6. I. S. Borisov, "Approximation of empirical fields, constructed with respect to vector observations with dependent coordinates," Sib. Mat. Zh., 23, No. 5, 31-41 (1982).
7. J. Komlos, P. Major, and G. Tusnady, "An approximation of partial sums of independent R.V.'-s and sample DF," Z. Wahrscheinlichkeitstheorie Verw. Gebiete, 32, Nos. 1/2, 111-133 (1975).
8. V. V. Yurinskii [V. V. Yurinskiǐ], "Exponential inequalities for sums of random vectors," J. Multivar. Anal., 6, No. 4, 473-499 (1976).
9. I. S. Borisov, "On the rate of convergence in the 'conditional' invariance principle," Teor. Veroyatn. Primen., 23, No. 1, 67-79 (1978).
10. I. I. Gikhman and A. V. Skorokhod, Theory of Random Processes [in Russian], Vol. 1, Nauka, Moscow (1972).

EMBEDDINGS OF FINITE CHEVALLEY GROUPS AND PERIODIC LINEAR GROUPS

A. V. Borovik

UDC 512.542+512.74

In 1965 V. P. Platonov formulated the classification problem for simple infinite periodic linear groups [1, problem 1.75]. The present paper is the first of two papers devoted to the solution of this problem for the case when the characteristic of the base field is $p > 2$. Assume that G is a simple periodic linear group. O. Kegel [2] showed that G is the union of an increasing series of finite simple groups:

$$1 = G_0 < G_1 < \ldots < G_n < \ldots$$

The main result of this paper is the following theorem.

THEOREM. Let G be an infinite simple periodic linear group over a field of characteristic $p > 2$. If in the notation introduced above all the groups G_i, $i = 1, 2, \ldots$ are known simple groups, then G is isomorphic to a Chevalley group or a twisted analog of a Chevalley group over some locally finite field of characteristic p.

Here the term "known simple groups" means any finite Chevalley group or twisted analog thereof, any alternating group, or any finite number of finite simple groups. A locally finite field is an algebraic extension of a finite field.

The proof is based on the construction of a BN-pair in the group G and on the study of embeddings of G into the automorphism group of the Tits building associated with the BN-pair [3, 4]. In the case where the BN-pair of G has rank $\leqslant 2$ the identification of G with a suitable Chevalley group is achieved by some more detailed study of embeddings of finite Chevalley groups (cf. Lemmas 3-6 below).

We use standard notation and terminology which may be found in [3, 5-7]. The term "Chevalley group" refers to simple Chevalley groups and their twisted analogs.

We will now prove the theorem. Since the ranks of all Sylow r-subgroups of G for all primes $r \neq p$ are bounded, the sequence $\{G_i\}$ contains no more than a finite number of alternating groups; if we eliminate them and all sporadic simple groups from $\{G_i\}$ we can say that all the groups G_i are Chevalley groups. Moreover, we may assume that all the groups G_i belong to the same series of Chevalley groups [5] (e.g.

$$G_i \simeq PSL_{n_i}(k_i),$$

where generally speaking both n_i and k_i depend on $i = 1, 2, \ldots$). Since the ranks of the Sylow r-subgroups for $r \neq p$ are bounded, the Lie ranks of the groups in $\{G_i\}$ are bounded, and therefore we may assume that they are all identical. If the sequence $\{G_i\}$ contains only a finite number of groups defined over fields of characteristic p then the ranks of all Sylow subgroups of groups from $\{G_i\}$ are finite and bounded. By Theorem 4.1 in [8] all Sylow subgroups of G satisfy the minimum condition. It follows from [9] that the group G is almost locally solvable — a contradiction. Consequently, $\{G_i\}$ contains infinitely many groups

Translated from Sibirskii Matematicheskii Zhurnal, Vol. 24, No. 6, pp. 26-35, November-December, 1983. Original article submitted July 21, 1981.

0037-4466/83/2406-0843$07.50

defined over fields of characteristic p; eliminating the remaining ones and changing the numbering we have the following

LEMMA 1. Under the hypotheses of the theorem we may assume, without loss of generality, that all the G_i belong to the same one of the 14 series of Chevalley groups over finite fields of characteristic p and are all of the same Lie rank.

After these preliminary remarks we begin the proof of the theorem by induction on the rank of the groups G_i, $i = 1, 2, \ldots$. We start by considering the case when the G_i are of rank $\leqslant 2$. To identify the group $G = \varinjlim G_i$ with a suitable Chevalley group over a locally finite field we will use the following lemma.

LEMMA 2. Let

$$1 \xrightarrow{\text{id}} H_1 \xrightarrow{\text{id}} H_2 \xrightarrow{\text{id}} \ldots$$

and

$$1 \xrightarrow{\text{id}} G_1 \xrightarrow{\text{id}} G_2 \xrightarrow{\text{id}} \ldots$$

be two directed systems whose index sets are the natural numbers, satisfying the following conditions:

(1) every automorphism of the group H_i, $i = 1, 2, \ldots$ can be extended to an automorphism of H_{i+1};

(2) there exist surjective isomorphisms $\varphi_i\colon H_i \to G_i$, such that $\varphi_i(H_{i-1}) = G_{i-1}$, $i \geqslant 2$. Then $\varinjlim H_i = \varinjlim G_i$.

Proof. We need only construct a family of surjective isomorphisms

$$\psi_i : H_i \to G_i,\ i = 1, 2, \ldots,$$

with $\psi_i|_{H_{i-1}} = \psi_{i-1}$ for $i \geqslant 2$. We will construct them inductively, putting

$$\psi_1 = \varphi_2|_{H_1},\ \psi_2 = \varphi_2.$$

Assume that $\psi_1, \ldots, \psi_{i-1}$ are already constructed. Extend the automorphism $\varphi_i^{-1} \circ \psi_{i-1}$ of H_{i-1} to an automorphism θ_i of H_i and put $\psi_i = \varphi_i \circ \theta_i$. Then it is clear that

$$\psi_i|_{H_{i-1}} = \varphi_i \circ \theta_i|_{H_{i-1}} = \varphi_i \circ \varphi_i^{-1} \circ \psi_{i-1} = \psi_{i-1}.$$

The lemma is established.

Lemma 2 will be applied as follows. The Chevalley groups of rank $\leqslant 2$ over fields of odd characteristic belong to one of the types PSL_2, PSL_3, PSU_3, PSU_4, PSU_5, PSp_4, G_2, ${}^3D_4$, ${}^2G_2$. We will say that a group of a given type X has no sporadic embeddings in characteristic p if whenever k_1 and k_2 are finite fields of characteristic p the existence of an embedding

$$\varphi\colon X(k_1) \to X(k_2)$$

implies k_1 can be embedded into k_2 and $\varphi(X(k_1))$ is conjugate to the natural embedding of the subgroup $X(k_1)$ into $X(k_2)$ by applying an element of the extension of $X(k_2)$ by the group of diagonal automorphisms. It appears that Chevalley groups of any type do not possess sporadic embeddings in any characteristic; however, for the purposes of this paper we verified this only for the case of the groups in odd characteristic enumerated above.

LEMMA 3. Assume that the conditions of Lemma 1 hold, and assume further that the groups $G_i \simeq X(k_i)$, $i = 1, 2, \ldots$ do not possess sporadic embeddings; then $G \simeq X(k)$, where $k = \varinjlim k_i$.

Proof. Consider two directed systems

$$1 \to X(k_1) \to X(k_2) \to \ldots$$

where all embeddings are canonical, and

$$1 \xrightarrow{\text{id}} G_1 \xrightarrow{\text{id}} G_2 \to .$$

By hypothesis there exist isomorphisms $\Psi_i: X(k_i) \to G_i$ mapping $X(k_{i-1})$ into G_{i-1}. We note also that it follows from the description of the automorphisms of the Chevalley groups (cf. [10]) that every automorphism of $X(k_{i-1})$ can be extended to an automorphism of $X(k_i)$. Therefore we find in view of Lemma 2:

$$\varinjlim G_i \simeq \varinjlim X(k_{i-1}) \simeq X(\varinjlim k_i).$$

The lemma is established.

LEMMA 4. The groups of types PSL_n, $n \geqslant 2$, PSU_n, $n \geqslant 3$, PSp_n, $n \geqslant 4$ do not have any sporadic embeddings in characteristic $p > 2$.

Proof. For groups of type PSL_2 this follows from Dickson's theorem about the subgroups of $PSL_2(q)$ [11, II.8.27]. Now assume that $n \geq 3$, $G_1 \simeq PSL_n(k_1)$, $PSU_n(k_1)$, or $PSp_n(k_1)$, and $G_2 \simeq PSL_n(k_2)$, $PSU_n(k_2)$, or $PSp_n(k_2)$, respectively; let $\varphi\colon G_1 \to G_2$ be an embedding and put $H = \varphi(G_1)$. We will assume that the group G_2 is uniquely embedded in $PGL_n(k_2)$. In G_i take an involution t_i which is the image of an involution of $GL_n(k_i)$ with eigenvalues $(-1, -1, 1, \ldots, 1)$.

Put $C_i = C_{G_i}(t_i)$, $i = 1, 2$. C_i contains a normal subgroup $K_i * L_i$ where $K_i \simeq SL_2({}^2k_i)$ in the case $G_i \simeq PSU_n(k_i)$ (here ${}^2k_i$ is the fixed field of an involutive automorphism of the field k_i), and $K_i \simeq SL_2(k_i)$ otherwise, and put $L_i = 1$ if $G \simeq PSU_3(k_i)$ or $PSL_3(k_i)$ and $L_i \simeq SL_{n-2}(k_i)$, $SU_{n-2}(k_i)$, $Sp_{n-2}(k_i)$ in the remaining cases.

Considering the structure of centalizers of involutions on G_2 it is easy to see that $\varphi(t_1)$ is conjugate in G_2 to t_2. We may assume that $\varphi(t_1) = t_2$. It is also easy to verify that $\varphi(L_1) \leqslant L_2$, $\varphi(K_1) \leqslant K_2$. It follows from the description of the subgroups of G_2 [11, II.8.27] that k_1 can be embedded into k_2. The unipotent elements of K_2 are images of transvections, therefore $\varphi(G_1)$ is generated by the projective images of transvections of $GL_n(k_2)$. From the description of groups generated by transvections [12] it follows now that the inverse image $\tilde{H}$ of H in $GL_n(k_2)$ is conjugate in $GL_n(k_2)$ to the subgroup of $\tilde{G}_2 = SL_n(k_2)$, $SU_n(k_2)$, $Sp_n(k_2)$, consisting of the matrices with coefficients from k_1. This concludes the proof of the lemma in the case $G_2 \simeq PSL_n(k_2)$.

Now assume $G_2 \simeq PSU_n(k_2)$ or $PSp_n(k_2)$. Then $\tilde{G}_2 = SU_n(k_2)$ or $Sp_n(k_2)$ consists of all matrices over k_2 leaving invariant a nondegenerate skew-symmetric or Hermitian form f. Therefore $GL_n(k_2)$ contains an element g such that $H^g = SU_n(k_1)$ or $Sp_n(k_1)$ preserves f. Hence, the subgroup $\tilde{H}$ preserves the form $f^{g^{-1}}$. On the other hand, $\tilde{H} \leqslant \tilde{G}_2$ preserves f. Since $\tilde{H}$ is conjugate in $GL_n(k_2)$ to $SU_n(k_1)$ or $Sp_n(k_1)$ it is absolutely irreducible, and the form preserved by $\tilde{H}$ is unique up to a scalar factor. Therefore $f^{g^{-1}} = \lambda f$, where $\lambda \in \bar{k}_2^*$, i.e., $g \in GU_n(k_2)$ or $GSp_n(k_2)$ — the groups of the standard forms f.

LEMMA 5. The groups of type G_2, ${}^3D_4$ do not have any sporadic embeddings in characteristic $p > 2$.

Proof. Put $H_1 \simeq G_2(k_1)$ or ${}^3D_4(k_1)$ and $H_2 \simeq G_2(k_2)$ or ${}^3D_4(k_2)$, respectively, where k_1 and k_2 are finite fields of characteristic $p > 2$, and let $\varphi: H_1 \to H_2$ be an embedding.

In the groups H_i, $i = 1, 2$ there is at most one class of conjugate involutions. If t_i is an involution of H_i then $C_i = C_{H_i}(t_i)$ contains two normal subgroups L_i and M_i such that $L_i \simeq SL_2(k_i)$ in the case $H_i \simeq G_2(k_i)$ and $L_i \simeq SL_2({}^3k_i)$ in the case $H_i \simeq {}^3D_4(k_i)$ (here ${}^3k_i$ is the fixed field of an automorphism of order 3 of the field k_i) and $M_i \simeq SL_2(k_i)$ in the other cases. The group L_iM_i has index 2 in C_i and

$$\langle t_i\rangle = Z(L_i) = Z(M_i) = Z(C_i).$$

As in the previous lemma we may assume that $\varphi(t_1) = t_2 = t$, $\varphi(L_1) \leqslant L_2$, $\varphi(M_1) \leqslant M_2$. It follows from Dickson's theorem about subgroups of the group $PSL_2(k_2)$ [11, II.8.27] that the field k_1 can be embedded into k_2. Therefore, there exists a Steinberg endomorphism σ of the simply connected Chevalley group $\mathfrak{H}$ of type G_2 or D_4 over the algebraic closure K of the field k_2, such that $H_1 = \mathfrak{H}_\sigma$, $H_2 = \mathfrak{H}_{\sigma^m}$ for a suitable natural number m [13].

Assume that $\rho: \mathfrak{H} \to GL(V)$ is a nontrivial rational representation over K of smallest dimension. It follows from Theorem 13.1 of [13] that the restriction of ρ to H_2 and $\rho\varphi$ are irreducible representations. According to the same theorem, the representation $\rho\varphi$ of H_1 can be lifted to an irreducible rational representation χ of $\mathfrak{H}$. We put $\mathfrak{H}_1 = \chi(\mathfrak{H})$, $\mathfrak{H}_2 = \rho(\mathfrak{H})$, and identify H_1 and H_2 with their images in $\mathfrak{H}_1$ and $\mathfrak{H}_2$.

Now we can prove the lemma for the case that $\mathfrak{H}_1 \cap \mathfrak{H}_2$ is an infinite group. By Theorem 1 of [14] every finite subgroup X of $\tilde{\mathfrak{H}} = \mathfrak{H}/Z(\mathfrak{H})$ containing $\tilde{\mathfrak{H}}_\sigma$ is of the form

$$\tilde{\mathfrak{H}}^u_{\sigma^n} \leqslant X \leqslant \tilde{\mathfrak{H}}_{\sigma^n}$$

(here and later Y^u denotes the subgroup generated in Y by all unipotent elements). Therefore, one finds without difficulty that if $\mathfrak{H}_1 \cap \mathfrak{H}_2$ is an infinite group then it contains the group of K-rational points of the group $\mathfrak{H}_1$ for some infinite subfield k of K and therefore is dense in $\mathfrak{H}_1$. But $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are closed in the Zariski topology, hence $\mathfrak{H}_1 \leqslant \mathfrak{H}_2$ and comparison of dimensions shows that $\mathfrak{H}_1 = \mathfrak{H}_2$. Since $\mathfrak{H}$ is simply connected there exists an automorphism α of $\mathfrak{H}$ such that $\rho = \chi\alpha$. Since $H_1 \leqslant H_2$ we find

$$\mathfrak{H}_\sigma \leqslant (\mathfrak{H}_{\sigma^m})^\alpha.$$

Again by Theorem 1 of [14]

$$(\mathfrak{H}_{\sigma^m})^\alpha = \mathfrak{H}_{\sigma^m}.$$

Since the Schur multiplier of H_2 is trivial [10] we may apply the homomorphism ρ to H_2 and find that the subgroup $(\mathfrak{H}_\sigma)^\rho \simeq G_2(k_1)$ or ${}^3D_4(k_1)$ which has a natural embedding into $H_2 \cong G_2(k_2)$ or ${}^3D_4(k_2)$ is conjugate to H_1 through the automorphism $\chi\rho^{-1}$ of H_2. The lemma is now a consequence of the description of the automorphisms of the groups $G_2(k_2)$ and ${}^3D_4(k_2)$ [10].

We will reduce the proof of the lemma to the case already considered. If the group $\mathfrak{H}$ is of type G_2 then $\mathfrak{H}$ has a nontrivial rational representation obtained by reduction modulo p of the seven-dimensional representation of the complex Lie algebra of type G_2 [15]. If $\mathfrak{H}$ is of type D_4, then $\mathfrak{H}$ has a natural eight-dimensional representation $\mathfrak{H} \to \Omega_8(K)$. Correspondingly to the two $\mathfrak{H}$, $\dim V \leqslant 7$ or 8. Let V^- and V^+ be the subspaces of V consisting of the eigenvectors of the involution t belonging to the eigenvalues -1 and 1, respectively. Since $t \in M_2$ and $t \in L_2$ the groups M_2, L_2 act faithfully on V^-, hence $\dim V^- \leqslant 4$. Furthermore, $\dim V \leqslant 8$ and $\det t = 1$, hence $\dim V^- = 4$ or 6. If $\dim V^- = 6$ then $\dim V^+ \leqslant 2$, and it is easy to see that L_2M_2 centralizes V^+. But then the multiplicity of the eigenvalue -1 for an involution from $L_2M_2 - \langle t\rangle$ equals 2, contradicting the fact that all involutions in H_2 are conjugate. Therefore $\dim V^- = 4$, and the same reasoning shows that M_2L_2 does not centralize V^+. Every nonidentity normal subgroup of M_2L_2 equals $\langle t\rangle$, L_2 or M_2. One verifies readily that one of the groups M_2, L_2 (we shall denote it by X_2) centralizes V^+ and the other (denoted by Y_2) has a nontrivial action on V^+. Put $X_1 = X_2 \cap C_1$, $Y_1 = Y_2 \cap C_1$.

We put

$$\mathfrak{C}_1 = C_{\mathfrak{H}_1}(t), \ \ \mathfrak{C}_2 = C_{\mathfrak{H}_2}(t).$$

It is clear that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ normalize V^- and V^+. Put

$$\mathfrak{G}^- = N_{GL(V)}(V^-) \cap C_{GL(V)}(V^+).$$

Denote by $\mathfrak{C}_1$ and $\mathfrak{C}_2$ the images of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ under the natural embedding into $\mathfrak{G}^-$. It follows from the description of centralizers of involutions in the groups $G_2(K)$ [16] and $\Omega_8(K)$ [17] that

$$\mathfrak{X}_i * \mathfrak{Z}_i \simeq SL_2(K) * SL_2(K)$$

has finite index in $\mathfrak{C}_i$. The group X_i is contained in one of $\mathfrak{X}_i$, $\mathfrak{Z}_i$. We shall assume that $X_i \leqslant \mathfrak{X}_i$, i = 1, 2, thus $\mathfrak{X}_i \leqslant \mathfrak{G}^- \cap \mathfrak{C}_i$.

Let Z_i be the image of Y_i in $\mathfrak{G}^-$; then $Z_i \leqslant \mathfrak{Z}_i$. It follows from the description of the irreducible representations of $SL_2(K)$ [10, Sec. 12] that the representation of $\mathfrak{X}_i\mathfrak{Z}_i$ on V^- is equivalent to the tensor product of two-dimensional representations of the groups $\mathfrak{X}_i$ and $\mathfrak{Z}_i$, i = 1, 2, which implies, in particular, that the groups $\mathfrak{X}_1\mathfrak{Z}_1$ and $\mathfrak{X}_2\mathfrak{Z}_2$ are conjugate in $\mathfrak{G}^- \simeq GL(V^-)$. Put

$$(\mathfrak{X}_1\mathfrak{Z}_1)^g = \mathfrak{X}_2\mathfrak{Z}_2, \ \ g \in \mathfrak{G}^-.$$

The group $\mathfrak{G}^-$ contains an element which interchanges the subgroups $\mathfrak{X}_2$ and $\mathfrak{Z}_2$ under conjugation; hence, we may assume that $\mathfrak{X}_1^g = \mathfrak{X}_2$, $\mathfrak{Z}_1^g = \mathfrak{Z}_2$. It follows from the description of finite subgroups of $SL_2(K)$ [11, Theorem II.8.27] that every subgroup of $\mathfrak{X}_2\mathfrak{Z}_2$ which is isomorphic to X_1Z_1 is conjugate in $\mathfrak{X}_2\mathfrak{Z}_2$ to E_1, therefore $E_1^{gh} = E_1$ for some $h \in \mathfrak{X}_2\mathfrak{Z}_2$. Moreover, again by Theorem 11.8.27 of [11]

$$N_{\mathfrak{X}_1\mathfrak{Z}_1}(E_1)/Z(E_1) \simeq PGL_2(k_1) \times PGL_2(k_1)$$

or $PGL_2(k_1) \times PGL(^3k_1)$; hence $N_{\mathfrak{G}^-}(E_1) \cap N_{\mathfrak{G}}^-(\mathfrak{X}_1\mathfrak{Z}_1)$ induces the full automorphism group of the group E_1. Consequently

$$N_{\mathfrak{G}^-}(E_1) \leqslant C_{\mathfrak{G}^-}(E_1) \cdot N_{\mathfrak{G}^-}(\mathfrak{X}_1\mathfrak{Z}_1).$$

But E_1 acts irreducibly on V^-, hence by Schur's lemma

$$C_{\mathfrak{G}^-}(E_1) = Z(\mathfrak{G}^-)$$

and

$$gh \in N_{\mathfrak{G}^-}(E_1) \leqslant N_{\mathfrak{G}^-}(\mathfrak{X}_1\mathfrak{Z}_1).$$

On the other hand

$$(\mathfrak{X}_1\mathfrak{Z}_1)^{gh} = \mathfrak{X}_2\mathfrak{Z}_2,$$

and therefore $\mathfrak{X}_1\mathfrak{Z}_1 = \mathfrak{X}_2\mathfrak{Z}_2$, and $\mathfrak{X}_1 = \mathfrak{X}_2$ is an infinite subgroup of $\mathfrak{H}_1 \cap \mathfrak{H}_2$. Now the lemma is established.

LEMMA 6. Subgroups of type $^2G_2$ do not possess sporadic embeddings.

Proof. Let

$$H \leqslant G,\ H \simeq {}^2G_2(k_1),\ G \simeq {}^2G_2(k_2),$$

and assume that t is an involution from H. Then it is known that

$$C_H(t) \simeq \langle t\rangle \times PSL_2(k_1),\ C_G(t) \simeq \langle t\rangle \times PSL_2(k_2).$$

It follows from the description of the subgroups of $PSL_2(k_2)$ that k_2 is a subfield of k_2. Assume that $F \simeq {}^2G_2(k_1)$ is the subgroup of k_1-points of $G = {}^2G_2(k_2)$. We will show that F and H are conjugate in G. Since H and G have at most one class of involutions, we may assume that $t \in F$. Moreover, $C_G(t)$ contains only one class of subgroups isomorphic to $PSL_2(k_1)$ [11, Theorem II.8.27]; therefore we may replace H by some subgroup of G conjugate to it and then assume that $C_F(t) = C_H(t)$. Let T be a Sylow 2-subgroup of $C_F(t)$. It is known that $N_G(T)$, $N_H(T)$, and $N_F(T)$ are extensions of $T \simeq Z_2 \times Z_2 \times Z_2$ by the Frobenius group of order 21; therefore

$$N_H(T) = N_F(T) \leqslant H \cap F,$$

hence $H \cap F > C_F(t)$.

By Lemma 12.2 in [18] $C_F(t)$ is a maximal subgroup of F; therefore $H \cap F = F$ and H = F. The lemma is now established.

The proof of the theorem is therefore complete for the case when the groups G_i have rank $\leqslant 2$. We will now deal with the case where the rank of G_i is $\geqslant 3$.

We choose in every group G_i a Sylow p-subgroup U_i such that $U_i \leqslant U_j$ for $i \leqslant j$, $i, j = 1, 2, \dots$.

LEMMA 7. There exists a natural number i_0 such that for every p-subgroup $Q \geqslant U_{i_0}$ of G the group $N_G(Q)$ is p-closed. (Recall that a group is called p-closed if it contains a normal Sylow p-subgroup.)

Proof. Assume that V is a vector space over $\overline{GF(p)}$ on which the group G acts faithfully. We define inductively for every p-subgroup Q of G

$$V^Q_{(0)} = C_V(Q)$$

and $V^Q_{(j+1)}$ as the inverse image in V of the space

$$C_{V/V^Q_{(j)}}(Q).$$

Put

$$\operatorname{Stab} Q=\bigcap_{j=0}^{\infty} N_G(V_{(j)}^Q).$$

Clearly,

$$\operatorname{Stab} Q=\bigcap_{j=0}^{n} N_G(V_{(j)}^Q),$$

where n = dim V.

Put

$$U=\bigcup_{i=0}^{\infty} U_i,$$

then U is a Sylov p-subgroup of G. Since the space V is finite-dimensional we find

$$V_{(j)}^{U_i}=V_{(j)}^{U}$$

for all $j \geqslant 0$ and all i larger than some i_0. It follows that

$$N_G(U_i)\leqslant \operatorname{Stab} U$$

for $i \geqslant i_0$. Note that U is a Sylov p-subgroup in Stab U. The subgroup of Stab U generated by all p-elements centralizes every factor of the series

$$\{V_j^U\}$$

and is therefore nilpotent. Consequently, Stab U is a p-closed group, Stab U = $N_G(U)$.

Now assume that the p-subgroup Q of G contains U_{i_0}. We construct inductively

$$Q_1=N_Q(U_{i_0}),\quad Q_{j+1}=N_Q(Q_j).$$

Since Q is a nilpotent group, the series $\{Q_j\}$ reaches Q in a finite number of steps. Clearly,

$$Q_1\leqslant \operatorname{Stab} U_{i_0},$$

and according to the previous paragraph, $Q_1 \leqslant U$, and Stab Q_1 = Stab U. Moreover,

$$Q_{j+1}\leqslant \operatorname{Stab} Q_j\leqslant \operatorname{Stab} U.$$

Therefore

$$\operatorname{Stab} Q_{j+1}=\operatorname{Stab} U$$

and

$$\operatorname{Stab} Q=\operatorname{Stab} U.$$

Consequently $N_G(Q) \leqslant$ Stab U is a p-closed group. The lemma is established.

LEMMA 8. The group G contains a BN-pair whose Weyl group is isomorphic to the Weyl group of the groups G_i, i = 1, 2,... .

Proof. We choose i_0 as in the previous lemma and change the sequence $\{G_i\}$ so that i_0 = 1. The theorem of Borel–Tits about parabolic subgroups [19] in G_{i+1} implies that there exists a parabolic subgroup P_{i+1} containing $N_{G_{i+1}}(U_i)$ where $U_i \leqslant R_u(P_i)$. In view of the previous lemma P_{i+1} is p-closed and therefore is a Borel subgroup of G_{i+1}. Therefore we may assume without loss of generality that the U_i are unipotent radicals of Borel subgroups $B_i = U_iH_i$ of the groups G_i (where the H_i are tori) such that $U_i \leqslant U_j$ and $H_i \leqslant H_j$ for $i \leqslant j$. Put

$$U=\bigcup_{i=1}^{\infty} U_i,\quad H=\bigcup_{i=1}^{\infty} H_i,\quad B=UH.$$

It follows from the classification of tori in finite groups of Lie type [20, Chap. II, Sec. 1] that H_i is a maximal torus in G_i and

$$N_G(H_i)=H\cdot N_{G_1}(H_1),$$

and, in particular, the Weyl group

$$W = N_{G_1}(H_1)/H_1$$

covers all Weyl groups

$$N_{G_i}(H_i), \quad i = 1, 2, \ldots.$$

Put $N_i = N_{G_i}(H_i)$. Then (B_i, N_i) is a BN-pair in the group G_i. This means that the following axioms hold [4, 3.2.1]:

(0) B_i and N_i generate G_i;

(1) $B_i \cap N_i = H_i \triangleleft N_i$;

(2) the group $W_i = N_i/H_i$ can be generated by a set R_i of involutions such that for all $r \in R_i$ and $w \in W_i$ we have

$$rBw \subset BwB \cup BrwB$$

and

$$rBr \neq B.$$

(We recall that the sets of the form rBw are correctly defined as $\bar{r}B\bar{w}$ where $\bar{r} \in r, \bar{w} \in w$ are representatives of the cosets of H_i and N_i, and are independent of the choice of representatives.)

We may therefore assume that all R_i lie in W_1. The group W_1 is finite, therefore infinitely many of the sets R_i are identical. Eliminating unnecessary terms from the sequence $\{G_i\}$ we see that all $R_i = R_1$, $i = 1, 2, \ldots$. It is now clear that for the groups B, $N = N_G(H)$ and the set $R_1 \subset N/H$ axioms (0), (1) and the first part of (2) are satisfied. Moreover, $rB_1r \neq B$ for $r \in R$, therefore $\langle rB_1r, B_1\rangle$ is not p-closed. Hence $\langle rBr, B\rangle$ is not p-closed and $rBr \neq B$. We have thus constructed a BN-pair in the group G with finite Weyl group W. The lemma is established.

We can now complete the proof of the theorem by induction on the rank n of the BN-pair of G. The case $n \leqslant 2$ has been dealt with in Lemmas 3-6, therefore we will assume that $n \geqslant 3$. We will apply the classification theorems from [4] to identify the group G with the group of k-rational points of a simple algebraic k-group for a suitable locally finite field k. For the remainder of the proof references like 3.2.6 refer to point 3.2.6 in [4]. The terminology of [4] is adapted to the one of [3].

Let Δ be the building corresponding to the BN-pair (B, N) of the group G (3.2.6), C a chamber in Δ. The Weyl group of the building Δ is isomorphic to the finite group $W = N/B \cap N$ (3.11), therefore the diagram diagr Δ of the building Δ (3.8) is one of the Coxeter diagrams A_n, C_n, $n \geqslant 3$, D_n, $n \geqslant 4$, E_6, E_7, E_8, F_4 (2.14).

We label the vertices of diagr Δ by numbers from $I = \{1, \ldots, n\}$ such that two outer vertices connected by an edge of multiplicity 1 have labels 1 and 2, for example:

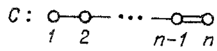

Diagram 1

Let X be a cell of C of type $I - \{1, 2\}$. Then the star StX (1.1) is isomorphic to a flag complex in the projective plane (3.12; 6.3).

On the other hand, let P be the stabilizer in G of the cell X under the natural action of G on Δ; then P is inductive limit of the parabolic subgroups $P \cap G_i$ of G_i, and the Levi factor $L = P/R_u(P)$ is inductive limit of the Levi factors

$$L_i = P_i/R_u(P_i), \quad i = 1, 2, \ldots.$$

Denote by L_i^u the group generated in L_i by the unipotent elements; then all L_i^u are of type A_2 (3.12) and by Lemmas 4 and 5 L^u is a group of type A_2 over some locally finite field k.

As in 5.2 one verifies that StX is canonically isomorphic to the building of the group L. Therefore StX is a flag complex in the projective plane over the field k. If diagr Δ is of type C_n this guarantees that the projective plane of the polar space associated with Δ (7.4, 7.9) is desarguesian. If diagr Δ is of type F_4 the same considerations enable us to exclude cases (iii) and (iv) of Theorem 10.2 which enumerates the possibilities for the structure of Δ.

Now we can use Theorems 6.6, 6.13, 8.32, 10.2, 10.4 to verify that the group of special automorphisms of the building Δ is isomorphic to an extension of the group $\mathfrak{G}$ of k-automorphisms of some absolutely simple k'-group (k' is some finite extension of the field k) by the automorphism group of k'. Since G is a simple group and the group of automorphisms of a locally finite field is Abelian, G can be embedded into $\mathfrak{G}$.

If we now show that $G=\mathfrak{G}^u$, the theorem follows from the classification of semisimple algebraic groups [21]. In $\mathfrak{G}$ we choose a BN-pair $\mathfrak{B}, \mathfrak{N}$ such that $B \leqslant \mathfrak{B}, N \leqslant \mathfrak{N}$ (which is possible by 3.11).

If $\mathfrak{P}$ is a parabolic subgroup of $\mathfrak{G}$, then $\mathfrak{P}$ is the stabilizer of some $X \in \Delta$ and P = $\mathfrak{P} \cap G$ is a parabolic subgroup of G. As pointed out already, the buildings of the Levy factors L(P) and $L(\mathfrak{P})$ of the groups P and $\mathfrak{P}$ are isomorphic to the building of StX. The group $L(P)^u$ is the inductive limit of the groups $L(P \cap G_i)^u$, i = 1, 2,... which are of smaller rank than G; therefore, by the inductive hypothesis, $L(P)^u$ is product of normal subgroups each of which is isomorphic to some Chevalley group (possibly with center different from the identity) over a suitable locally finite field. By 5.8 $L(P)^u \simeq L(\mathfrak{P})^u$ and $P=G \cap \mathfrak{P}$ covers the factor group

$$L(\mathfrak{P})^u=\mathfrak{P}^u/R_u(\mathfrak{P}^u).$$

Since this holds for every parabolic subgroup of $\mathfrak{P}$ and $\mathfrak{G}$ one obtains readily that $G=\mathfrak{G}^u$.

Indeed, let Φ^+ be a system of positive roots related to $\mathfrak{B}, \mathfrak{N}$, and $\mathfrak{U}_\alpha$ be the root subgroup corresponding to the root $\alpha \in \Phi^+$. Put $\mathfrak{P}_\alpha=\langle \mathfrak{B}, \mathfrak{U}_{-\alpha}\rangle$ and $\mathfrak{Z}_\alpha=\langle \mathfrak{U}_\alpha, \mathfrak{U}_{-\alpha}\rangle$ (cf. [22, 5.12, 4.2]). The group $\mathfrak{N}$ is generated by all its subgroups of the form $\mathfrak{N} \cap \mathfrak{P}_\alpha$ for $\alpha \in \Phi^+$. Since the image of $N \cap \mathfrak{P}_\alpha^u$ in

$$L\left(\mathfrak{P}_\alpha^u\right)=\mathfrak{P}_\alpha^u/R_u\left(\mathfrak{P}_\alpha^u\right)$$

covers the image of $\mathfrak{N} \cap \mathfrak{P}_\alpha^u$ in $L(\mathfrak{P}_\alpha)^u$, it follows that $N=\mathfrak{N}$ and H is a maximal torus in $\mathfrak{N}$. Let H_α = ker α; then $\mathfrak{Z}_\alpha=C_{\mathfrak{G}}(H_\alpha)$ is a reductive group, and in particular

$$C_{R_u(\mathfrak{P}_\alpha)}(H_\alpha)=1.$$

Put $P_\alpha=G \cap \mathfrak{P}_\alpha$. Then

$$C_{P_\alpha/R_u(P_\alpha)}(H_\alpha)$$

covers

$$C_{\mathfrak{P}_\alpha/R_u(\mathfrak{P}_\alpha)}(H_\alpha).$$

On the other hand, H_α and $R_u(P_\alpha)$ are locally finite groups in which the periods of elements are relatively prime; therefore it is easy to see that $C_{P_\alpha}(H_\alpha)$ covers

$$C_{P_\alpha/R_u(P_\alpha)}(H_\alpha).$$

Consequently $C_{P_\alpha}(H_\alpha)$ covers

$$\mathfrak{Z}_\alpha R_u(\mathfrak{P}_\alpha)/R_u(\mathfrak{P}_\alpha).$$

But

$$C_{P_\alpha}(H_\alpha) \leqslant H_\alpha \mathfrak{Z}_\alpha,$$

therefore

$$C_{P_\alpha}(H_\alpha)=H_\alpha \mathfrak{Z}_\alpha$$

and $\mathfrak{Z}_\alpha \leqslant G$. Since $\mathfrak{G}^u$ is generated by all subgroups $\mathfrak{Z}_\alpha$ for $\alpha \in \Phi^+$, we have $G=\mathfrak{G}^u$. This concludes the proof of the theorem.

LITERATURE CITED

1. The Kourovka Notebook (Unsolved Problems in Group Theory), Amer. Math. Soc. Translations, Ser. 2, Vol. 121, Providence (1983).
2. O. H. Kegel, "Über einfache lokal endliche Gruppen," Math. Z., 95, No. 3, 169-195 (1967).
3. N. Bourbaki, Groupes et Algebres de Lie, Chaps. IV-VI, Hermann, Paris (1968).
4. J. Tits, Buildings of spherical type and finite BN-pairs, Springer-Verlag, Berlin (1974).
5. R. W. Carter, "Simple groups and simple Lie algebras," J. London Math. Soc., 40, 193-240 (1965).
6. Yu. I. Merzlyakov, Rational Groups [in Russian], Nauka, Moscow (1980).
7. J. E. Humphreys, Linear Algebraic Groups, Springer-Verlag, New York (1975).
8. N. Blackburn, "Some remarks on Cernikov p-groups," Illinois J. Math., 6, No. 3, 421-433 (1962).
9. I. I. Pavlyuk and V. P. Shchunkov, "Locally finite groups satisfying min-p for all p," in: Proceedings of the Seventh All-Union Symposium on Group Theory [in Russian], Krasnoyarsk (1980), pp. 84-85.
10. R. Steinberg, Lectures on Chevalley Groups, Mimeographed Notes, Yale (1967).
11. B. Huppert, Endliche Gruppen, I, Springer-Verlag, Berlin (1967).
12. A. E. Zalesskii and V. N. Serezhkin, "Linear groups generated by transvections," Izv. Akad. Nauk SSSR, Ser. Mat., 40, No. 1, 26-49 (1976).
13. R. Steinberg, "Endomorphisms of Linear Algebraic Groups," Mem. Am. Math. Soc., No. 80, Providence (1968).
14. N. Burgoine, R. Griess, and R. Lyons, "Maximal subgroups and automorphisms of Chevalley groups," Pac. J. Math., 71, No. 2, 365-403 (1977).
15. A. Borel, "Properties and linear representations of Chevalley groups," in: Seminar on Algebraic Groups and Related Finite Groups, Lecture Notes in Mathematics, 131, 1-55 (1970).
16. R. Ree, "A family of simple groups associated with the simple Lie algebra G_2," Am. J. Math., 83, No. 3, 432-462 (1961).
17. J. Dieudonne, La Geometrie des Groupes Classiques, Springer-Verlag, Berlin (1955).
18. T. M. Gagen, Topics in Finite Groups, London Math. Soc. Lecture Notes Series, Vol. 16, Cambridge Univ. Press (1976).
19. A. Borel and J. Tits, "Elements unipotents et sous-groupes paraboliques de groupes reductifs," Inv. Math., 12, 95-104 (1971).
20. T. A. Springer and R. Steinberg, "Conjugacy classes," in: Lecture Notes in Mathematics, Vol. 131, Springer-Verlag, Berlin (1970).
21. J. Tits, "Classification of algebraic semisimple groups," in: Algebraic Groups and Discontinuous Subgroups, Am. Math. Soc., Providence (1966), pp. 33-62.
22. A. Borel and J. Tits, "Groupes reductifs," Publ. IHES, 27, 55-150 (1965).