example, [14, Chap. 4, Theorems 3, 4]), to solve a Dirichlet problem in D_3 with the data f and g (an estimate of the solution is given, for example, by Theorem 1.2 and estimate (1.11) from [12, pp. 147, 148]), and to then return to the variables x. Solution u_1 on $B_{2\varepsilon}^+$ is constructed similarly.

Function $u_2 = u - u_1$ satisfies the conditions

$$-\Delta u_2 = \rho_1 q_1 + \rho_2 q_2 \text{ on } B_{2\varepsilon}^-(0)\cup B_{2\varepsilon}^+(1), u_2 = 0 \text{ on } \sigma_{2\varepsilon}(0)\cup\sigma_{2\varepsilon}(1).$$

If we put $v = \partial u_2/\partial x_n$, then

$$-\Delta v = 0 \text{ on } B_{2\varepsilon}^-(0)\cup B_{2\varepsilon}^+(1),\ v = h - \partial u_1/\partial x_n \text{ on } \sigma_{2\varepsilon}(0)\cup\sigma_{2\varepsilon}(1). \tag{7}$$

In addition, it follows from the equation for u_2 for $x_n = 0$ and $x_n = -1$ that

$$-\partial v/\partial x_n = \rho_1 q_1 + \rho_2 q_2 \text{ on } \sigma_{2\varepsilon}(0)\cup\sigma_{2\varepsilon}(1). \tag{8}$$

Using a known estimate [10, Theorem 8.2] for the solution of the problem (7), to which we apply the dilation $x \to \varepsilon x$, we obtain

$$\varepsilon|v|^{1+\lambda}(B_{3\varepsilon/2}^-(0)\cup B_{3\varepsilon/2}^+(1))\leqslant C\left(\varepsilon|h|^{1+\lambda}(\sigma_{2\varepsilon}(0)\cup\sigma_{2\varepsilon}(1)) + \right.$$
$$\left. + \varepsilon^2|f|^\lambda(B_{2\varepsilon}^-(0)\cup B_{2\varepsilon}^+(1)) + |g|^{2+\lambda}(\sigma_{2\varepsilon}(0)\cup\sigma_{2\varepsilon}(1)) + |v^0|(B_{2\varepsilon}^-(0)\cup B_{2\varepsilon}^+(1))\right). \tag{9}$$

Taking condition (8) into account, we have

$$\rho_1(x',0)q_1(x')+\rho_2(x',0)q_2(x') = -\partial v/\partial x_n(x',0),$$
$$\rho_1(x',-1)q_1(x')+\rho_2(x',-1)q_2(x') = -\partial v/\partial x_n(x',-1)$$

for $|x'| < \varepsilon$. In view of condition (2), we establish, in terms of the right-hand side of inequality (9), the estimate $|q_j|^\lambda(\sigma_{3\varepsilon/2}(0))$.

Applying a known Schauder estimate [11, p. 245] for problem (4), (5) in domains $B_{2\varepsilon}^-(0)$, $B_{2\varepsilon}^+(1)$, we arrive at the statement of Lemma 1 from the resulting estimate for q_j.

We put $\Omega_2 = \Omega_1 \cap \{0 < x_1\}$ and $\sigma_\varepsilon^+(\tau) = \sigma_\varepsilon(\tau) \cap \{0 < x_1\}$, $\tau = 0, 1$.

LEMMA 2. Lemma 1 remains valid if Ω_1, $\sigma_\varepsilon(\tau)$ and $B_\varepsilon^-(0)$, $B_\varepsilon^+(1)$ are replaced, respectively, by Ω_2, $\sigma_\varepsilon^+(\tau)$ and $B_\varepsilon^-(0) \cap \Omega_2$, $B_\varepsilon^+(1) \cap \Omega_2$.

Proof. As was done in Lemma 1 we can limit ourselves to the case $f = 0$ on Ω_2 and $g = 0$ on $\overline{\Gamma} = \partial\Omega\backslash(\Gamma_0 \cup \Gamma_H)$. From Eq. (4) and the equations $u = 0$ on Γ, $u = g$ on $\Gamma_0 \cup \Gamma_H$, we deduce that $(\rho_1 q_1 + \rho_2 q_2)(0, x_2, \ldots, x_{n-1}, -\tau) = -\Delta' g(0, x_2, \ldots, x_{n-1}, -\tau)$. As was done at the start of the proof of Lemma 1, we construct a function u_3 such that $-\Delta u_3 = -\Delta' g(0, x_2, \ldots, x_{n-1}, -\tau)$ on $\Omega_2 \cap B_{2\varepsilon}^{\pm}(\tau)$, $u_3 = 0$ on Γ, estimatable in $C^{2+\lambda}(\Omega_2 \cap B_{2\varepsilon}^{\pm}(\tau))$ in terms of the indicated norm g. We put $u_4 = u - u_3$ on $\Omega_2 \cap B_{2\varepsilon}^{\pm}(\tau)$. Then $\Delta u_4 = 0$ on $\overline{\Gamma} \cap \overline{\Gamma}_\tau$, and since Δu_4 does not depend on x_n on $B_{2\varepsilon}^-(0)$ and $B_{2\varepsilon}^+(1)$, it follows that $\Delta u_4 = 0$ on $\{x_1 = 0\} \cap (B_{2\varepsilon}^-(0) \cup B_{2\varepsilon}^+(1))$ also. It is obvious that on the last set $u_4 = 0$ also, so that on this set $\partial^2 u_4/\partial x_1^2 = 0$. Now by an odd extension with respect to x_1, we reduce an estimate of the norm of u_4 to Lemma 1. The required estimate of u then follows from estimates of u_3 and u_4. Thus we have established Lemma 2.

Proof of Theorem 1. By virtue of the conditions on domain Ω' there exists for it a covering $\{w_\varepsilon\}$ of set $\overline{\Omega}'$ such that if $w_\varepsilon \subset \Omega'$, then w_ε is a ball of radius ε in $\mathbf{R}^{n-1}$, but if the intersection of w_ε with $\partial\Omega'$ is nonempty, then w_ε is the image of a ball under a mapping, which $\sigma_\varepsilon \cap \{x_1 = 0\}$ takes into $\partial\Omega' \cap w_\varepsilon$, while the half-ball σ_ε maps into $\Omega' \cap w_\varepsilon$, where the $C^{2+\lambda}$-norms of this mapping and its inverse are bounded independently of w_ε.

Suppose initially that $w_\varepsilon \subset \Omega'$. We introduce a function $\varphi_\varepsilon \in C_0^\infty(w_\varepsilon)$, equal to one in a ball of half the radius and the same center as w_ε, $0 \le \varphi_\varepsilon \le 1$. We extend φ_ε to be constant with respect to x_n. We define $u_\varepsilon = u\varphi_\varepsilon$ and $q_{j\varepsilon} = q_j\varphi_\varepsilon$. Let A_ε be the principal part of operator A, the coefficients of which have been calculated at some fixed point $(x', 0)$, where $x' \in w_\varepsilon$. Let $A_{\varepsilon 0} = A - A_\varepsilon$. Let $\rho_{j\varepsilon}(-\tau) = \rho_j(x', -\tau)$, and let us introduce functions $\rho_{j\varepsilon}$, equal to $\rho_{j\varepsilon}(0)$ on $\Omega' \times (-2\varepsilon, 0)$ and to $\rho_{j\varepsilon}(-1)$ on $\Omega' \times (-1, -1 + 2\varepsilon)$, the norms of which on $C^\lambda(\mathbf{R}^n)$ are estimated in terms of the norms of ρ_j in $C^\lambda(\Omega)$. Let $\rho_{j\varepsilon 0} = \rho_j - \rho_{j\varepsilon}$. From relations (4) and (5) we have

$$A_\varepsilon u_\varepsilon = \rho_{1\varepsilon} q_{1\varepsilon} + \rho_{2\varepsilon} q_{2\varepsilon} + (\rho_{1\varepsilon 0} q_{1\varepsilon} + \rho_{2\varepsilon 0} q_{2\varepsilon} - \varphi_\varepsilon A_{\varepsilon 0} u + f\varphi_\varepsilon + A_{1\varepsilon} u), \quad \partial q_{j\varepsilon}/\partial x_n = 0 \text{ on } \Omega, \tag{10}$$

$$u_\varepsilon = g\varphi_\varepsilon \text{ on } \partial\Omega \text{ and } \partial u_\varepsilon/\partial x_n = h\varphi_\varepsilon + g\partial\varphi_\varepsilon/\partial x_n \text{ on } \Gamma_0 \cup \Gamma_1, \tag{11}$$

where $A_{1\varepsilon}$ is a first-order operator, depending on ε, the norms of whose coefficients in $C^\lambda(\Omega)$ do not exceed $C(\varepsilon)$.

Using a linear change in variables, the norms of which and their inverses are uniformly bounded with respect to w_ε, and for which the halfspace $x_n < 0$ maps into itself, we convert operator A_ε to the Laplace operator in the above fixed points from w_ε. Let $\tilde{w}_\varepsilon^+$ and $\tilde{w}_\varepsilon^-$ denote the cylinders $w_\varepsilon \times (-\varepsilon, 0)$ and $w_\varepsilon \times (-1, -1 + \varepsilon)$.

According to Lemma 1 and relations (10) and (11), we obtain

$$\begin{aligned} \varepsilon^2 |q_{1\varepsilon}|^\lambda(w_\varepsilon) + \varepsilon^2 |q_{2\varepsilon}|^\lambda(w_\varepsilon) \leqslant C(\varepsilon^2 |\rho_{1\varepsilon 0} q_{1\varepsilon} + \rho_{2\varepsilon 0} q_{2\varepsilon} - f\varphi_\varepsilon + A_{1\varepsilon} u|^\lambda (\tilde{w}_{C\varepsilon}^- \cup \tilde{w}_{C\varepsilon}^+) + |g\varphi_\varepsilon|^{2+\lambda}(w_{C\varepsilon} \times \{-1, 0\}) + \\ + \varepsilon |h\varphi_\varepsilon + g\partial\varphi_\varepsilon/\partial x_n|^{1+\lambda}(w_{C\varepsilon} \times \{-1, 0\}) + |u_\varepsilon|^1 (\tilde{w}_{C\varepsilon}^- \cup \tilde{w}_{C\varepsilon}^+)) \leqslant \\ \leqslant C(\varepsilon^2 |\rho_{1\varepsilon 0} q_{1\varepsilon}|^\lambda (\tilde{w}_{C\varepsilon}^- \cup \tilde{w}_{C\varepsilon}^+) + \varepsilon^2 |\rho_{2\varepsilon 0} q_{2\varepsilon}|^\lambda (\tilde{w}_{C\varepsilon}^- \cup \tilde{w}_{C\varepsilon}^+) + \varepsilon^2 |A_{\varepsilon 0} u_\varepsilon|^\lambda (\tilde{w}_{C\varepsilon}^- \cup \tilde{w}_{C\varepsilon}^+)) + C(\varepsilon)(M + |u|^2(\Omega)), \end{aligned} \tag{12}$$

where M is the sum of the norms of functions f, g, and h, indicated in relation (6). Noting that $\rho_{j\varepsilon 0}$ and the coefficients of $A_{\varepsilon 0}$ vanish at one of the points of $\tilde{w}_\varepsilon$, we conclude that the sum of the norms of $\rho_{j\varepsilon_0} q_{j\varepsilon}$ and $A_{\varepsilon 0} u_\varepsilon$ on the right-hand side of inequality (12) does not exceed the quantity

$$C\varepsilon^\lambda (|q_{1\varepsilon}|^\lambda (w_\varepsilon) + |q_{2\varepsilon}|^\lambda (w_\varepsilon) + |\nabla^2 u_\varepsilon|^\lambda (w_{C\varepsilon}^- \cup w_{C\varepsilon}^+)) + C(\varepsilon)|u|^2(\Omega).$$

Here we have taken into account the fact that $q_{j\varepsilon}$ does not depend on x_n and is equal to zero outside of w_ε. Choosing ε so that $C\varepsilon^\lambda < 1/2$, using the preceding inequality for an estimate of the right-hand side of relation (12), and carrying terms with $q_{1\varepsilon}$, $q_{2\varepsilon}$ to the left-hand side of relation (12), we have

$$|q_{1\varepsilon}|^\lambda (w_\varepsilon) + |q_{2\varepsilon}|^\lambda (w_\varepsilon) \leqslant C\varepsilon^\lambda |\nabla^2 u_\varepsilon|^\lambda (w_{2\varepsilon}^- \cup w_{2\varepsilon}^+) + C(\varepsilon)(M + |u|^2(\Omega)). \tag{13}$$

By virtue of the definition of the Hölder norms from relations (11), $|q_{j\varepsilon}|^\lambda(\Omega') \le \varepsilon^{-\lambda}|q_{j\varepsilon}|^\lambda(w_\varepsilon) + C(\varepsilon)|q_j|^0(\Omega')$, and the first norm on the right-hand side of relation (13) does not exceed $C\varepsilon^\lambda|\nabla^2 u_\varepsilon|^\lambda(\Omega) + C(\varepsilon)|u|^2(\Omega)$; therefore,

$$|q_{1\varepsilon}|^\lambda(\Omega')+|q_{2\varepsilon}|^\lambda(\Omega')\le C\varepsilon^\lambda|u|^{2+\lambda}(\Omega)+C(\varepsilon)(M+|q_1|^0(\Omega')+|q_2|^0(\Omega')+|u|^2(\Omega)). \quad (14)$$

The case in which w_ε intersects with $\partial\Omega'$ is considered analogously. Here we first "rectify" the boundary of Ω' with the aid of the change of variables indicated at the start of the proof of Theorem 1, and we then convert operator A_ε, whose coefficients coincide with the corresponding coefficients of A at some point of $\partial\Omega' \times \{0\}$, to a Laplace operator with the aid of linear changes in variables, leaving invariant the halfspaces $0 < x_1$, $0 < x_n$ and uniformly bounded over w_ε along with their derivatives. The existence of such linear changes of variables follows from condition (3) relating to the coefficients a^{jn}. Further, the estimate (14) may be derived from Lemma 2 in exactly the same way it was derived above from Lemma 1.

Thus, estimate (14) was obtained for all sets w_ε. It is now not difficult to conclude from the definition of the covering w_ε and the functions $q_{j\varepsilon}$ that $q_{j\varepsilon}$ on the left-hand side of inequality (14) can be replaced by q_j.

By virtue of Eq. (4) the sums

$$\rho_1(x',0)q_1(x')+\rho_2(x',0)q_2(x'); \quad \rho_1(x',-1)q_1(x')+\rho_2(x',-1)q_2(x')$$

may be estimated in the $C^0(\Omega')$ norm in terms of $C(|u|^2(\Omega) + M)$. Using condition (2), we also estimate, in terms of $C(|u|^2(\Omega) + M)$, also the quantity $|q_j|^0(\Omega')$. Then from relation (14) we have

$$|q_1|^\lambda(\Omega')+|q_2|^\lambda(\Omega')\le C\varepsilon^\lambda|u|^{2+\lambda}(\Omega)+C(\varepsilon)(M+|u|^2(\Omega)). \quad (15)$$

Now the known Schauder estimates of solutions of the Dirichlet problem (4), (5) yield

$$|u|^{2+\lambda}(\Omega)\le C\varepsilon^\lambda|u|^{2+\lambda}(\Omega)+C(\varepsilon)(M+|u|^2(\Omega)).$$

Selecting and fixing ε so that $C\varepsilon^\lambda < 1/2$, we rid ourselves of the first term on the right-hand side. Applying known iterational inequalities [11, 12], $|u|^2(\Omega)$ can be replaced by $|u|^0(\Omega)$. Using, in addition, estimate (15), we complete the proof of Theorem 1.

COROLLARY. We assume that domain Ω is bounded and that the solution of problem (4), (5) is unique. Estimate (6) is then valid without the last term $|u|^0(\Omega)$.

This corollary may be derived from estimate (6) using a well-known method [10] based on compactness of the imbedding of $C^\lambda(\Omega)$ in $C^{\lambda/2}(\Omega)$ for bounded domains Ω.

2. THEOREM OF UNIQUENESS

Let Ω be a domain in $\mathbf{R}^n$ with a piecewise-smooth boundary, each smooth piece of which is a portion of the boundary of some domain of class $C^{2+\lambda}$. Similar domains are described in more detail in [12, p. 212]. Assume, in addition, that Ω is convex with respect to x_n and bounded. We assume that the coefficients of operator A do not depend on x_n, $a^{jk} \in C^{2+\lambda}$, $a^j \in C^{1+\lambda}$, $a \in C^\lambda$, and the coefficient a and coefficient a_* of the adjoint operator A^* nonnegative.

THEOREM 2. Let us assume that the weight function ρ satisfies the condition

$$0 < \partial\rho/\partial x_n \text{ almost everywhere on } \Omega,\ \rho,\ \partial\rho/\partial x_n \in C^\lambda(\bar\Omega). \quad (16)$$

If functions $u \in C^2(\bar\Omega)$, $q \in C^\lambda(\bar\Omega)$, $f \in C(\bar\Omega)$ are solutions of the problem

$$Au = \rho q + f,\ \partial q/\partial x_n = 0,\ \partial f/\partial x_n = 0 \text{ on } \Omega, \quad (17)$$

$$u = 0,\ \partial u/\partial x_n = 0 \text{ on } \partial\Omega, \quad (18)$$

then $u = 0$, $q = 0$, and $f = 0$ on Ω.

Let Γ be the interior on $\partial\Omega$ of a set of points $x \in \partial\Omega$, such that $\langle n, e_n\rangle = 0$, $e_n = (0, \ldots, 0, 1)$, ν is the conormal to $\partial\Omega$.

LEMMA 3. If u satisfies conditions (17), (18), then

$$\int_{\Omega} v(\rho q + f)\,dx - \int_{\Gamma} (\partial u/\partial \nu)\, v\,d\Gamma = 0 \tag{19}$$

for an arbitrary function $v \in C^2(\Omega)$ such that

$$A^* v = 0 \text{ on } \Omega. \tag{20}$$

Proof. From Eqs. (17) and (20) we have, according to Green's formula,

$$\int_{\Omega} v(\rho q + f)\,dx = \int_{\Omega} (vAu - uA^*v)\,dx = \int_{\partial\Omega} ((v\,\partial u/\partial\nu - u\,\partial v/\partial\nu) + a_1 uv)\,d\Gamma = \int_{\Gamma} v(\partial u/\partial\nu)\,d\Gamma,$$

since $u = 0$ on $\partial\Omega$ by virtue of condition (18), and it follows from this very condition that the first order derivatives of u on $\partial\Omega\backslash\Gamma$ are equal to zero. This establishes Lemma 3.

Proof of Theorem 2. We assume that $q \not\equiv 0$. Let $q \geq 0$. Differentiating Eqs. (17) with respect to x_n and using conditions (16) and (17), we obtain

$$A\partial u/\partial x_n = q\partial\rho/\partial x_n \geqslant 0 \text{ on } \Omega,\ \partial u/\partial x_n = 0 \text{ on } \partial\Omega.$$

According to the maximum principle, $0 \leq \partial u/\partial x_n$ on Ω. Since $u = 0$ on $\partial\Omega$ according to condition (18) and $0 \leq \partial u/\partial x_n$, it follows that $u = 0$ on Ω. Therefore, $q\partial\rho/\partial x_n = 0$ on Ω, which contradicts our assumption and condition (16). The case $q \leq 0$ is similarly not possible.

Let Ω' be the projection of Ω on the hyperplane $\{x_n = 0\}$. Let $\Omega_+' = \{x \in \Omega' \colon\ 0 < q(x)\}$, $\Omega_-' = \{x \in \Omega' \colon\ q(x) < 0\}$. In view of what was presented above, we can assume that

$$\Omega_+' \neq \varnothing;\ \Omega_-' \neq \varnothing. \tag{21}$$

Since the coefficients of A do not depend on x_n, then along with v we will have, as a solution of Eq. (20), also the function $\partial v/\partial x_n$. By Lemma 3, for an arbitrary such function v with $\partial v/\partial x_n \in C^2(\overline{\Omega})$, we have

$$\int_{\Omega} (\partial v/\partial x_n)(\rho q + f)\,dx - \int_{\Gamma} (\partial u/\partial\nu)\,dv/\partial x_n\,d\Gamma = 0. \tag{22}$$

We integrate Eq. (22) by parts. Then

$$\int_{\partial\Omega} v(\rho q + f)\,n_n\,d\Gamma - \int_{\Omega} (\partial\rho/\partial x_n)\,vq\,dx + \int_{\overline{\Gamma}} v\,d\mu = 0 \tag{23}$$

for an arbitrary indicated function v, where μ is a measure on Γ which is constructed on Γ and the function u.

We extend Eq. (23) to solutions of Eq. (20) continuous on $\overline{\Omega}$. We extend v from $\partial\Omega$ onto $\mathbf{R}^n$ by a continuous function $\tilde{v}$. We approximate domain Ω by a monotonically decreasing sequence of bounded domains $\Omega(k)$ of class $C^{2+\lambda}$, so that $\bigcap_k \Omega(k) = \overline{\Omega}$. Let $v(k)$ be the solution of a Dirichlet problem for Eq. (20) on $\Omega(k)$ with boundary data $\tilde{v}$ on $\partial\Omega(k)$. It is known that the $v(k)$ exist and belong to $C^2(\overline{\Omega})$. Since the coefficients of A do not depend on x_n, then also $\partial v/\partial x_n \in C^2(\overline{\Omega})$ [12, Theorems 10.1, 12.1]. It is obvious that Eq. (23) is valid for $v(k)$. As is well-known from the theory of the stability of a Dirichlet problem solution [15, 16], the functions $v(k)$ converge pointwise on $\overline{\Omega}$ to the function v. Since $v(k)$, by the maximum principle, is uniformly bounded, then using Lebesgue's theorem on passage to the limit under the integral sign, we obtain from Eq. (23) for $v(k)$ as $k \to +\infty$ this equation for arbitrary solutions of Eq. (20) continuous on $\overline{\Omega}$.

Let ψ be a function, measurable on Ω, constant with respect to x_n on $\Omega \cup (\partial\Omega\backslash\Gamma)$, equal to zero on Γ for $x \in \partial\Omega\backslash\Gamma$, $x' \in \Omega_+'$, and to one for $x \in \partial\Omega\backslash\Gamma$, $x' \in \Omega'\backslash\Omega_+'$.

We select a sequence of functions $\psi_k \in C(\partial\Omega)$ so that $0 \leq \psi_k \leq 1$, $\psi_0 \leq \psi_k$, $\psi_k \to \psi$ in $L_1(\partial\Omega)$, $\psi_0 \equiv 0$, $\psi_k = 0$ on Γ, $k = 0, 1, \ldots$. Let v_k be a solution of the following Dirichlet problem:

$$A^* v_k = 0 \text{ on } \Omega,\quad v_k = \psi_k \text{ on } \partial\Omega.$$

According to the maximum principle, from the inequalities $0 \leqslant \psi_0 \leqslant \psi_k \leqslant 1$, $\psi_0 \neq 0$, it follows that

$$0 < v_0 \leqslant v_k \leqslant 1 \text{ on } \Omega. \tag{24}$$

Putting $v = v_k$ in Eq. (23), letting $\Omega_\pm = \{x \in \Omega: \ x' \in \Omega_\pm'\}$, and taking inequalities (24) into account, we write

$$0 \geqslant \int_{\partial\Omega\setminus\Gamma} v_k \rho q n_n \, d\Gamma - \int_{\Omega_+} q (\partial\rho/\partial x_n)\, dx - \int_{\Omega_-} v_0 q (\partial\rho/\partial x_n)\, dx + \int_{\partial\Omega\setminus\Gamma} v_k f n_n \, d\Gamma. \tag{25}$$

Passing to the limit for $k \to +\infty$ in relations (24) and introducing the notation $\Gamma_+ = \{x \in \partial\Omega: x' \in \Omega_+, \ n_n(x) \neq 0\}$, we obtain

$$0 \geqslant \int_{\Gamma_+} \rho q n_n \, d\Gamma - \int_{\Omega_+} q (\partial\rho/\partial x_n)\, dx - \int_{\Omega_-} v_0 q (\partial\rho/\partial x_n)\, dx + \int_{\partial\Omega\setminus\Gamma} \psi f n_n \, d\Gamma. \tag{26}$$

By virtue of the integration by parts formula, the difference of the first integrals is zero. The last integral is equal to the integral over $\partial\Omega$ since $n_n = 0$ on Γ. Again, according to the integral by parts formula, this integral is equal to the integral of $\partial(\psi f)/\partial x_n$ over Ω, i.e., is equal to zero. Relation (26) now leads to a contradiction since $0 < \partial\rho/\partial x_n$ almost everywhere on Ω, $0 < v_0$ on Ω_- in view of relation (24), and $q < 0$ on Ω_- by virtue of the choice of the set $\Omega_- \neq \emptyset$ according to assumption (21). The contradiction establishes that $q = 0$.

We show that $f = 0$ also. Let $f \neq 0$. We can consider $f > 0$ in some neighborhood w of point $x \in \partial\Omega$ with $n_n(x) > 0$. Let $\varphi \in C_0^\infty(w)$, $\varphi \not\equiv 0$ and $0 \leq \varphi$. Selecting as v a solution of the Dirichlet problem for Eq. (20) with boundary data $v = \varphi$ on $\partial\Omega$, and using relation (23), we obtain

$$0 = \int_{\partial\Omega\setminus\Gamma} v f n_n \, d\Gamma = \int_{\partial\Omega} v f n_n \, d\Gamma > 0$$

according to the choice of φ. Thus, $f = 0$. Hence, $u = 0$, $f = 0$, $q = 0$, which completes the proof of Theorem 2.

Remark. Theorem 2 can be proved with the condition of convexity of Ω with respect to x_n replaced by the condition $\partial\Omega = \partial\overline{\Omega}$. In this connection, we need to modify only the choice of function φ.

As an application of Theorem 2 we consider the problem of finding a triple of functions (u, q, f) satisfying the following conditions:

$$Au + qu = f, \ \partial q/\partial x_n = \partial f/\partial x_n = 0, \ 0 \leqslant q \text{ on } \Omega, \tag{27}$$

$$u = g \text{ on } \partial\Omega, \tag{28}$$

$$\partial u/\partial x_n = h \text{ on } \partial\Omega\setminus\Gamma, \tag{29}$$

where Ω is the cylindrical domain $\Omega' \times (-H, 0)$ considered in Theorem 1. As for operator A, we assume, in addition, that $a^{jn} = 0$ and $a^n = 0$ on Ω for $j < n$, $a^{nn} = 1$.

We also introduce the conditions:

$$g \in C^{2+\lambda}, \ \partial g/\partial x_n \geqslant 0, \ \partial g/\partial x_n \neq 0 \text{ on } \Gamma, \ h \in C^{1+\lambda}(\overline{\Omega}), \ f \in C^\lambda(\overline{\Omega}), \tag{30}$$

$$0 \leqslant A'g + qg - f \text{ on } \Gamma_0, \ \ A'g + qg - f \leqslant 0 \text{ on } \Gamma_H, \ \ Ag = f \text{ on } \partial\Omega \times \{-H, 0\}, \tag{31}$$

$$0 \leqslant h, \ \partial g/\partial x_n = h \text{ on } \partial\Omega \times \{-H, 0\}. \tag{32}$$

Using the well-known theory of the solvability and regularity for second-order elliptic equations [12, Secs. 4, 5, 10, 12] and also symmetric extensions relative to the hyperplanes $x_n = 0$ and $x_n = -H$, where additional conditions are needed on the coefficients of A, we can prove existence of a solution u of problem (30), (31) of class $C^{2+\lambda}(\overline{\Omega})$.

LEMMA 4. If function $u \in C^{2+\lambda}(\overline{\Omega})$ satisfies conditions (27), (29), where g and h satisfy either the conditions (30), (32) or conditions (30), (31), then $\partial u/\partial x_n > 0$ on Ω.

Proof. Consider the case of conditions (30) and (31). Denoting $\partial u/\partial x_n$ by v and differentiating Eq. (27) and boundary conditions (28) on Γ, we obtain

$$Av+qv=0 \text{ on } \Omega,\quad v=\partial q/\partial x_n \text{ on } \Gamma.$$

By virtue of Eq. (27),

$$\partial v/\partial x_n=\partial^2 u/(\partial x_n)^2=A'u+qu-f=A'g+qg-f \text{ on } \Gamma_0\cup\Gamma_H$$

in view of boundary condition (28). Thus, $0 \le \partial v/\partial x_n$ on Γ_0 and $\partial v/\partial x_n \le 0$ on Γ_H. By the extremum principle a negative minimum cannot be attained on $\Gamma_0 \cup \Gamma_H$. It can therefore be attained only on $\overline{\Gamma}$ or v is constant. Therefore, by virtue of condition (30) v is nonnegative. If v were to vanish at at least one point of Ω, it would then be constant, which would contradict one of the conditions (30). Thus, $0 < v$ on Ω. A statement relating to v under conditions (30) and (32) follows directly from the maximum principle. This establishes Lemma 4.

THEOREM 3. When conditions (30), (31) or (30), (32) are satisfied, problem (27)-(29) has at most one solution of class $C^{2+\lambda}(\overline{\Omega}) \times C^{\lambda}(\overline{\Omega}) \times C^{\lambda}(\overline{\Omega})$.

Proof. Suppose the problem has two solutions, (u_1, q_1, f_1) and (u_2, q_2, f_2). Letting $u = u_2 - u_1$, $q = q_1 - q_2$, $f = f_2 - f_1$ and subtracting from Eqs. (27)-(29) for the second solution the same equations for the first solution, we obtain

$$Au+q_2u=u_1q+f,\ \partial q/\partial x_n=0,\ \partial f/\partial x_n=0 \text{ on } \Omega,$$
$$u=0 \text{ on } \partial\Omega,\quad \partial u_1/\partial x_n=0 \text{ on } \partial\Omega.$$

By Lemma 4, $0 < \partial u/\partial x_n$ on Ω. Putting $\rho = u_1$, we find from Theorem 2 that $q = 0$, $f = 0$ and $u = 0$. Thus, $u_1 = u_2$, $q_1 = q_2$, $f_1 = f_2$, and Theorem 3 is thereby established.

3. THEOREM OF UNIQUENESS OF A SOLUTION

By the method of continuation with respect to a parameter [10; 11; 12, p. 149] we can deduce from Schauder estimates and a uniqueness theorem a theorem for the existence of a solution of the problem concerning finding a triple of functions (u, q_1, q_2) satisfying the conditions (4), (5). In Theorem 4, presented below, we consider a cylindrical domain Ω, described in Sec. 1, and we assume that operator A satisfies the same conditions as in Theorem 2, in particular, that its coefficients do not depend on x_n and that coefficients a, a_* are nonnegative, $a^{nn} = 1$.

THEOREM 4. Let the weight functions ρ_1, ρ_2 satisfy the condition

$$0<\varepsilon_0<\partial\rho_1/\partial x_n,\ \rho_2=1 \text{ on } \Omega. \tag{33}$$

Then for arbitrary functions $f \in C^{\lambda}(\overline{\Omega})$, $g \in C^{2+\lambda}(\partial\Omega)$, satisfying the compatibility conditions

$$Ag=f \text{ on } \partial\Omega'\times\{-H,\ 0\},\quad \partial g/\partial x_n=h \text{ on } \partial\Omega'\times\{-H,\ 0\}, \tag{34}$$

there exists a unique solution of problem (4), (5) of class $C^{2+\lambda}(\overline{\Omega}) \times C^{\lambda}(\overline{\Omega}) \times C^{\lambda}(\overline{\Omega})$.

We note that by subtracting from u the solution of the Dirichlet problem with the data f, g, we can reduce the general case to the case of zero f, g; we will consider this case later.

We preface the proof of Theorem 4 with two lemmas concerning solvability of the problem in the simplest case and concerning the approximation of functions from Hölder classes.

LEMMA 5. Theorem 4 is true for the case $A = -\Delta$, $\rho_1 = x_n$, $h \in C^2(\overline{\Gamma}_0 \cup \overline{\Gamma}_H)$.

Proof. Let $\{v_k\}$ be a complete orthonormalized set of characteristic functions of the Dirichlet problem for the operator $-\Delta' = -\partial^2/\partial x_1^2 - \partial^2/\partial^2 x_2^2 - \ldots - \partial^2/\partial x_{n-1}^2$ in Ω'; we denote the corresponding characteristic values by λ_k^2, $0 < \lambda_k$. It is known that in the given case such a system exists and that $v_k \in C^{2+\lambda}(\overline{\Omega}')$ [12, Theorems 17.1, 12.1, 10.1]. We introduce the class of functions $W_\Delta = \{h = \sum_{k=1}^{N} h_k v_k$, where N, h_k are some numbers}. We show that the functions

$$u(x)=\sum_{k=1}^{N}\left(C_{1k}e^{\lambda_k x_n}+C_{2k}e^{-\lambda_k(x_n+H)}+\lambda_k^{-2}(x_n q_{1k}+q_{2k})\right)v_k(x),$$

$$q_j(x)=\sum_{k=1}^{N} q_{jk}v_k(x'),\ j=1,2,$$

are a solution of the inverse problem (4), (5) if h_0 (h on Γ_0) and h_H (h on Γ_H) belong to the class W_Λ and

$$q_{1k}=\lambda_k\left(1-e^{-\lambda_k H}\right)(h_{0k}+h_{Hk})\Big/\left(-H\left(1+e^{-\lambda_k H}\right)+2\lambda_k^{-1}\left(1-e^{-\lambda_k H}\right)\right),$$

$$q_{2k}=\lambda_k\left(1-e^{-\lambda_k H}\right)^{-1}\Big(\left(\left(\lambda_k^{-1}-H\right)-\left(\lambda_k^{-1}+H\right)e^{-2\lambda_k H}\right)h_{0k}+$$

$$+\left(\lambda_k^{-1}\left(1-e^{-2\lambda_k H}\right)+2He^{-\lambda_k H}\right)h_{Hk}\Big)\Big/\left(-H\left(1+e^{-\lambda_k H}\right)+2\lambda_k^{-1}\left(1-e^{-\lambda_k H}\right)\right),$$

where C_{1k}, C_{2k} are defined in terms of q_{1k}, q_{2k} as solutions of the following system of linear equations:

$$C_{1k}+C_{2k}e^{-\lambda_k H}+\lambda_k^{-2}q_{2k}=0,\ C_{1k}\lambda_k-C_{2k}\lambda_k e^{-\lambda_k H}+\lambda_k^{-2}q_{1k}=h_{0k}. \tag{35}$$

Here h_{0k}, h_{Hk} are coefficients in the expansion of h_0 and h_H with respect to the basis $\{v_k\}$.

Actually, validity of the equation $-\Delta u = x_n q_1 + q_2$ may be verified by a direct substitution. The boundary conditions on Γ follow from the definition of characteristic functions. The remaining boundary conditions (5) are equivalent, by virtue of the definition of u and q_j, to Eqs. (35) (conditions $u = 0$ on Γ_0 and $\partial u/\partial x_n = 0$ on Γ_0) and to the equations

$$C_{1k}e^{-\lambda_k H}+C_{2k}+\lambda_k^{-2}(-Hq_{1k}+q_{2k})=0\quad (u=0 \text{ on } \Gamma_H),$$

$$C_{1k}\lambda_k e^{-\lambda_k H}-\lambda_k C_{2k}+\lambda_k^{-2}q_{1k}=h_{Hk}\quad (\partial u/\partial x_n=0 \text{ on } \Gamma_H),$$

which result from the formulas for q_{1k} and q_{2k} of Eqs. (35). Thus the problem is solvable for arbitrary data h_0, $h_H \in W_\Lambda$.

We approximate functions h_0, h_H of class C^2, equal to zero on $\partial\Gamma_0$, $\partial\Gamma_H$, by functions from W_Λ according to $C^{1+\lambda}$-norm. Let $p > n(1 - \lambda)$. Since $\partial\Omega' \in C^{2+\lambda}$, the characteristic functions are then dense in $L_p(\Omega')$; therefore there exists a sequence of functions $f(k)$ from W_Λ, converging in $L_p(\Omega')$ to $-\Delta' h_0$. We note that if $f \in W_\Lambda$, then the solution of the Dirichlet problem $-\Delta' h = f$ on Ω', $h = 0$ on $\partial\Omega'$ also belongs to W_Λ. Therefore the solution $h(k)$ of the Dirichlet problem with the data $f(k)$ belongs to W_Λ. By virtue of known Schauder L_p-estimates of solutions of elliptic boundary value problems [10, Chap. 5], we have

$$\|h(k)-h_0\|_{W_p^2(\Omega')}\leqslant C\|f(k)+\Delta' h_0\|_{L_p(\Omega')}\to 0$$

as $k \to +\infty$. By a Sobolev imbedding theorem [11, p. 230], taking the choice of p into ac-account, we obtain

$$|h(k)-h_0|^{1+\lambda}(\Omega')\leqslant C\|h(k)-h_0\|_{W_p^2(\Omega')}.$$

Thus, $h(k)$ converges to h_0 in $C^{1+\lambda}(\Omega')$. We consider h_H similarly.

Lemma 5 is then a consequence of the Schauder estimate (6), the corollaries to Theorems 1 and 2, the possibility of the indicated approximation, and the solvability of the problem with data from W_Λ.

LEMMA 6. Let $h \in C^{1+\lambda}(\bar\Omega')$ and $h = 0$ on $\partial\Omega'$. Then there exists a sequence of functions $h(k) \in C^2(\bar\Omega')$, $h(k) = 0$ on $\partial\Omega'$ such that $h(k)$ converges to h in $C^{1+\lambda}(\Omega')$ and $|h(k)|^{1+\lambda}(\Omega') \le C(h)$.

Proof. By virtue of the conditions on domain Ω' there exists a finite set of balls $w_1, \ldots, w_m$ in R^{n-1} such that their union contains the closure of Ω', and if $\partial\Omega' \cap w_j$ is nonempty, then a diffeomorphism $y(x; j)$ of class $C^{2+\lambda}(\bar w_j)$ may be found, which maps w_j onto a domain v_j of class $C^{2+\lambda}$, such that its inverse belongs to $C^{2+\lambda}(v_j)$, while $\Omega' \cap w_j$, $\partial\Omega' \cap w_j$ are mapped, respectively, onto $\{0 < y_1\} \cap v_j$, $\{0 = y_1\} \cap v_j$. Let φ_j be a C^∞-partition of unity corresponding to the covering w_j. We have $h = h\varphi_1+\ldots+h\varphi_m$ on $\bar\Omega'$, the supports of $h\varphi_j$ lie in w_j, and $h\varphi_j$ satisfy the conditions of the lemma with respect to h; it is therefor sufficient to approximate each $h\varphi_j$.

Initially, let $\bar{w}_j \subset \Omega'$. We introduce the standard averaging kernel $\psi_\varepsilon(x) = \psi(|x|/\varepsilon)$, where $\psi \in C_0^\infty(\mathbf{R}^{n-1}), \int \psi dx = 1$, $0 \le \psi$ and $\operatorname{supp}\psi = \{x: |x| \le 1\}$ (see, for example, [12, p. 68]). Then the functions $(h\varphi_j)_\varepsilon$, which are convolutions of $h\varphi_j$ with ψ_ε, belong to $C^2(\mathbf{R}^{n-1})$, their supports belong to w_j for sufficiently small ε, and $(h\varphi_j)_\varepsilon \to h\varphi_j$ in $C^1(\Omega')$ as $\varepsilon \to 0$. We extend $h\varphi_j$ through zero on $\mathbf{R}^{n-1}\backslash w_j$. It is not hard to see that if the Hölder constant $[u]^\lambda$ is not larger than C_0, then also $[u_\varepsilon]^\lambda \le C_0$; therefore, $|(h\varphi_j)_\varepsilon|^{1+\lambda} \le C$. Thus $(h\varphi_j)_\varepsilon$ approach $h\varphi_j$ as $\varepsilon \to 0$.

Consider now the case in which the intersection of w_j with $\partial\Omega'$ is nonempty. Going over to variable y, we reduce the problem to the case of the halfspace $0 < y_1$. We denote $h\varphi_j$ in the variables y by g_j. We extend g_j through zero onto $\{0 \le y_1\}\backslash v_j$. It is obvious that $g_j \in C^{1+\lambda}(\{0 \le y_1\})$ and that $g_j = 0$ for $y_1 = 0$. We define $g_j(-y_1, y_2, \ldots, y_{n-1}) = -g_j(y_1, y_2, \ldots, y_{n-1})$ for $0 \le y_1$. Averages of $g_{j\varepsilon}$ are equal to zero for $y_1 = 0$ by virtue of symmetry (oddness) of the extension and choice of the averaging kernel. As was the case above, for small ε we have: $(\operatorname{supp} g_{j\varepsilon}) \cap \{0 \le y_1\}$ belongs to v_j, and as above $g_{j\varepsilon}$ approaches g_j as $\varepsilon \to 0$. Returning to the variables x and extending the function in variables x through zero onto $\Omega'\backslash w_j$, we obtain the required approximations of function $h\varphi_j$. This completes the proof of Lemma 6.

Proof of Theorem 4. We note, by virtue of Theorems 1 and 2 and also the corollary to Theorem 1, that estimate (6) is valid without the term $|u|^0(\Omega)$ on the right-hand side. We introduce a one-parameter family of inverse problems (4), (5), where in place of A we take the operator $A_t = -\Delta + t(A + \Delta)$, and in place of ρ_1 and ρ_2 we take the weights $\rho_{1t} = (1 - t)x_n + t\rho_1$, $\rho_{2t} = 1$.

The set τ of values of parameter t for which the problem is solvable is nonempty since, by Lemmas 5 and 6, it contains $t = 0$. Proof that τ' is closed proceeds according to the known scheme [11, 12] with use of estimate (6) without the term $|u|^0(\Omega)$ and the usual properties of Hölder norms.

We show that τ is open in [0, 1]. Let $t_0 \in \tau$. We denote by B_t the linear operator which to the pair of functions (q_1, q_2) from $C_0{}^\lambda(\bar{\Omega}) \times C_0{}^\lambda(\bar{\Omega}')$ makes correspond the pair of functions $\partial u/\partial x_n$ on Γ_0 and $\partial u/\partial x_n$ on Γ_H, where u is the solution of the Dirichlet problem (4), (5) with $A = A_t$ (without the conditions $\partial u/\partial x_n = h$ on $\Gamma_0 \cup \Gamma_H$). In view of the Schauder estimates of solutions of the Dirichlet problem and the compatibility conditions (34) the operator B_t is continuous from $X = C_0{}^\lambda(\bar{\Omega}') \times C_0{}^\lambda(\bar{\Omega}')$ into $Y = (C^{1+\lambda}(\bar{\Gamma}_0) \cap C_0(\Gamma_0)) \times (C^{1+\lambda}(\bar{\Gamma}_H) \cap C_0(\bar{\Gamma}_H))$, where $C_0{}^\lambda(\bar{\Omega}')$, $C_0(\bar{\Gamma}_0)$, $C_0(\bar{\Gamma}_H)$ are sets of Hölder or continuous functions, equal to zero on the boundary of the indicated sets. According to the definition of the set τ, for an arbitrary $h \in Y$ there exists a solution u, q_1, q_2 of the inverse problem with $t = t_0$. By the remark made at the beginning of the proof, operator B_{t_0} has a continuous inverse from Y to X. Since operators, which are close in the uniform operator norm to invertible in Banach spaces, are also invertible, operator B_t will be invertible for t close to t_0. Thus the set τ is open.

By virtue of the principle of continuity $\tau = [0, 1]$; therefore, the initial problem corresponding to $\tau = 1$ is solvable. Uniqueness of its solution is a consequence of Theorem 2. This concludes the proof of Theorem 4.

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PERIODIC GROUPS WITH THE PRIMARY MINIMALITY CONDITION FOR CERTAIN SYSTEMS OF SUBGROUPS

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UDC 519.45

The primary minimality condition was introduced by Chernikov [1]. Polovitskii [2] described the periodic, locally solvable groups satisfying the primary minimality condition [1]. Pavlyuk, Shafiro, and Shunkov [3] proved that locally finite groups with the primary minimality condition for locally solvable subgroups are almost locally solvable. Later an analogous result was obtained for binary-finite groups: Sedova [4] considered groups without involutions, and Pavlyuk [5] treated the general case. In the present paper we study extensions of binary-finite groups by binary-finite groups in which certain systems of locally solvable subgroups satisfy the primary minimality condition.

Notation and Definitions

Definition 1 (Chernikov). A group G satsifies the p-minimality condition for some $p \in \pi(G)$ if any descending chain of subgroups $H_1 \supset H_2 \supset \ldots \supset H_k \supset \ldots$ in which $H_i - H_{i+1}$ $(i = 1, 2, \ldots)$ contains at least one p-element is finite. A group G satisfies the primary minimality condition if G satisfies the p-minimality condition for each $p \in \pi(G)$ [1].

Definition 2. If in a periodic group G all divisible* Abelian subgroups (divisible Abelian p-subgroups) generate a divisible Abelian subgroup T (divisible Abelian p-subgroup T) and the factor group G/T contains no infinite divisible Abelian subgroup (infinite divi-

*The author uses the term complete, but divisible is more common in English, and more suggestive — Translator.

Krasnoyarsk. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 31, No. 4, pp. 160-165, July-August, 1990. Original article submitted November 19, 1987.

 0037-4466/90/3104-0666 $12.50