A RANDOM DIFFERENCE EQUATION

A. Grincevičius

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In the present note we consider the difference equation with random coefficients

$$\zeta_n = \xi_n \zeta_{n-1} + \eta_n, \qquad n = 1, 2, \ldots, \tag{1}$$

where $\{\xi_n, \eta_n\}_1^\infty$ is a sequence of independent identically distributed random vectors with the initial condition $\zeta_0 = \eta_0$. The interest in studying this difference equation is explained by the fact that it arises in certain problems of physics, economics, etc. There is a survey of papers on the given difference equation and questions connected with it in Vervaat [1].

From (1), by induction we get

$$\zeta_n = \eta_n + \eta_{n-1}\xi_n + \eta_{n-2}\xi_{n-1}\xi_n + \ldots + \eta_1\xi_2\xi_3 \ldots \xi_n + \eta_0\xi_1\xi_2 \ldots \xi_n.$$

In what follows it will be assued that the random variable η_0 is independent of the random vectors (ξ_n, η_n), $n = 1, 2, \ldots$. Along with the random variable ζ_n we consider the random variable

$$\psi_n = \eta_1 + \eta_2\xi_1 + \eta_3\xi_1\xi_2 + \ldots + \eta_n\xi_1\xi_2 \ldots \xi_{n-1}.$$

From the independence and identical distribution of the random vectors (ξ_n, η_n) it obviously follows that the random variables ζ_n and $\psi_n + \eta_0\xi_1 \ldots \xi_n$ are identically distributed. The fundamental result of the paper is

THEOREM 1. Suppose for any real number c $\mathbf{P}\{\eta_1 = c(1 - \xi_1)\} < 1$. Then one of the following holds:

1) the distribution of the random variable ψ_n diverges to $\pm\infty$ as $n \to \infty$;

2) the random variable ψ_n converges almost surely as $n \to \infty$.

In the proof of Theorem 1 we need two auxiliary assertions. If nothing is said to the contrary, we shall assume that limits are taken as $n \to \infty$.

Proposition 1. The following conditions are equivalent:

a) for some Borel function $f(\cdot)$ almost surely $\eta_1 + \eta_2\xi_1 = f(\xi_1\xi_2)$;

b) for some real number c almost surely either $\eta_1 = c(1 - \xi_1)$, or $(\xi_1, \eta_1) = (1, c)$.

Proof. Condition a) obviously follows from condition b). Let condition a) hold. Since the random vectors $(\eta_1 + \eta_2\xi_1, \xi_1\xi_2)$ and $(\eta_2 + \eta_1\xi_2, \xi_2\xi_1)$ are identically distributed, one also has $\eta_2 + \eta_1\xi_2 = f(\xi_2\xi_1)$ almost surely. The event $\{\xi_1 = 0\}$ implies the events $\{\eta_1 = f(0)\}$ and $\{\eta_2 = f(0)\xi_2 = f(0)\}$, so if $\mathbf{P}\{\xi_1 = 0\} > 0$, then $\eta_2 + f(0)\xi_2 = f(0)$ almost surely, since the random variables ξ_1 and (ξ_2, η_2) are independent. Now if $\mathbf{P}\{\xi_1 = 0\} = 0$, then

$$\begin{pmatrix} \xi_1 & \eta_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_2 & \eta_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \xi_1\xi_2 & \eta_1 + \eta_2\xi_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \xi_2 & \eta_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 & \eta_1 \\ 0 & 1 \end{pmatrix}$$

almost surely, so the support of the distribution of the random matrix $\begin{pmatrix} \xi_1 & \eta_1 \\ 0 & 1 \end{pmatrix}$ is contained in a commutative subgroup of the group of real matrices $\left\{\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}, x \neq 0\right\}$, isomorphic to the group of linear transformations of the line. As is well known any such commutative subgroup is contained either in a subgroup $\left\{\begin{pmatrix} x & c(1-x) \\ 0 & 1 \end{pmatrix}, x \neq 0\right\}$ (c is a fixed real number), or in the subgroup $\left\{\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\right\}$. Whence condition b) also follows.

Proposition 2. Let $f_n(t)$ and $g_n(t)$ be the characteristic functions of the random variables X_n and Y_n, respectively, $|f_n(t)| \le g_n(t)$, $n = 1, 2, \ldots$, and the distribution of the random variable Y_n diverge to $\pm\infty$ as

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$n \to \infty$. Then the distribution of the random variable X_n also diverges to $\pm\infty$.

Proof. Let Z be a random variable, for example with standard normal distribution, and not depend on the remaining random variables. Then

$$0 \leqslant \mathbf{M} \exp(-t^2 X_n^2/2) = \mathbf{M} \exp(itX_n Z) = \mathbf{M} f_n(tZ) \leqslant \mathbf{M} g_n(tZ) = \mathbf{M} \exp(-t^2 Y_n^2/2)$$

where the last term in the chain of inequalities tends to zero for all $t \neq 0$. We have proved that the characteristic function of the random variable $X_n Z$ tends to zero for all $t \neq 0$, hence its distribution diverges to $\pm\infty$ (cf., e.g., Tutubalin [5, Lemma 5]). Consequently, the distribution of the random variable X_n also diverges to $\pm\infty$.

Proof of Theorem 1. If $\mathbf{P}\{\xi_1 = 0\} > 0$, then 2) obviously holds, so let us assume that $\mathbf{P}\{\xi_1 = 0\} = 0$. Further, let $|\xi_1| = 1$ almost surely. It is easy to see that it is sufficient to prove the theorem only for even n. We set $n = 2k$, so

$$\psi_n = (\eta_1 + \eta_2 \xi_1) + (\eta_3 + \eta_4 \xi_3)(\xi_1 \xi_2) + \ldots + (\eta_{2k-1} + \eta_{2k} \xi_{2k-1})(\xi_1 \xi_2) \ldots (\xi_{2k-1} \xi_{2k}).$$

By virtue of Proposition 1, either for some number $c \neq 0$ almost surely $(\xi_1 \eta_1) = (1, c)$ [then 1) holds], or the random variable $\eta_1 + \eta_2 \xi_1$ is not a Borel function of $\xi_1 \xi_2$. In the latter case we want to symmetrize the random variable ψ_n under the condition $\xi_1\xi_2, \xi_3\xi_4, \ldots, \xi_{2k-1}\xi_{2k}$. In order not to complicate the notation, let us assume that the random variable η_1 itself is not almost surely a Borel function of ξ_1.

We write

$$f(t, x) = \mathbf{M}\left(\exp(it\eta_1) \mid \xi_1 = x\right).$$

We have

$$\mathbf{M}\left(\exp(it\psi_n) \mid \xi_1, \ldots, \xi_n\right) = \mathbf{M}\left(\exp(it\psi_{n-1}) \mid \xi_1, \ldots, \xi_{n-1}\right) f(t\xi_1 \ldots \xi_{n-1}, \xi_n), \tag{2}$$

and since f(t, x) is an even function in t, and the random variables ξ_j assume only values ± 1, one has

$$|f(t\xi_1 \ldots \xi_{n-1}, \xi_n)|^2 = |f(t, \xi_n)|^2,$$

and consequently, from (2) by induction we get

$$|\mathbf{M} \exp(it\psi_n)|^2 \leqslant \mathbf{M}\,|\mathbf{M}\left(\exp(it\psi_n) \mid \xi_1, \ldots, \xi_n\right)|^2 = \mathbf{M} \prod_{j=1}^{n} f(t, \xi_j) = \prod_{j=1}^{n} \mathbf{M} f(t, \xi_j).$$

The right side of the latter inequality is the characteristic function of the sum of independent identically distributed nondegenerate random variables, so its distribution diverges to $\pm\infty$ (cf. Petrov [4, Chap. III, Sec. 2, Theorem 6, p. 63]).

From Proposition 2 it now follows that the distribution obtained by symmetrization of the distribution of the random variable ψ_n diverges to $\pm\infty$, and all the more the distribution of the random variable ψ_n itself diverges to $\pm\infty$.

Let us assume now that $|\xi_1| \neq 1$ with positive probability, and we consider the random walk

$$S_0 = 0, \; S_1 = \ln|\xi_1|, \; \ldots, \; S_n = \ln|\xi_1| + \ln|\xi_2| + \ldots + \ln|\xi_n|, \; \ldots \tag{3}$$

Three cases are possible (cf. Feller [3, Chap. XII, Sec. 2, Theorem 1]): S_n tends to $+\infty$, S_n tends to $-\infty$ or S_n oscillates between $-\infty$ and $+\infty$. We consider each case separately.

I. S_n tends to $-\infty$ almost surely. Let 1) not hold. Then one can find a sequence $n_k \uparrow \infty$ such that the distribution of the random variable ψ_{n_k} as $k \to \infty$ converges to a (possibly improper) distribution F, and here $F(+\infty) - F(-\infty) > 0$. But then the distribution of the random variable ψ_{n_k+1} also converges to the distribution F. Passing to the limit in the equation

$$\psi_{n_k+1} = \eta_1 + \xi_1 \bar{\psi}_{n_k},$$

where $\bar{\psi}_{n_k} = \eta_2 + \eta_3\xi_2 + \ldots + \eta_{n_k+1}\xi_2 \ldots \xi_{n_k}$ is distributed just as ψ_{n_k} is, and normalizing the limit distribution F, we get that if the random variable ψ_0 does not depend on the remaining random variables and has distribution $F/(F(\infty) - F(-\infty))$, then $\eta_1 + \xi_1\psi_0$ too is distributed just as ψ_0 is. By induction we get that

$$\eta_1 + \eta_2 \xi_1 + \eta_3 \xi_1 \xi_2 + \ldots + \eta_n \xi_1 \ldots \xi_{n-1} + \xi_1 \ldots \xi_n \psi_0$$

is distributed just as ψ_0 is. But $\xi_1 \ldots \xi_n \psi_0$ tends to zero, so the distribution of the random variable ψ_n tends to the distribution of the random variable ψ_0.

From the convergence of the random variable ψ_n in distribution follows the convergence in probability. For the proof of convergence almost surely we apply the method of Jessen and Wintner [2]. Let ψ_n converge in probability to the random variable ψ, and σ_n be the σ-algebra generated by the random variables $(\xi_1, \eta_1), \ldots, (\xi_n, \eta_n)$. It is easy to prove

$$\mathbf{M}(\exp(it\psi) \mid \sigma_n) = \exp(it(\eta_1 + \ldots + \eta_n \xi_1 \ldots \xi_{n-1})) f(t\xi_1 \ldots \xi_n),$$

where $f(t) = \mathbf{M} \exp(it\psi)$. By Theorem A of Jessen and Wintner [2] (or by the convergence theorem for semimartingales) $\mathbf{M}(\exp(it\psi) \mid \sigma_n)$ converges almost surely, and since $\xi_1 \ldots \xi_n$ converges almost surely to zero, one also has $\eta_1 + \eta_2\xi_1 + \ldots + \eta_n\xi_1 \ldots \xi_{n-1}$ converges almost surely.

II. S_n tends to $+\infty$ almost surely. By virtue of what has already been proved the distribution of the random variable $\psi_n / \xi_1 \ldots \xi_n = \eta_n / \xi_n + \eta_{-1} / \xi_{n-1}\xi_n + \ldots + \eta_1 / \xi_1 \ldots \xi_n$ either diverges to $\pm\infty$ or converges weakly; in addition the limit distribution is continuous (cf. [8]). Since $|\xi_1 \ldots \xi_n|$ tends to $+\infty$, in both cases condition 1) of Theorem 1 holds.

III. S_n oscillates almost surely between $-\infty$ and $+\infty$. Then the increasing stepwise renewal process tends to $+\infty$. We denote by $L_0 = 0$; $L_1, L_2, \ldots$, the first, second, ... upper stepwise moments of the random walk (3) and Q, the number of these stepwise moments to time n inclusive. Just as earlier, let us assume for simplicity of notation, that η_1 is not a Borel function of ξ_1. We have

$$|\mathbf{M} \exp(it\psi_n)|^2 = |\mathbf{M}[\mathbf{M}(\exp(it\eta_1) \mid \xi_1) \ldots \mathbf{M}(\exp(it\eta_n\xi_1 \ldots \xi_{n-1}) \mid \xi_1, \ldots, \xi_n)]|^2 =$$
$$= |\mathbf{M} f(t, \xi_1) f(t\xi_1, \xi_2) \ldots f(t\xi_1 \ldots \xi_{n-1}, \xi_n)|^2 \leqslant \mathbf{M}[|f(t, \xi_1)|^2 \ldots |f(t\xi_1 \ldots \xi_{L_m-1}, \xi_{L_m})|^2] + \mathbf{P}\{Q < m\} \tag{4}$$

[the function f(t, x) is defined by (1)].

If along with the random vectors (ξ_n, η_n) we consider a sequence of independent random vectors $(\xi_n, \eta_n, \bar{\eta}_n)$, $n = 1, 2, \ldots$, such that

$$\mathbf{M} \exp(i(t\eta_j + s\bar{\eta}_j + u\xi_j)) = \mathbf{M}[f(t, \xi_j) f(s, \xi_j) \exp(iu\xi_j)],$$

then the random variable equal to

$$(\eta_1 - \bar{\eta}_1) + (\eta_2 - \bar{\eta}_2)\xi_1 + \ldots + (\eta_{L_m} - \bar{\eta}_{L_m})\xi_1 \ldots \xi_{L_m-1}, \tag{5}$$

if $Q \leq m$, and equal to zero if $Q > m$, will have the characteristic function standing on the right side of (4). The random variable (5) can be represented in the form

$$\delta_1 + \delta_2\varphi_1 + \ldots + \delta_m\varphi_1 \ldots \varphi_{m-1},$$

where

$$\delta_j = (\eta_{L_{j-1}+1} - \bar{\eta}_{L_{j-1}+1}) + \ldots + (\eta_{L_j} - \bar{\eta}_{L_j})\xi_{L_{j-1}+1} \ldots \xi_{L_j-1},$$
$$\varphi_j = \xi_{L_{j-1}+1} \ldots \xi_{L_j}.$$

As is well known from the theory of random walks, the random vectors (φ_j, δ_j) are independent and identically distributed. Further, it is easy to prove that $\mathbf{P}\{\delta_j = (1 - c)\varphi_j\} < 1$ (cf. [7]), and since $|\varphi_1 \ldots \varphi_m|$ tends to $+\infty$ almost surely as $m \to \infty$, by virtue of what has already been proved (case II) the distribution of the random variable (5) diverges to $\pm\infty$. By Proposition 2 the distribution of the random variable ψ_n also diverges to $\pm\infty$. The theorem is proved.

We note that the hypothesis of the theorem is essential. In fact, if for some real number c almost surely $\eta_1 = c(1 - \xi_1)$, then $\psi_n = c(1 - \xi_1\xi_2 \ldots \xi_n)$, and if $\ln \xi_1$ will be the distribution such that $\mathbf{M} \times \ln \xi_1 = 0$, but the distribution of the random variable $\sum_1^n \ln \xi_j$ diverges to $-\infty$ (cf. [3]), then ψ_n will converge in probability, but almost surely diverge.

From Theorem 1 there follows directly the following:

COROLLARY. If the hypothesis of Theorem 1 holds, then the convergence in distribution of the random series

$$\eta_1 + \eta_2\xi_1 + \ldots + \eta_n\xi_1 \ldots \xi_{n-1} + \ldots$$

is equivalent with the convergence almost surely.

With the additional assumption $-\infty < \mathbf{M} \ln |\xi_1| < 0$, this assertion was obtained by Vervaat [1].

THEOREM 2. Let the hypothesis of Theorem 1 hold. Then independent of the initial random variable η_0 the distribution of the random variable ζ_n as $n \to \infty$ either

a) diverges to $\pm\infty$, or

b) converges weakly, and here almost surely $\xi_1 \ldots \xi_n$ converges to zero.

Although Theorem 2 does not follow directly from Theorem 1, to prove it only slight changes in the proof of Theorem 1 are necessary.

Theorems 1 and 2 can be used to prove a limit theorem for the random variables ζ_n and ψ_n. Let the hypothesis of Theorem 1 hold, the random walk S_n go to $+\infty$ almost surely. Then, for example, the distribution of the random variable $\zeta_n / \xi_1 \ldots \xi_n$ either diverges to $\pm\infty$ or converges, and the function of its limit distribution is continuous (cf. [8]), so the sequence $\ln|\zeta_n| - S_n$ is stochastically bounded below. If for some constants a_n and b_n the distribution of the random variable $(S_n - a_n)/b_n$ converges weakly, then

$$\lim_{n\to\infty} \mathbf{P}\{(\ln|\zeta_n| - a_n)/b_n - (S_n - a_n)/b_n > -\varepsilon\} = 1 \tag{6}$$

for any positive number ε. An upper estimate for the difference of the random variables standing in (6) is obtained more simply. For this it suffices only to require the finiteness of the expectation of some power of the random variable $\ln^+|\eta_1|$ (cf. Kalenskii [6]).

Moreover, the method of proof of Theorem 1 allows us to get a limit theorem for the random variables ζ_n and ψ_n also in the more complicated case when the random walk S_n oscillates between $-\infty$ and $+\infty$.

Proposition 3. Let the hypothesis of Theorem 1 hold, and S_n oscillate almost surely between $-\infty$ and $+\infty$. Then the sequence $\ln|\psi_n| - \sup_{0\leq j\leq n} S_j$ is stochastically bounded below.

Proof. For simplicity of notation again we assume, without loss of generality, that η_1 is not a Borel function of ξ_1. We have $\exp(\sup_{0\leq j\leq n} S_j) = |\xi_1 \ldots \xi_{L_Q}|$ almost surely, since L_Q is the moment of the first maximum of the sequence $S_0, S_1, \ldots, S_n$. Whence it is easy to get the inequality

$$|\mathbf{M}\exp(it\psi_n/\exp\sup_{0\leq j\leq n} S_j)|^2 \leq \mathbf{M}\prod_{j=L_{Q-m+1}}^{L_Q} |f(t\xi_j^{-1} \ldots \xi_Q^{-1}, \xi_j)|^2 + \mathbf{P}\{Q<m\}. \tag{7}$$

The right side of (7) is the characteristic function of the random variable equal to

$$(\eta_{L_{Q-m+1}} - \bar{\eta}_{L_{Q-m+1}})/\xi_{L_{Q-m+1}} \ldots \xi_{L_Q} + \ldots + (\eta_{L_Q} - \bar{\eta}_{L_Q})/\xi_{L_Q} \tag{8}$$

for $Q \geq m$ and zero for $Q < m$. The random variable (8) can be represented in the form of a sum

$$\delta_{Q-m+1}/\varphi_{Q-m+1} \ldots \varphi_Q + \ldots + \delta_Q/\varphi_Q,$$

which has the same distribution as the random variable equal to

$$\delta_1/\varphi_1 + \delta_2/\varphi_1\varphi_2 + \ldots + \delta_m/\varphi_1 \ldots \varphi_m \tag{9}$$

for $Q \geq m$ and zero for $Q < m$ (cf. [7]).

As $m \to \infty$ the distribution of the random variable (9) either diverges to $\pm\infty$ or converges weakly and has continuous distribution.

In the latter case, in order to apply Proposition 2, it is necessary to multiply both sides of (7) by the positive characteristic function of a distribution diverging to $\pm\infty$ sufficiently slowly. Whence we get that the limit distribution of the random variable $\psi_n/\exp\sup_{0\leq j\leq n} S_j$ cannot have an atom at zero, i.e., the random variable $\ln|\psi_n| - \sup_{1\leq j\leq n} S_j$ is stochastically bounded below.

Analogously, one can prove the stochastic boundedness below of the sequence $\ln|\zeta_n| - \sup_{0\leq j\leq n}(S_n - S_j)$.

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LARGE DEVIATIONS OF ORDER n FOR SUMS OF RANDOM VARIABLES RELATED TO A MARKOV CHAIN

P. Gudynas

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In this paper we prove a limit relation for probabilities of large deviations of order n of sums of real random variables (r. r. v.) related to a homogeneous Markov chain. An analogous result for independent r. r. v. can be obtained from more general assertions of Cramer [1], Chernoff [7], and Petrov [2]. The hypotheses of the fundamental theorem of the present paper are somewhat unusual, since they are formulated in terms of the spectral theory of linear operators. Hence there are also given below two more concrete examples of sequences of r. r. v. related to a Markov chain and satisfying the hypotheses of the fundamental theorem.

Let $(X, \mathscr{F})$ be a measurable space and let $p(x, A)$ be the transition probability from $(X, \mathscr{F})$ to $(X, \mathscr{F})$.

We consider a homogeneous Markov chain ξ_i, $i = 0, 1, 2, \ldots$, where ξ_i assume values in the measurable space $(X, \mathscr{F})$ and for any $x \in X$, $A \in \mathscr{F}$,

$$\mathbf{P}(\xi_i \in A \mid \xi_{i-1} = x) = p(x, A), \quad i = 1, 2, \ldots \tag{1}$$

Let $f(x)$ be a real measurable function, defined on $(X, \mathscr{F})$, and $\mathfrak{M}$ be the Banach space of bounded complex-valued measurable functions $(X, \mathscr{F})$. If for a real number h

$$\sup_{x \in X} \int_X e^{hf(y)} p(x, dy) < \infty, \tag{2}$$

then we define the bounded linear operator $P(h + iv)$, mapping $\mathfrak{M}$ into $\mathfrak{M}$, in the following way:

$$\big(P(h+iv)g\big)(x) = \int_X g(y) e^{(h+iv)f(y)} p(x, dy). \tag{3}$$

It will be assumed below that (2) holds for all $h > 0$.

Let $\mathfrak{N}$ be some closed subspace of the Banach space $\mathfrak{M}$ such that for any z, $\operatorname{Re} z > 0$, the operator $P(z)$ carries $\mathfrak{N}$ into itself, and the function φ, identically equal to one, is an element of $\mathfrak{N}$.

We write $S_n = f(\xi_1) + f(\xi_2) + \ldots + f(\xi_n)$. We shall be interested in the asymptotic behavior of the probability

$$\mathbf{P}_x(S_n \geqslant na) = \mathbf{P}(S_n \geqslant na \mid \xi_0 = x). \tag{4}$$

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