coefficient can be bounded above by a constant, independent of the map η. Let V be a ball in M with respect to the Kobayashi metric, such that the closure of V in M is compact (the existence of such a ball follows from the condition that M is hyperbolic), U be a ball of the same radius in Δ with respect to the hyperbolic metric. Since a holomorphic map from Δ to M does not increase the distance with respect to these metrics [7, p. 311], the inequality $\|f\circ\eta\|_U \leqslant \|f\|_V$ is valid (the norms are defined as in Corollary 1). It follows from the explicit expressions for the Bloch seminorm (1) and the element of length $((1-|z|^2)|dz|)$ that the Lipschitz coefficient coincides with the Bloch seminorm b. Using Banach's open mapping theorem, it is easy to prove that the norms $\|\cdot\|_U$ and b on the space of Bloch functions on Δ, equal to zero at zero, are equivalent, from which we get the assertion formulated above.

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ON POTENTIALS OF MEASURES IN BANACH SPACES

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Let $(X, |\cdot|)$ be a real or complex Banach space, K a continuous function on the semiaxis $t \geqslant 0$, and μ a charge on X (here and below, charges and measures are assumed to be regular, Borel, and of bounded variation). Under the assumption of absolute convergence, we consider the potentials

$$u^{\mu}_{K,\alpha}(x) = \int_X K(\alpha|x-\xi|)\,d\mu(\xi), \quad x \in X, \quad \alpha \in \mathbf{R}, \quad \alpha \geqslant 0.$$

The problem consists in the elucidation of the connections between X and K under which the charges are uniquely determined by their potentials; i.e., from the equalities $u^{\mu}_{K,\alpha}(x) = 0$ for all $x \in X$ and $\alpha \geqslant 0$ there follows that $\mu = 0$ (weak uniqueness). We are interested also in the conditions under which one value $\alpha = 1$ is sufficient for the unique determination of the charge (strong uniqueness). When $K(t) = t^{\lambda}$, $\lambda \in \mathbf{R}$, the weak and the strong uniquenesses coincide; the values of λ and the functions K, for which uniqueness fails, are said to be exceptional. We note that for $X = \mathbf{R}^n$ and $-n < \lambda < 0$ the problem reduces to the classical uniqueness problem for Riesz potentials (see, for example, [1]).

The reason of the recent strengthened interest in the considered uniqueness problem has been initially the relationship between this question and the problem of the description of the isometric operators on the subspaces of L^p, observed for the first time in 1970 by Plotkin [2, 3], who has proved that in the one-dimensional case (both real and complex) uniqueness takes place for all $\lambda > 0$, with the exception of even numbers.

Subsequently (starting with [4]), in some investigations [5-8], by various methods one has also studied applications to the description of L^p-isometries, known earlier from [9, 10].

The generalizations of the one-dimensional uniqueness theorems which have appeared later have been connected with the attempts of describing the isometries in Orlicz spaces (see

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[11, 12], where in the one-dimensional case one has proved strong and weak uniqueness for certain classes of kernels K, containing power functions), and also in vector-valued L^p-spaces. It has been established in [13, 14] that for $X=\mathbf{R}^n$ and for an infinite-dimensional Hilbert space X only the even values of λ are exceptional (in 1985, a new proof for an infinite-dimensional Hilbert space has been communicated to the author by W. Linde). In [15] it has been proved that uniqueness holds for all noninteger $\lambda > 0$ in the case of real spaces C(Q) of continuous functions on a metric compactum Q.

In this paper we obtain uniqueness theorems that strengthen the mentioned results. We show that for a finite-dimensional ℓ_p space ($X = \ell_p^n$, $1 \leqslant p < \infty$), those values of the exponent λ are exceptional for which $\lambda/p \in \mathbf{N}$ and, in addition, one of the following three conditions holds: $\lambda/p < n$, p is even, p and $\lambda/p - n$ are odd numbers. For complex spaces ℓ_∞^n the exceptional values of λ are even, while in the real case they are those for which $\lambda + n$ is odd. We prove a theorem on the weak uniqueness for measures on a line, showing that for $X=\mathbf{R}$, among the functions K(t) with powerlike growth order at infinity, whose Fourier transforms are locally summable outside zero, only the polynomials with even powers of t are exceptional. In the case when X is the p-sum of an infinite number of smooth Banach spaces (in particular, for infinite-dimensional L^p spaces), the exceptional values of λ are the multiples of p (i.e., $\lambda/p \in \mathbf{N}$). One obtains strong uniqueness also for the kernels K which are linear combinations of power functions. Finally, if X = C(Q), where Q is an infinite metric compactum without isolated points, weak convergence holds for a wide class of kernels K, including all the convex nonconstant functions.

In the finite-dimensional case the proofs are based on the methods of the harmonic analysis of distributions (i.e., generalized functions over S); moreover, the potential is considered as a convolution of distributions, connected in the usual manner with the Fourier transform. A fundamental technical step is the description of the zeros of the Fourier transforms of distributions of a special form.

In the infinite-dimensional case, the method based on the Fourier transforms of distributions cannot be applied. So far one has not succeeded to develop a unique approach which would allow to reduce the problem to the solving of concrete technical questions. In the obtained solutions for infinite-dimensional spaces one has used in an essential manner special properties of the norms of these spaces. Thus, the short proof of the uniqueness theorem for infinite p-sums of Banach spaces has been possible due to a simple property of the norm, communicated to the author by W. Linde in the Hilbert case (recently, W. Linde has communicated that he has also obtained the proof of the theorem for infinite-dimensional L^p-spaces). Hope for the appearance of a unique approach is connected with the estimates of the potentials of measures in an arbitrary Banach space, obtained with the aid of an infinite-dimensional variant of a lemma by Cartan on coverings (see [16-18]), given in the last section of the paper.

The simplicity of the proof in the infinite-dimensional case, as well as the width of the class of kernels in the theorem for X = C(Q), can be explained, apparently, by the fact that the extent of the information on the measure (the values of the potentials are known at all the points of the space) exceeds significantly the "dimension" of the measure (in an infinite-dimensional space the measure has a thin support). In connection with this the following question is of interest: is the measure uniquely determined by the values of the potential at the points belonging to the support of the measure?

The discussed uniqueness problem is directly related with problems of completeness of special systems of functions and also with the question of the unique determination of a random process by its deviations from the elements of a Banach space [15]. From our uniqueness theorems one obtains directly the equimeasurability of functions and of their images under certain linear isometric mappings of subspaces of vector-valued L^p-spaces (see [13, 14]).

The fundamental results of this paper have been communicated in [19].

1. Some Preliminary Facts from the Harmonic Analysis of Distributions

As usual, let $S=S(\mathbf{R}^n)$ be the space of fast decreasing functions in $\mathbf{R}^n$. By definition, two distributions $f_1, f_2 \in S'$ coincide on an open set $\Omega \subset \mathbf{R}^n$, if $(f_1 - f_2)|_{D(\Omega)} = 0$, where $D(\Omega)$ is the space of functions $\varphi \in S$ with compact supports, contained in Ω. A distribution has the type of a locally summable function g on Ω if $\langle f, \varphi\rangle = \int_{\mathbf{R}^n} \bar{g}\, \varphi\, dx$ for all $\varphi \in D(\Omega)$. If $f \in S', \varphi \in S$, then the convolution $f*\varphi$ is defined in the usual manner and is a distribution of the type of the $C^\infty(\mathbf{R}^n)$-functions; in addition $(f*\varphi)^\wedge=\hat{f}\cdot\hat{\varphi}$, where $\hat{f}$ is the Fourier transform [20]).

Let R be a subset in S. The R-convolution of the distributions $f_1, f_2\in S'$ is defined to be the set (possibly empty) of all distributions f such that

$$(f*(\varphi_1*\varphi_2))(x)=\int_{\mathbf{R}^n}(f_1*\varphi_1)(\xi)(f_2*\varphi_2)(x-\xi)\,d\xi \tag{1}$$

for all $\varphi_1, \varphi_2\in R$ and $x\in\mathbf{R}^n$.

The defined convolution is connected in the usual manner with the Fourier transform.

LEMMA 1. Let Ω be an open set in $\mathbf{R}^n$ and let f_1, f_2 be distributions whose Fourier transforms $\hat{f}_i$ coincide on Ω with the functions $g_i\in L^2(\Omega)$, i = 1, 2. Then the $D(\Omega)^\wedge$-convolution of these distributions is nonempty and the Fourier transform of any element of the convolution coincides on Ω with g_1g_2 [here $R = D(\Omega)^\wedge$ is the space of the Fourier transforms of the functions from $D(\Omega)$].

Proof. We consider arbitrary $\varphi_1, \varphi_2\in D(\Omega)^\wedge$. Then $(f_i*\varphi_i)^\wedge=\hat{f}_i\hat{\varphi}_i=g_i\hat{\varphi}_i\in L^2(\Omega)$, i = 1, 2. By the Plancherel theorem [20] $f_i*\varphi_i\in L^2(\mathbf{R}^n)$, the integral in the right-hand side of (1) is absolutely convergent and represents a convolution of L^2 functions. Therefore the distribution f belongs to the $D(\Omega)^\wedge$-convolution of f_1 and f_2 if and only if $\hat{f}\hat{\varphi}_1\hat{\varphi}_2=((f_1*\varphi_1)*(f_2*\varphi_2))^\wedge=g_1g_2\hat{\varphi}_1\hat{\varphi}_2$ for any $\varphi_1, \varphi_2\in D(\Omega)^\wedge$. The last equality is equivalent with the equality of $\hat{f}$ and g_1g_2 on Ω.

LEMMA 2. Let μ be a charge on $\mathbf{R}^n$ and let f be Borel function on $\mathbf{R}^n$, such that $|f(x)|\leqslant C(1+|x|)^\rho$ for all $x\in\mathbf{R}^n$, where C, $\rho>0$, $\int_{\mathbf{R}^n}|f(a-x)|\,d|\mu|(x)<\infty$ and $\int_{\mathbf{R}^n}f(a-x)\,d\mu(x)=0$ for all $a\in\mathbf{R}^n$. If the function $\hat{\mu}$ does not have zeros in an open ball $B\subset\mathbf{R}^n$ and $\hat{f}/B\in L^1(B)$, then $(\hat{f},\varphi)=0$ for each function $\varphi\in D(B)$, i.e., $\hat{f}=0$ on B.

Proof. We consider a δ-sequence $\{\omega_i\}_{i=1}^\infty$, $\omega_i\in D(\mathbf{R}^n)$, such that $|\hat{\omega}_i(x)|\leqslant 1$ for all $x\in\mathbf{R}^n$ and $i\in\mathbf{N}$. Then $\lim_{i\to\infty}\hat{\omega}_i(x)=1$ for all $x\in\mathbf{R}^n$. We set $h_i=(\hat{\omega}_if)*\mu$, then $\hat{h}_i=(\hat{\omega}_if)^\wedge\hat{\mu}$, since $\hat{\omega}_if\in L^1(\mathbf{R}^n)$. By the theorem on the dominated convergence under the integral sign, we have $\lim_{i\to\infty}\langle\hat{h}_i,\psi\rangle=0$ for any $\psi\in S$. Indeed,

$$\int_{\mathbf{R}^n}\overline{(\hat{\omega}_if)^\wedge(t)\hat{\mu}(t)}\,\psi(t)\,dt=\langle\hat{h}_i,\psi\rangle=\langle h_i,\hat{\psi}\rangle=\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\overline{(\hat{\omega}_if)(a-x)}\,\hat{\psi}(a)\,\overline{d\mu(x)}\,da\xrightarrow[i\to\infty]{}$$

$$\xrightarrow[i\to\infty]{}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\overline{f(a-x)}\hat{\psi}(a)\,\overline{d\mu(x)}\,da=0$$

for any $\psi\in S$. Further, $\hat{\mu}(t)\neq 0$ for all $t\in B$, and, therefore, the set of functions $\bar{\hat{\mu}}\cdot\psi$, $\psi\in S$, is dense in the space $C_0(B)$ of all continuous complex functions with supports in B. For sufficiently large i and $h\in C_0(B)$ we have

$$\left|\int_{\mathbf{R}^n}(\hat{\omega}_if)^\wedge(t)\,h(t)\,dt\right|=\left|\int_B\hat{f}(t)(\omega_i*h)(t)\,dt\right|\leqslant\|h\|\int_B|\hat{f}(t)|\,dt.$$

By the Banach–Steinhaus theorem, for any $\varphi\in D(B)$ $\lim_{i\to\infty}\int_{\mathbf{R}^n}(\hat{\omega}_if)^\wedge(t)\,\varphi(t)\,dt=0$ and thus $\langle\hat{f},\varphi\rangle=\lim_{i\to\infty}\langle f,\hat{\omega}_i\hat{\varphi}\rangle=\lim_{i\to\infty}\langle(\hat{\omega}_if)^\wedge,\varphi\rangle=0$ for $\varphi\in D(B)$.

The lemma regarding the connection of the convolution with the Fourier transform reduces the uniqueness problem in the finite-dimensional case to the description of the zeros of the Fourier transform of the distribution $K(|x|)$. Indeed, Lemma 2 shows that strong uniqueness in the finite-dimensional case takes place if and only if $\widehat{K(|x|)}$ is not equal to zero on an open set. In order to describe the degeneracy cases of the Fourier transform of the function $|x|^\lambda$ on an open set we need some auxiliary statements. First we prove that the Fourier of the norm in ℓ_p^n (and of a series of other functions) is a real analytic function outside the coordinate planes in $\mathbf{R}^n$.

LEMMA 3. Suppose that $f\in C(\mathbf{R}^n)$ and extends to an analytic function on the domain $K_\varepsilon = \{z\in\mathbf{C}^n\colon |\mathrm{Im}\, z_k| < \varepsilon\, |\mathrm{Re}\, z_k|,\ 1\leqslant k\leqslant n\}$, where $\varepsilon > 0$, with the estimate $|f(z)|\leqslant A(1+|z|)^\rho$, A, $\rho>0$. Then $\hat{f}$ is a real analytic function outside the coordinate planes.

Proof. We consider the function $f_h(x)=f(x)e^{-h\langle x,x\rangle}$, $h>0$, $f_h\in L^1(\mathbf{R}^n)$. Then

$$\hat{f}_h(\xi) = \int_{\mathbf{R}^n} f_h(x)\, e^{-i\langle x,\xi\rangle}dx = \sum_\delta (\delta_1\cdot\ldots\cdot\delta_n)\int_0^\infty\ldots\int_0^\infty f_h(x)\, e^{i(\delta_1\cdot\xi_1 x_1+\ldots+\delta_n\xi_n x_n)}dx_1\ldots dx_n, \tag{2}$$

where the summation is carried out over all collections $\delta=\{\delta_k\}_{k=1}^n$, $\delta_k=\pm1$.

We fix a number σ, $0 < \sigma < \varepsilon$, and let $\xi=(\xi_1,\ldots,\xi_n)\in\mathbf{R}^n$, $\xi_k\neq0$ for $k = 1,\ldots,n$. For each choice of signs for δ, we consider the ray H_δ in $\mathbf{C}^n$: $H_\delta=\{z\in\mathbf{C}^n\colon \mathrm{Im}\, z_k=\sigma\delta_k\, \mathrm{sgn}(\xi_k)\,\mathrm{Re}\, z_k,\ k=1,\ldots,n\}$. By virtue of the analyticity of the function $f_h(z)=f(z)e^{-h\langle z,z\rangle}$ in the cone K_ε and the condition $\lim\limits_{|z|\to\infty} f_h(z)=0$, we can take in (2) integrals along the rays H_δ:

$$\hat{f}_h(\xi) = \sum_\delta(\delta_1\cdot\ldots\cdot\delta_n)\int_0^\infty\ldots\int_0^\infty f_h(x_1+i\sigma\delta_1\,\mathrm{sgn}(\xi_1)\,x_1,\ldots$$

$$\ldots, x_n+i\sigma\delta_n\,\mathrm{sgn}(\xi_n)\,x_n)\prod_{k=1}^n e^{i\delta_k\xi_k x_k}\prod_{k=1}^n e^{-\delta|\xi_k|x_k}dx_1\ldots dx_n. \tag{3}$$

For $h \geqslant 0$ the last integrals converge absolutely and extend to analytic functions in the cone $K_\sigma=\{\zeta=\xi+i\eta\in\mathbf{C}^n\colon |\eta_k|<\sigma|\xi_k|,\ k = 1,\ldots,n\}$. In addition, the functions $\hat{f}_h$ converge for $h \to 0$, uniformly on compacta lying inside the octants in $\mathbf{R}^n$, to the function defined by righthand side in (3) for $h = 0$ and thus, $\hat{f}$ coincides, outside the coordinate axes in $\mathbf{R}^n$, with a real analytic function.

We assume that the function f satisfies the conditions of Lemma 3 and, in addition, it is even with respect to each of the variables. Then either $\hat{f}$ does not vanish on any open set in $\mathbf{R}^n$, or $\hat{f} = 0$ inside all the octants in $\mathbf{R}^n$. The following lemma characterizes the distributions which are equal to zero outside the coordinate planes in $\mathbf{R}^n$.

LEMMA 4. Let g be a multiplier on S. If f is a distribution such that $f = 0$ on $\{x\colon g(x) \neq 0\}$, then $g^m f = 0$ for some positive integer m.

Proof. Let $\alpha(t)$ be an infinitely differentiable function on $[0, +\infty[$ such that $\alpha(t) = 0$ for $t \leqslant 1$ and $\alpha(t) = 1$ for $t \geqslant 2$. Let N be the order of the distribution f and let $m > N$. We set $g_{k,m}(x) = g^m(x)\alpha(kg^2(x))$. Then for any multiindex β, $|\beta|\leqslant N$, the functions $D^\beta(g_{k,m})$ converge uniformly on compacta as $k \to \infty$ to the function $D^\beta(g^m)$ and, consequently, the functions $D^\beta(g_{k,m}\varphi)$ converge uniformly as $k \to \infty$ to $D^\beta(g^m\varphi)$ for any $\varphi\in D(\mathbf{R}^n)$. In addition, $g_{k,m}f = 0$. Therefore, $\langle g^m f,\varphi\rangle=\langle f, g^m\varphi\rangle=\lim\limits_{k\to\infty}\langle f, g_{k,m}\varphi\rangle=\lim\limits_{k\to\infty}\langle g_{k,m}f,\varphi\rangle=0$ for any $\varphi\in D(\mathbf{R}^n)$.

We consider the multiplier $g(\xi)=\xi_1\cdot\ldots\cdot\xi_n$. If $\hat{f} = 0$ outside the coordinate planes, then $(\xi_1\cdot\ldots\cdot\xi_n)^m\hat{f}=0$ for some positive integer m. If $T=\partial^n/\partial x_1\ldots\partial x_n$, then by virtue of the known relation between differentiation and Fourier transform of distributions, we have $T^m f = 0$ everywhere in $\mathbf{R}^n$.

Let f be a continuous positive function in some ball $B(a,\varepsilon)=\{x\in\mathbf{R}^n\colon |x-a|<\varepsilon\}$. For each $\lambda \in \mathbf{C}$ the function f^λ is well defined in $B(a, \varepsilon)$.

LEMMA 5. If f extends to an analytic function on some domain $\Omega\subset\mathbf{C}^n$ and if this domain contains a point z such that $f(z) = 0$ but $\partial f/\partial z_k(z) \neq 0$ for all k, then the equality $T^m f^\lambda = 0$ in the ball $B(a, \varepsilon)$ is possible only for nonnegative integers $\lambda < mn$.

Proof. We assume that $T^m f^\lambda = 0$ in $B(a, \varepsilon)$. Then for any $x\in B(a,\varepsilon)$ we have

$$0=(T^m f^\lambda)(x)=\lambda(\lambda-1)\cdot\ldots\cdot(\lambda-mn+1)\prod_{j=1}^n\left(\frac{\partial f}{\partial x_j}(x)\right)^m f^{\lambda-mn}(x)+\sum_{k=1}^{mn-1}q_k f^{\lambda-k}(x),$$

where q_k are polynomials in partial derivatives. Multiplying this equality by $f^{mn-\lambda}$, we obtain

$$\lambda(\lambda-1)\cdot\ldots\cdot(\lambda-mn+1)\prod_{j=1}^n\left(\frac{\partial f}{\partial x_j}(x)\right)^m+\sum_{k=1}^{mn-1}q_k f^{mn-k}(x)=0.$$

By virtue of the analyticity of the function f, the last equality is satisfied everywhere in Ω and, in particular, at the point z. Therefore, $\lambda(\lambda-1)\cdot\ldots\cdot(\lambda-mn+1)=0$.

Thus, for a function f, satisfying the conditions of Lemmas 3 and 5, $\hat{f}^{\lambda}$ can be equal to zero on an open set only for nonnegative integers λ.

We consider a class of functions f, containing, in particular, the ℓ_p^n norm.

A function f on $\mathbf{R}^n$ is said to be quasihomogeneous if $f(t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x_n) = t^{\alpha} f(x_1, \ldots, x_n)$ for some $\alpha_i > 0$ and $\alpha > 0$ and for all $t > 0$ and $x \in \mathbf{R}^n$.

Let $n = n_1 + \ldots + n_r$ be a partition into positive integer terms and let $\mathbf{R}^n = \mathbf{R}^{n_1} \oplus \ldots \oplus \mathbf{R}^{n_r}$ be the corresponding partition of $\mathbf{R}^n$ with respect to the standard coordinates. An element of the space $\mathbf{R}^{n_k}$ will be denoted by y_k.

Let λ be a positive integer and let $f_k = f_k(y_k)$ be continuous quasihomogeneous functions on $\mathbf{R}^{n_k}$, such that the Fourier transforms of their positive integer powers (up to the λ-th) are real analytic in the balls $B(a_k, \varepsilon) \subset \mathbf{R}^{n_k}$. We set $q = q_1 \cdot \ldots \cdot q_r$, where q_k is a polynomial in n_k variables with complex coefficients, q(D) is the corresponding differential operator and $f(x) = \sum_{k=1}^{r} f_k(y_k)$.

LEMMA 6. If $q(D)f^{\lambda} = 0$ everywhere on $\mathbf{R}^n$, then for each partition $\lambda = \lambda_1 + \ldots + \lambda_r$ into nonnegative integer terms we have $(f_k^{\lambda_k})^{\wedge} = 0$ in $B(a_k, \varepsilon)$ for at least one k.

Proof. We apply induction on r. For r = 1 we have $q_1(D_1)(f_1^{\lambda}) = 0$, i.e., $q_1(\xi)\widehat{f_1^{\lambda}}(\xi) = 0$ for any $\xi \in B(a_1, \varepsilon)$, and $\widehat{f_1^{\lambda}} = 0$ everywhere in $B(a_1, \varepsilon)$. Further,

$$q(D)\left(\sum_{k=1}^{r} f_k\right)^{\lambda} = \sum_{\lambda_1=0}^{\lambda} C_{\lambda}^{\lambda_1} q_1(D_1)(f_1^{\lambda_1})(q_2(D_2)\cdot \ldots \cdot q_r(D_r))\left(\sum_{k=2}^{r} f_k\right)^{\lambda-\lambda_1} = 0. \quad (4)$$

Thus, some linear combination of distributions $q_1(D_1)(f_1^{\lambda_1})$ for $\lambda_1 = 0, \ldots, \lambda$ is equal to zero. Considering the Fourier transform of the sum in (4) as a distribution in $\mathbf{R}^{n_1}$, we obtain that a linear combination of the real analytic functions $(f_1^{\lambda_1})^{\wedge}$, $\lambda_1 = 0, \ldots, \lambda$ is equal to zero in the ball $B(a_1, \varepsilon)$. Further we note that the Fourier transform of a quasihomogeneous distribution with exponents $\alpha_1, \ldots, \alpha_n$ and α is also a quasihomogeneous distribution with exponents $\alpha_1, \ldots, \alpha_n$ and $-\alpha_1 - \ldots - \alpha_n - \alpha$. Thus $(f_1^{\lambda_1})^{\wedge}$ are quasihomogeneous distributions with the same exponents $\alpha_1, \ldots, \alpha_n^1$ but with different α. Consequently, the distributions $(f_1^{\lambda_1})^{\wedge}$ are linearly independent (if they are considered in an arbitrary ball). If a linear combination of these distributions is equal to zero in $B(a_1, \varepsilon)$, then for each $\lambda_1 = 0, \ldots, \lambda$ either $(f_1^{\lambda_1})^{\wedge} = 0$ in $B(a_1, \varepsilon)$ or the coefficient of $(f_1^{\lambda_1})^{\wedge}$ in the linear combination is equal to zero, i.e., $q_2(D_2)\cdot \ldots \cdot q_r(D_r)\left(\sum_{k=2}^{r} f_k\right)^{\lambda-\lambda_1} = 0$, and, by the induction hypothesis, for any partition $\lambda - \lambda_1 = \lambda_2 + \ldots + \lambda_r$ there exists $k \geqslant 2$ such that $(f_k^{\lambda_k})^{\wedge} = 0$ in $B(a_k, \varepsilon)$.

We set, in particular, $f_k(y_k) = |y_k|^{p_k}$, $p_k > 0$ (the Euclidean norm). The Fourier transforms of f_k as distributions in $\mathbf{R}^{n_k}$ can be easily computed [21]: for even p_k everywhere outside the coordinate planes we have $\hat{f}_k = 0$ and for other values of p_k we have $\hat{f}_k(\xi) = C|\xi|^{-p_k-n_k}$, $C \in \mathbf{R}$, $C \neq 0$, i.e., $\hat{f}_k$ is not equal to zero on any open set in $\mathbf{R}^{n_k}$.

LEMMA 7. The Fourier transform of the function $f(x) = \left(\sum_{k=1}^{r} |y_k|^{p_k}\right)^{\lambda}$, where $r \geqslant 2$, $p_k > 0$, $\lambda > 0$, is equal to zero on some open set in $\mathbf{R}^n$ if and only λ is a positive integer and, in addition, at least one of the following three conditions holds: $\lambda < r$; among the p_k's one has an even number; all the p_k's and $\lambda - r$ are odd numbers.

Proof. From Lemma 3 there follows that $\hat{f}$ is a real analytic function outside the coordinate planes $\xi_j = 0$, $1 \leqslant j \leqslant n$. We assume that $\hat{f} = 0$ everywhere inside an open set in $\mathbf{R}^n$. Then $\hat{f} = 0$ inside at least one octant. Since f is even with respect to each variable, $\hat{f}$ is also even with respect to each variable, and, consequently, $\hat{f} = 0$ everywhere outside the coordinate planes. By Lemma 4, this is possible only if $(\xi_1 \cdot \ldots \cdot \xi_n)^m \hat{f} = 0$ for some positive integer m, i.e., $T^m f = 0$ in $\mathbf{R}^n$, where $T = \partial^n/\partial x_1 \ldots \partial x_n$. For $r \geqslant 2$ the function f satisfies the conditions of Lemma 5 and, therefore, λ is a positive integer. Finally, by virtue of Lemma 6, for every partition of the number λ into nonnegative integers $\lambda = \lambda_1 + \ldots + \lambda_r$, at least for one k we have $(|y_k|^{\lambda_k p_k})^\wedge = 0$ on an open set, i.e., $\lambda_k p_k$ is an even number. If we assume now that $\lambda > r$ and among the numbers p_k there are no even ones, then it is easy to prove by induction on r (starting with r = 2 and considering partitions with $\lambda_1 = 1$ and $\lambda_1 = 2$) that all p_k and $\lambda - r$ are odd natural numbers.

Conversely, if λ is a positive integer and one of the three conditions of the lemma holds, then for any partition $\lambda = \lambda_1 + \ldots + \lambda_r$ into nonnegative integer terms, there exists an even number among the numbers $\lambda_k p_k$ and, therefore,

$$\hat{f}(\xi) = \sum_{\lambda=\lambda_1+\ldots+\lambda_r} \frac{\lambda!}{\lambda_1!\cdot\ldots\cdot\lambda_r!} \prod_{k=1}^{r} \left(|y_k|^{\lambda_k p_k}\right)^\wedge(\eta_k) = 0$$

for any $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$, $\xi_j \neq 0$, $j = 1, \ldots, n$, $\left(\xi = (\eta_1, \ldots, \eta_r),\ \eta_k \in \mathbf{R}^{n_k}\right)$.

2. A Uniqueness Theorem for Measures on a Line

First we consider the uniqueness problem on the line, i.e., for $X = \mathbf{R}$. As it follows from Lemma 2, the strong uniqueness on the line is equivalent to the fact that $\hat{K}$ does not degenerate on any open set. At the same time, weak uniqueness holds for all functions K of powerlike growth at infinity with regular Fourier transform, with the exception of polynomials in even powers.

THEOREM 1. Let K be a Borel function on $\mathbf{R}$ (on $\mathbf{C}$), admitting the estimate $|K(x)| \leqslant A(1+|x|^\rho)$, where $A, \rho > 0$, and $\hat{K}$ is locally summable outside zero. There exists a nontrivial charge μ on $\mathbf{R}$ (on $\mathbf{C}$), for which

$$\int |K(tx+a)|\, d|\mu|(x) < \infty,\ \int K(tx+a)\, d\mu(x) = 0$$

for all t, $a \in \mathbf{R}$ ($\in \mathbf{C}$), if and only if K is a polynomial.

Proof. We assume that there exists a nontrivial charge μ with the indicated properties. Then the continuous function $\hat{\mu}$ is different from zero on some open set H. Let $K_t(x) = K(tx)$ for $t \neq 0$. By Lemma 2, for any $\varphi \in D(H)$ and for any $t \neq 0$ we have $0 = \langle \hat{K}_t, \varphi\rangle = \langle \hat{K}, \varphi(x/t)\rangle$. Thus, for any number $t \neq 0$ and any $\psi \in D(tH)$ we have $\langle K, \hat{\psi}\rangle = 0$ and, consequently, K is a distribution with support in $\{0\}$. By Lemma 4 (or [20]), $\hat{K}$ is a linear combination of derivatives of δ-functions, i.e., K is a polynomial.

Conversely, if K is a polynomial of degree n, then for μ one can take any change for which the first n moments are equal to zero.

COROLLARY 1. Let X be an arbitrary real (complex) Banach space and let K be a Borel function on $\mathbf{R}$ (on $\mathbf{C}$), satisfying the conditions of the theorem. There exists a nontrivial charge μ on X for which

$$\int_X |K(\langle\xi, x\rangle + a)\, d|\mu|(x) < \infty,\ \int_X K(\langle\xi, x\rangle + a)\, d\mu(x) = 0$$

for all $\xi \in X^*$ and $a \in \mathbf{R}$ ($a \in \mathbf{C}$), if and only if K is a polynomial.

Proof. For any scalars t, a and for any $\xi \in X^*$ we have $\int_X K(t\langle\xi, x\rangle + a)\, d\mu(x) = 0$, and, after the change of variable $y = \langle\xi, x\rangle$, we obtain $\int K(ty+a)\, d\xi(\mu)(y) = 0$. If K is not a polynomial, then by Theorem 1 we have $\xi(\mu) = 0$ for any ξ and, therefore, the Fourier transform of the charge μ is equal to zero and $\mu = 0$.

3. Finite-Dimensional Spaces of Type ℓ_p

We consider the n-dimensional real space $\ell_p^{(n_1,\ldots,n_r)}$ with the norm $|x| = \left(\sum_{k=1}^{r} |y_k|^p\right)^{1/p}$, where $p \geqslant 1$ (we preserve the notations introduced in Sec. 1). We prove a uniqueness theorem for the kernel $K(t) = t^\lambda$, $\lambda > 0$.

THEOREM 2. Let $1 \leqslant p < \infty$, $r \geqslant 2$ and $\lambda > 0$. There exists a nontrivial charge μ on $X = \ell_p^{(n_1,\ldots,n_r)}$, for which $\int_X |x|^\lambda d|\mu|(x) < \infty$ and $\int_X |x-\xi|^\lambda d\mu(\xi) = 0$ for all $x \in X$, if and only if λ/p is a positive integer and, moreover, at least one of the following three conditions holds: $\lambda/p < r$; p is even; p and $\lambda/p - r$ are odd numbers.

Proof. By virtue of Lemma 3, the Fourier transform of the distribution $f(x) = |x|^\lambda$ is a real analytic function outside the coordinate planes. By Lemma 1, $\hat{f}(\xi)\hat{\mu}(\xi) = 0$ for any $\xi \in \mathbf{R}^n$, $\xi_j \neq 0$, $j = 1, 2, \ldots, n$. If $\hat{f}$ is not equal to zero on any open set, then the continuous function $\hat{\mu}$ is equal to zero everywhere in $\mathbf{R}^n$. If, however, $\hat{f} = 0$ on an open set, then $\hat{f} = 0$ everywhere inside the octants and as a nontrivial charge μ, satisfying the conditions of the theorem, one can take $d\mu(x)=\hat{\varphi}(x)dx$, where φ is an even function from $D(\mathbf{R}^n)$ with support inside the octants. Conditions for the equality $\hat{f} = 0$ on an open set are described in Lemma 7.

Thus, the answer does not depend on the magnitude of the n_k but on their number r. For example, in the real and the complex ℓ_p^n the sets of the exceptional exponents coincide.

The case r = 1, not covered by Theorem 2 (when X is a Euclidean space), has been considered earlier (see the introduction) and is derived from Lemma 1 by taking into account the known formula for the Fourier transform of the distribution $|x|^\lambda$.

From Theorem 2 one derives easily a theorem on strong uniqueness for $X = \ell_p^{(n_1,\ldots,n_r)}$ and kernels K which are linear combinations of power functions. Indeed, if $K(t)=\sum_{i=1}^{m}\alpha_i t^{\lambda_i}$, $\alpha_i \in \mathbf{C}$, $\alpha_i \neq 0$, $0<\lambda_1<\ldots<\lambda_n$, then $\widehat{K(|x|)}$ is a linear combination of homogeneous functions of different orders. If $\widehat{K(|x|)}$ vanishes on an open set, then $\widehat{|x|^{\lambda_i}}=0$ on an open set for each $i = 1,\ldots,m$, i.e., all the exponents λ_i are exceptional in the sense of Theorem 2. Thus, strong uniqueness holds for the kernel K if and only if one has uniqueness for at least one of the kernels $K_i(t) = t^{\lambda_i}$.

We consider now the case $p = \infty$. Here the singularities of the Fourier transform of the distribution $f(x)=|x|^\lambda$, where $|x|=\max_{k=1,\ldots,r}|y_k|$ is the norm in $\ell_\infty^{(n_1,\ldots,n_r)}$, are not restricted to the coordinate planes. In this case, in order to describe the Fourier transforms, we make use of explicit formulas connected with the Bessel functions.

THEOREM 3. Let $r \geqslant 1$ and $\lambda > 0$. There exists a nontrivial charge μ on $X = \ell_\infty^{(n_1,\ldots,n_r)}$, for which $\int_X |x|^\lambda d|\mu|(x)<+\infty$, $\int_X |x-\xi|^\lambda d\mu(\xi)=0$ for all $x\in X$, if and only if at least for one k, $1\leqslant k\leqslant r$, the number $\lambda + n - n_k$ is even.

Proof. The Fourier transform of the distribution $f_\lambda = |x|^\lambda$ admits the representation ($\xi = \xi_1+\ldots+\xi_r$, $\xi_k \in \mathbf{R}^{n_k}$, $\xi_k \neq 0$)

$$\hat{f}_\lambda(\xi) = -\frac{(2\pi)^{n/2}\lambda}{\prod_{k=1}^{r}|\xi_k|^{\frac{1}{2}n_k}}\int_0^\infty t^{\frac{1}{2}n+\lambda-1}\prod_{k=1}^{r}J_{\frac{1}{2}n_k}(|\xi_k|t)\,dt$$

(as usually, first we consider the case $-n < \lambda < 0$, when the integrals converge and then we extend analytically the obtained formulas for all values of λ). Making use of the known formulas and asymptotics for the Bessel functions J_ν [22], it is easy to show that $\hat{f}_\lambda(\xi)$ is a real analytic function in the domains where $\xi_k \neq 0$ and $\sum_{k=1}^{r}\pm|\xi_k|\neq 0$, while in the domains $\xi_k\neq 0$, $|\xi_j|>\sum_{k\neq j}|\xi_k|$, the Fourier transform has the form

$$\hat{f}_\lambda(\xi) = \frac{-2\lambda(2\pi)^{n/2}}{\prod_{k=1}^{r}|\xi_k|^{\frac{1}{2}n_k}}\cos\left(\frac{\pi}{2}(n-n_j+\lambda-1)\right)\int_0^\infty t^{\frac{1}{2}n-\lambda-1}K_{\frac{1}{2}n_j}(|\xi_j|t)\prod_{k\neq j}I_{\frac{1}{2}n_k}(|\xi_k|t)\,dt,$$

where I_ν, K_ν are the modified Bessel function of the first and third kind. From here it follows that the degeneracy of the Fourier transform on an open set is possible if and only if $n - n_k + \lambda - 1$ is an odd number for at least one k. Now the assertion of the theorem is derived from Lemma 1 in the same way as it has been done at the proof of Theorem 2. Thus, as in Theorem 2, one constructs an example of nontrivial charge in the exceptional cases.

In particular, for complex ℓ_∞^n, the even values of λ are exceptional, while for real ℓ_∞^n those values of λ are exceptional for which $\lambda + n$ is an odd number.

4. Infinite-Dimensional Case: Spaces of Type C( )

Let Q be a metric compactum, containing an infinite number of points and having no isolated points; X is a closed subspace of the space C(Q) of all continuous real (or complex) functions, such that for arbitrary $s_1, \ldots, s_n \in Q$ and constants $\zeta_1, \ldots, \zeta_n$, $|\zeta_k| = 1$, there exists a function $a \in X$, such that $a(s_k) = \zeta_k$ and $|a(s)| < 1$ for all $s \neq s_k$, $k = 1, \ldots, n$.

Let K(t) be a nonnegative nondecreasing convex function on the semiaxis $t \geqslant 0$, differentiable at the point t = 1 and such that $K'(1) > 0$.

For the space X and the function K we have a theorem on weak uniqueness.

THEOREM 4. If μ is a charge on X such that $\int_X K(t|x|)\,d|\mu|(x) < \infty$ for all $t \geqslant 0$ and $\int_X K(t|x-a|)\,d\mu(x) = 0$ for all $t \geqslant 0$, $a \in X$, then μ = 0.

Proof. We need the following known fact (see [23]): if a, b are arbitrary elements of a Banach space X, then

$$\lim_{0<t\to 0} \frac{|a+tb|-|a|}{t} = \sup_{x^* \in S(a)} \operatorname{Re} x^*(b) \leqslant |b|, \tag{5}$$

where $S(a) = \{x^* \in X^*\colon \|x^*\| = 1,\ x^*(a) = |a|\}$ is the set of the support functionals at the point a.

We consider arbitrary $a, b \in X$, $|a| = 1$ and $t \geqslant 0$. From the assumptions of the theorem we have

$$\int_X K(|t(x-b)+a|)\,d\mu(x) = 0. \tag{6}$$

Since K is a convex function, making use of the theorem on monotone convergence under the integral sign and of formula (5), we can differentiate the equality (6) with respect to t:

$$0 = \frac{d}{dt}\int_X K(|a+t(x-b)|)\,d\mu(x)|_{t=0} = K'(1)\int_X \sup_{x^* \in S(a)} \operatorname{Re} x^*(x-b)\,d\mu(x) \tag{7}$$

(the integral in the right-hand side is absolutely convergent since $|x| \leqslant \frac{K(|x|)-K(1)}{K'(1)} + 1$ for $|x| > 1$).

For functions $a \in X$, corresponding by virtue of the properties of the space X to the points $s_1, \ldots, s_n$ and to the numbers $\zeta_1, \ldots, \zeta_n$, equality (7) takes the form

$$\int_X \max_{k=1,\ldots,n} \operatorname{Re} \bar{\zeta}_k (x(s_k) - b(s_k))\,d\mu(x) = 0. \tag{8}$$

For each of the points s_k we select a sequence $\{s_{jk}\}_{j=1}^{\infty} \subset Q$, such that $\lim_{j\to\infty} s_{jk} = s_k$ (and, moreover, all the points s_{jk}, $k = 1, \ldots, n$, $j = 1, 2, \ldots$ are distinct). For each positive integer m we define the numbers $\varepsilon_1, \ldots, \varepsilon_m$: in the complex case they are the roots of order m of 1 and in the real case $\varepsilon_k = (-1)^k$. We set $\zeta_{jk} = \varepsilon_j$ for $k = 1, \ldots, n$.

Now we apply equality (8) to the points s_{jk} (instead of s_k) and to the points ζ_{jk} (instead of ζ_k), where $k = 1, \ldots, n$, $j = 1, \ldots, m$, and then we let m go to infinity. We obtain

$$\int_X \max_{k=1,\ldots,n} |x(s_k) - b(s_k)|\,d\mu(x) = 0,$$

where, as before, b is an arbitrary function from X.

We also note that the property of the subspace X guarantees the existence for any $s_1, \ldots, s_n \in Q$ and numbers $\xi_1, \ldots, \xi_n$ of a function $b \in X$, such that $b(s_k) = \xi_k$, $k = 1, \ldots, n$.

Thus, the charge, generated by the charge μ on ℓ_∞^n as a result of the mapping $x \mapsto \{x(s_k)\}_{k=1}^n$, satisfies the conditions of Theorem 3 and, consequently, it is equal to zero (in the real case one has to take an odd n). At the same time, the collection of finite-dimensional distributions, corresponding to the finite collections of points from Q, determines completely a probability measure on C(Q) (see, for example, [24, p. 33 of the Russian edition]; we note that the intersection of the balls in C(Q) is a limit of finite-dimensional sets). Therefore, μ = 0.

The convexity of the function K has been required only for the justification of the differentiation under the integral sign. The theorem is valid also for other classes of functions, for example, for functions that satisfy the Lipschitz condition, or for concave functions (under the additional condition $\int_X |x|\, d|\mu|(x) < \infty$).

5. Infinite-Dimensional L^p Spaces

If one performs in Theorem 2 a formal limiting passage as $n \to \infty$, then one can assume that for infinite-dimensional L^p-spaces the exceptional exponents will be the numbers that are multiple of p. We prove a somewhat more general fact: such exponents are exceptional for p-sums of an infinite number of smooth Banach spaces.

We make some preliminary remarks. If $p \geqslant 1$, $X_1, X_2, \ldots$ are Banach spaces, then $X = (X_1 \oplus X_2 \oplus \ldots)_p \doteq \{x = (x_1, x_2, \ldots) : x_i \in X_i, |x|^p = \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ is the p-sum of the spaces X_i. A Banach space is said to be smooth if at each point of its unit sphere there exists a unique support functional. It is easy to see that the p-sum of smooth spaces is also a smooth space for $p > 1$. In [25] one has formulated the following question: is a finite Borel measure, defined on a Banach space, uniquely determined by its values on balls? A positive answer to this question has been obtained in [25] for smooth spaces, for spaces C(Q) and L^1. However, the problem is not solved in the general case (see [26]).

THEOREM 5. Let $\lambda > 0$, $p \geqslant 1$, $X_1, X_2, \ldots, X_n, \ldots$ be an infinite sequence of smooth Banach spaces, $\dim X_i \geqslant 1$ for all $i \in \mathbf{N}$. There exists a nontrivial charge μ on $X = (X_1 \oplus X_2 \oplus \ldots)_p$, for which

$$\int_X |x|^\lambda\, d|\mu|(x) < \infty, \quad \int_X |x-a|^\lambda\, d\mu(x) = 0 \quad \forall a \in X$$

if and only if λ/p is a positive integer.

The proof is based on a simple property of the norm in X, which we have established by generalizing the method applied by W. Linde in the Hilbert case.

LEMMA 8. If the charge μ satisfies the conditions of Theorem 5, then $\int_X (|x-a|^p + c)^{\lambda/p} \times d\mu(x) = 0$ for arbitrary $a \in X$ and $c > 0$.

Proof. We fix a sequence $\{e_k\}_{k=1}^{\infty}$, $e_k \in X_k$, $|e_k| = 1$. For each $k \in \mathbf{N}$ we consider an element $a_k \in X$, for which the k-th coordinate is equal to e_k and the remaining ones are equal to zero. For any x, $a \in X$ and $c > 0$ we have $\lim_{k\to\infty} |x - a + ca_k|^p = |x-a|^p + c^p$. By the theorem on dominated convergence under the integral sign, we have

$$\int_X (|x-a|^p + c^p)^{\lambda/p}\, d\mu(x) = \lim_{k\to\infty} \int_X (|x-a+ca_k|^p)^{\lambda/p}\, d\mu(x) = 0.$$

The Proof of Theorem 5. Let $\lambda/p \notin \mathbf{N}$. We fix $a \in X$, we make use of Lemma 8, and we perform the change of variable $y = |x-a|$. Let $\mu_a(y) = \mu B(a, y)$ ($B(a, y)$ is the ball in X with center at the point a and radius y). Then $\int_0^\infty (y+c)^{\lambda/p}\, d\mu_a(y) = 0$ for every $c > 0$. Differentiating this equality with respect to c (which is possible by virtue of the dominant convergence), we obtain $\int_0^\infty (y+c)^{\lambda/p - n}\, d\mu_a(y) = 0$ for all $c > 0$ and $n \in \mathbf{N}$.

Now we fix $c > 0$. The function $k(\zeta) = \int_0^\infty (y+c)^{\lambda/p - \zeta}\, d\mu_a(y)$ is holomorphic in the semiplane $\operatorname{Re}\zeta > \lambda/p$, it is equal to zero for $\zeta = 0, 1, 2, \ldots$. In addition, $|k(\zeta)| \leqslant c^{\lambda/p - \operatorname{Re}\zeta} |\mu|(X)$ for $\operatorname{Re}\zeta > \lambda/p$. By Carleson's theorem (see [18, p. 7] of [27]) we have $k(\zeta) = 0$ for all ζ, $\operatorname{Re}\zeta > [\lambda/p] + 1$.

We perform the change of variables $\tau = \ln(y+c)$ in the integral which defines the function $k(\zeta)$ and we consider the values $\zeta = 1 + \lambda/p + it$, $t \in \mathbf{R}$, lying on a line parallel to the imaginary axis. By virtue of the standard uniqueness theorem for the Fourier transform, we have $\mu_a = 0$, i.e., the charge μ is equal to zero on all the balls with center at the point a. By the Hoffman–Jorgensen theorem [25], we have $\mu = 0$.

Finally, if λ/p is an integer, then we consider the positive integer $n > \lambda/p$ and arbitrary $e_i \in X_i$. Then the charge μ, equal to the sum of 2^n point charges, each of which is concentrated at the point $(\varepsilon_1 e_1, \ldots, \varepsilon_n e_n)$ and equal to $\varepsilon_1 \cdot \varepsilon_2 \cdot \ldots \cdot \varepsilon_n$, where $\varepsilon_i = \pm 1$, $i = 1, \ldots, n$, yields us the required example (after raising the norm to the power λ and opening the parentheses, at least one variable will be missing in each term).

We give another, somewhat more complex proof of Theorem 5, applied, however, to kernels which are linear combinations of power functions.

THEOREM 6. Let $n \in \mathbf{N}$, $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$, $\alpha_1, \ldots, \alpha_n \in \mathbf{C}\setminus\{0\}$, $K(t) = \sum_{k=1}^{n} \alpha_k t^{\lambda_k}$, $p \geq 1$, and let X be the p-sum of an infinite sequence of smooth Banach spaces. There exists a nontrivial charge μ on X, for which

$$\int_X K(\|x\|)\, d|\mu|(x) < \infty, \int_X K(\|x-a\|)\, d\mu(x) = 0$$

for all $a \in X$, if and only if all the numbers λ_k/p, $k = 1, \ldots, n$, are positive integers.

Proof. We consider an arbitrary $a \in X$, we apply Lemma 8 and we perform the change of variable $y = \|x-a\|$. We obtain that for all $s > 0$ we have

$$\sum_{k=1}^{n} \alpha_k \int_0^\infty (y+s)^{\lambda_k/p}\, d\mu_a(y) = 0. \tag{9}$$

We denote $z_k = \lambda_k/p$, $c_k = \frac{\alpha_k}{\pi} \sin \pi z_k \Gamma(z_k+1)$. We differentiate the equality (9) m times with respect to the variable s, $m \in \mathbf{N}$, $m > z_1$, and we make use of the known properties of the Γ-function [28, p. 773]:

$$0 = \sum_{k=1}^{n} \alpha_k z_k (z_k - 1) \cdot \ldots \cdot (z_k - m + 1) \int_0^\infty (y+s)^{z_k - m}\, d\mu_a(y)$$

$$= (-1)^m \sum_{k=1}^{n} \frac{\alpha_k}{\pi} \sin \pi z_k \Gamma(z_k+1)\, \Gamma(m - z_k) \int_0^\infty (y+s)^{z_k - m}\, d\mu_a(y)$$

for every $s > 0$. Since $m > z_k$ for all $k = 1, \ldots, n$, we can replace the Γ-function by an integral:

$$0 = \sum_{k=1}^{n} c_k \int_0^\infty x^{m - z_k - 1} e^{-x} dx \int_0^\infty (y+s)^{z_k - m}\, d\mu_a(y) = \sum_{k=1}^{n} c_k \int_0^\infty \int_0^\infty x^{m - z_k - 1} e^{-x} (y+s)^{z_k - m}\, dx\, d\mu_a(y).$$

After the substitution $t = x/(y + s)$ we obtain

$$0 = \sum_{k=1}^{n} c_k \int_0^\infty \int_0^\infty t^{m - z_k - 1} e^{-t(y+s)} dt\, d\mu_a(y) = \int_0^\infty \left(\sum_{k=1}^{n} c_k t^{m - z_k - 1} \right) \check{\mu}_a(t)\, e^{-ts} dt,$$

where $\check{\mu}_a$ is the Laplace transform of the charge μ_a. Since s is an arbitrary positive number, the Laplace transform of the function $\sum_{k=1}^{n} c_k t^{m - z_k - 1} \check{\mu}_a(t)$ is equal to zero; it follows that either $\check{\mu}_a(t) = 0$ for any $t > 0$ and thus $\mu_a = 0$, or $c_k = 0$ for all $k = 1, \ldots, n$, i.e., $\sin \pi z_k = 0$, $k = 1, \ldots, n$, and all numbers $z_k = \lambda_k/p$ are positive integers. It remains to note that from the equalities $\mu_a = 0$ for all $a \in X$ there follows $\mu = 0$ by virtue of the Hoffman–Jorgensen theorem.

We note that the necessity of the equality $\nu = 0$ for the charge ν under the condition that $\int_0^\infty (y+c)^p\, d\nu(y) = 0$ for all $c > 0$ and noninteger p has been established for the first time by Plotkin in [10]. We have given here other proofs of this fact, based on a theorem by Carleson and on the Laplace transform. The Carleman criterion for determinate moment problems (see [29]) also allows us to obtain a simple proof. We note that in order to establish the equality $\nu = 0$ with the aid of Carleson's theorem, it is sufficient to have the values of c belonging to any interval.

6. An Infinite-Dimensional Variant of Cartan's Lemma on Coverings; Estimates for the Potentials of Measures

Let X be a complete metric space and let μ be a measure on X (as before, we consider finite Borel measures). We consider a continuous, strictly increasing, nonnegative function φ on $[0, +\infty[$, $\varphi(0)=0$, $\lim\limits_{x\to\infty}\varphi(x)>\mu(X)$. The function φ will be called a majorant.

For each point $x\in X$ we set $\tau(x)=\sup\{t: \mu B(x, t)\geqslant\varphi(t)\}$, where B(x, t) is the closed ball in X with center x and radius t. It is easy to see that $\mu B(x, \tau(x))=\varphi(\tau(x))$ and $\sup\limits_x\tau(x)\leqslant\varphi^{-1}(\mu(X))<\infty$.

A point $x\in X$ is said to be regular (with respect to μ and φ) if $\tau(x) = 0$, i.e., $\mu B(x, t)<\varphi(t)$ for all $t > 0$. The purpose of introducing these concepts consists in the fact that for the values of the potentials at the regular points one can obtain an estimate in terms of the functions K and φ. Indeed, if K(t) is a continuously differentiable, decreasing function for $t > 0$ and $\lim\limits_{t\to 0}K(t)\varphi(t)=\lim\limits_{t\to\infty}K(t)\varphi(t)=0$, then, under the assumption of the convergence of the integrals, we have

$$\int\limits_X K(|x-\xi|)\,d\mu(\xi)=\int\limits_0^\infty K(t)\,d\mu_x(t)\leqslant-\int\limits_0^\infty K'(t)\varphi(t)\,dt, \tag{10}$$

where $\mu_x(t) = \mu B(x, t)$ and x is a regular point with respect to μ and φ.

We show that the set of regular points is sufficiently large for an arbitrary majorant φ. For this, first we cover all the irregular points by a sequence of balls whose radii tend to zero.

LEMMA 9. Let $0 < \gamma < 1/2$. There exists a sequence of balls $B_k = B(x_k, t_k)$, k = 1, 2,..., which collectively cover all the irregular points and which are such that $\sum\limits_{k=1}^\infty\varphi(\gamma t_k)\leqslant\mu(X)$ (i.e., $t_k\to 0$).

Proof. Let $0 < \alpha < 1$, $\beta > 2$ but $\gamma < \alpha/\beta$. We set $B_0 = \emptyset$ and we assume that the balls $B_0,\ldots,B_{k-1}$ have been constructed. If $\tau_k=\sup\{\tau(x): x\notin B_0\cup\ldots\cup B_{k-1}\}$, then there exists a point $x_k\notin B_0\cup\ldots\cup B_{k-1}$, such that $\tau(x_k)\geqslant\alpha\tau_k$. We set $t_k = \beta\tau_k$ and $B_k = B(x_k, t_k)$.

Clearly, the sequence τ_k (and thus also t_k) does not increase. The balls $B(x_k, \tau_k)$ are pairwise disjoint. Indeed, if $l > k$, then $x_l\notin B_k$, i.e., the distance between x_l and x_k is greater than $\beta\tau_k>2\tau_k\geqslant\tau_k+\tau_l$. Then,

$$\sum_{k=1}^\infty\varphi(\gamma t_k)\leqslant\sum_{k=1}^\infty\varphi(\alpha\tau_k)\leqslant\sum_{k=1}^\infty\varphi(\tau(x_k))=\sum_{k=1}^\infty\mu B(x_k, \tau_k)\leqslant\mu(X);$$

consequently, $\tau_k\to 0$, i.e., for each point x, not belonging to the union of the balls B_k, $\tau(x) = 0$, x is a regular point. In addition, $t_k = \beta\tau_k\to 0$.

The following lemma shows that an infinite-dimensional space cannot be covered by a sequence of balls whose radii tend to zero.†

LEMMA 10. If $Y_1\subset Y_2\subset\ldots$ are proper closed subspaces of the Banach space X and $0 < t_k\to 0$, then each ball of radius $\rho > \max t_k$ contains a point a such that $d(a, Y_k)=\inf\limits_{y\in Y_k}|a-y|>t_k$ for all k = 1, 2,... .

Proof. We shall assume (increasing some t_k if necessary) that the sequence t_k decreases and $t_1 = \max t_k$. Let $\varepsilon > 0$ be such that $\rho > (1 + 2\varepsilon)t_1$. We select a subsequence $s_k = t_{n_k}$, $n_1 = 1$, for which $\frac{\varepsilon}{1+\varepsilon}s_k>\sum\limits_{j>k}s_j$ for all $k\in\mathbf{N}$, and we set $Z_k = Y_{n_{k+1}}$. It is sufficient to prove the lemma for Z_k and s_k (we replace all Y_j and t_j, $n_k < j \leq n_{k+1}$, by $Y_{n_{k+1}}$ and t_{n_k}).

We consider functionals $\xi_k\in X^*$, such that $\|\xi_k\|=1$, $\xi_k(Z_k)=0$. For each $k\in\mathbf{N}$ there exists $u_k\in X$, such that $|u_k| = 1$ and $\xi_k(u_k) = 1$.

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†We thank O. G. Smolyanov and E. T. Shavgulidze for letting us know the proof of this fact.

We consider an arbitrary $B(a_0, \rho)$ and for $n \geqslant 1$ we set $a_n = a_{n-1} + (1+\varepsilon)\delta_n s_n u_n$, where $\delta_n = 1$ or $\delta_n = -1$ is selected in such a manner that $|\xi_n(a_{n-1} + \delta_n s_n u_n)| \geqslant s_n$. Then $|a_m - a_n| = (1+\varepsilon) \times \left|\sum_{k=n+1}^{m} \delta_k s_k u_k\right| \leqslant (1+2\varepsilon) s_{n+1}$, $m > n$, and there exists the limit $a = \lim_{n\to\infty} a_n$, and, moreover, $|a - a_0| \leqslant (1 + 2\varepsilon) s_1 < \rho$, i.e., $a \in B(a_0, \rho)$. Finally

$$d(a, Y_n) \geqslant |\xi_n(a)| \geqslant |\xi_n(a_{n-1} + \delta_n s_n u_n)| + \varepsilon s_n - \sum_{k>n} (1+\varepsilon) |\xi_n(\delta_k s_k u_k)| \geqslant (1+\varepsilon) s_n - \sum_{k>n} (1+\varepsilon) s_k > s_n,$$

and, consequently, the point a satisfies the requirements of the lemma.

Now we can prove the existence of a regular point in each ball of sufficiently large radius.

THEOREM 7. Suppose that X is an infinite-dimensional Banach space, μ is a measure on X, and φ is a majorant. Then each ball of radius $\rho > 2\varphi^{-1}(\mu(X))$ contains a regular point.

Proof. By Lemma 9, all the irregular points can be covered by the balls $B_k = B(x_k, t_k)$, where $0 < t_k \to 0$ and $\max t_k = t_1 < \rho$ [one can take $\gamma < 1/2$ such that $\frac{1}{\gamma}\varphi^{-1}(\mu(X)) < \rho$].

We consider the subspaces Y_k in X, $Y_k = \operatorname{span}\{x_i\colon i = 1,\ldots,k\}$. By virtue of Lemma 10, in each ball of radius ρ there exists a point a satisfying the inequality $d(a, Y_k) > t_k$ for all $k \in \mathbf{N}$. In particular, $a \notin B_k$ for all $k \in \mathbf{N}$, and the point a is regular.

We note that the set of points which are regular for the majorants $n\varphi$ for at least one $n \in \mathbf{N}$ is dense in X. Thus, the set of points at which the values of the potentials admit the estimate (10) (with a constant in the right-hand side independent of the kernel K) is dense in the space X.

We consider the majorant $\varphi(t) = 2|\mu|(X) e^{-\left(\frac{1}{t}\right)^M}$, where $M > 1$. If the point a is regular relative to $|\mu|$ and φ, then the inequality (10) allows us to obtain for s, $\operatorname{Re} s > 0$, the following estimates:

$$\int_X |x-a|^{-s} d|\mu|(x) = \int_0^\infty y^{-s} d|\mu|_a(y) \leqslant \int_0^s s y^{s-1}\varphi(y)\,dy = 2|\mu|(X)\, s\int_0^\infty z^{s-1} e^{-z^M} dz = 2|\mu|(X)\frac{s}{M}\Gamma\left(\frac{s}{M}\right). \tag{11}$$

Now we assume that $\int_X |x-a|^{-n} d\mu(x) = 0$ for every $n \in \mathbf{N}$. Representing the charge μ_a in the usual manner in the form of a difference of measures $\mu_a = \mu_a^+ - \mu_a^-$, we obtain that for all $n \in \mathbf{N}$ we have $s_n = \int_0^\infty y^{-n} d\mu_a^+(y) = \int_0^\infty y^{-n} d\mu_a^-(y)$. The moment problem which occurs after the change of variable $z = 1/y$ is determinate according to Carleman's criterion [29]. Indeed, from (11) it follows that $\sum_{n=1}^\infty \frac{1}{\sqrt[n]{s_n}} \geqslant \alpha \sum_{n=1}^\infty \left(\frac{M}{n}\right)^{1/M}$ for some $\alpha > 0$ and the series in the right-hand side diverges since $M > 1$. Thus, $\mu_a^+ = \mu_a^-$, i.e., $\mu_a = 0$, and the values of the charge on balls with center at the point a are equal to zero.

Thus, if for all $n \in \mathbf{N}$ the equalities $\int_0^\infty |x-a|^{-n} d\mu(x) = 0$ hold for any $m\varphi$-regular point a for all $m \in \mathbf{N}$, then the values of the charge μ are equal to zero on the balls with centers in a dense subset of X and thus, on all the balls in X. If, in addition, for the space X the Hoffman–Jorgensen theorem holds, then $\mu = 0$.

The given reasoning allows us to hope for a solution in the future of the uniqueness problem for an arbitrary Banach space of infinite dimension.

Note Added in Proof. The following remark is due to the first of the authors. Let $\omega(t)$ be a continuous positive function on the semiaxis $t \geqslant 0$ such that $\alpha(t)^{-1} < \omega(x+t)/\omega(x) < \alpha(t)$ for all t and x, where $\log\alpha(t)/\alpha(t) \to 0$, $t \to \infty$. Let $K(z)$ be holomorphic in the angle $|\arg z| < \theta$, $|K(z)| \leqslant \omega(|z|)$ and let μ be a charge such that $S_0^\infty \omega(t)\,d|\mu|(t) < \infty$. If, under these conditions, $\mu \neq 0$ but $S_0^\infty K(x+t)\,d\mu(t) = 0$ for all $x \geqslant 0$, then the Fourier–Laplace transform $\hat{K}(\zeta)$ extends to a meromorphic function in $\mathbf{C}^*$, while if this holds with the replacement of $K(t)$ by

$K(\lambda t)$, $\lambda > 0$, then it extends to a holomorphic function. The proof is obtained by a standard deformation of the contour. Under more qualified estimates, the growth of $\hat{K}(\zeta)$ is made more accurate. For example, if $\omega'/\omega = O(1/t)$, then K is a polynomial. In particular, this remark allows us to generalize Theorem 6 (we note that the results of Sec. 6 ensure the correctness of the formulation of the problem within very wide limits).

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