VARIATIONAL APPROACH TO THE PERIODIC SOLUTIONS PROBLEM

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UDC 517.988.63

Introduction. In this work we investigate the periodic boundary-value problem for the linear

$$\frac{d}{dt}\left(P(t)\frac{dz}{dt}\right)+Q(t)z=f(t)\quad (0\leqslant t\leqslant \omega) \tag{1}$$

and nonlinear

$$\frac{d}{dt}\left(P(t)\frac{dz}{dt}\right)+F(t,z)=0\quad (0\leqslant t\leqslant \omega) \tag{2}$$

abstract equations in a Hilbert space. The existence of various classes of solutions of the periodic problem for Eq. (1) is studied in [1-3]. In these works the assumption is made that the spectrum of the operators Q(t) for every $t\in[0,\omega]$ is located in the left half-plane. However, the case when Q(t) has spectrum points in the right half-plane seems to be more interesting. In this case the periodic problem does not always have a solution (which is clear even for scalar equations). Conditions of the existence of periodic solutions of second-order linear equations of type (1), under the assumption that Q(t) has spectrum points in the right half-plane, were investigated in [4]. However, only normal solvability is established there and the question of solvability for an arbitrary right-hand side from a certain class is not answered. This question is considered by the semigroup theory methods in [5] for the case when P(t) = I and the function Q(t) is semibounded from above. In this paper we apply the variational approach, when the existence of a solution of the periodic problem for Eqs. (1) and (2) is equivalent to the existence of a stationary point for the energy functional. The important aspect of this approach is that we determine specific classes of equation coefficients which generate regular differential operators. These classes are determined in such a way that the space in which we seek a solution is decomposed into the direct sum of two subspaces. On the first subspace the corresponding functional is convex, on the second subspace it is concave. Thus the stationary point is a saddle point. The variational approach allowed us to investigate the equations under milder restrictions on the coefficients (in comparison with [5]). It also allowed us to consider a larger class of solutions. From our results the assertions of [6-11] for the finite-dimensional case follow as a particular case.

It is worth mentioning that a general scheme introduced in Sec. 1 can be applied not only to the periodic problem for abstract operator equations in a Hilbert space, but also for other boundary-value problems.

1. Solvability of the Equations with Potential Operators

1.1. Theorem on Stationary Point of a Functional. Let $\mathcal{Z}$ be a real Hilbert space with a scalar product [,] and with a norm $|\cdot|$. Let $\mathfrak{D}$ be a dense set in $\mathcal{Z}$. A functional $\mathcal{I}:\mathfrak{D}\to\mathbf{R}$ is called *Gateaux differentiable* if for every z, h $\in\mathfrak{D}$ the limit

$$D\mathcal{I}(z;h)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\mathcal{I}(z+\varepsilon h)-\mathcal{I}(z)\}$$

exists and is a continuous linear functional with respect to h. By the Riesz theorem this functional has the following representation: $D\mathcal{I}(z;h)=[\Phi z,h]$. The operator $\Phi:\mathfrak{D}\to\mathcal{Z}$ is called the *gradient* of the functional $\mathcal{I}$; it is denoted as $\Phi=\operatorname{grad}\mathcal{I}$. The functional $\mathcal{I}$ is called the *potential* of the operator Φ; we denote it as $\mathcal{I}=\operatorname{pot}\Phi$. An operator Φ which is the gradient of some functional is called a *potential* operator.

THEOREM 1.1. Let $\mathcal{Z}=X\oplus Y$ be a direct sum of two subspaces. Let $\mathcal{I}:\mathcal{Z}\to\mathbf{R}$ be a differentiable functional and $\Phi=\operatorname{grad}\mathcal{I}$. Let for every $z,h\in\mathcal{Z}$ and for some α, β > 0 the

Translated from Sibirskii Matematicheskii Zhurnal, Vol. 25, No. 1, pp. 106-119, January-February, 1984. Original article submitted November 23, 1981.

 0037-4466/84/2501-0088$8.50

inequalities

$$A(h, h) \leqslant [\Phi(z+h)-\Phi z, h] \leqslant B(h, h) \tag{1.1}$$

hold. Here A, B: $\mathscr{Z}\times\mathscr{Z}\to\mathbf{R}$ are bilinear continuous functionals which satisfy the following conditions:

$$A(x, x) \geqslant \alpha|x|^2, \qquad x\in \mathrm{X}, \tag{1.2}$$

$$B(y, y) \leqslant -\beta|y|^2, \qquad y\in \mathrm{Y}. \tag{1.3}$$

Then the equation

$$\Phi z = 0 \tag{1.4}$$

has the unique solution $\varphi\in\mathscr{Z}$ and the estimate

$$|\varphi| \leqslant |\Phi(0)|/\min(\alpha, \beta) \tag{1.5}$$

holds.

The proof of this theorem can be found, for example, in [12].

1.2. Generalized Solution. Let $\mathscr{Z}$ be dense and let it be imbedded continuously in a Hilbert space $\mathfrak{H}$ with the scalar product (,) and with the norm $\|\cdot\|$. Let $\mathfrak{D}$ be a linear set, which is dense in $\mathscr{Z}$, and let $\mathfrak{N}$: $\mathfrak{D}\to\mathfrak{H}$ be a potential operator.

LEMMA 1.1. Let $\mathscr{I}:\mathfrak{D}\to\mathbf{R}$ be a differentiable operator and

$$D\mathscr{I}(z; h) = (\mathfrak{N}z, h), \qquad z, h\in\mathfrak{D}.$$

Let the inequalities

$$A(h, h) \leqslant (\mathfrak{N}(z+h)-\mathfrak{N}z, h) \leqslant B(h, h), \quad z, h\in\mathfrak{D}, \tag{1.6}$$

hold, where A, B: $\mathscr{Z}\times\mathscr{Z}\to\mathbf{R}$ are bilinear symmetric continuous functionals.

Then the functional $\mathscr{I}$ can be extended by continuity to the whole $\mathscr{Z}$. The extension (which we denote also by $\mathscr{I}$) is a differentiable functional and the following inequalities hold:

$$A(h, h) \leqslant D\mathscr{I}(z+h; h) - D\mathscr{I}(z; h) \leqslant B(h, h), \quad z, h\in\mathscr{Z}. \tag{1.7}$$

Proof. The form $(\mathfrak{N}z, h)$ determines on $\mathscr{Z}$ a linear functional which is continuous with respect to h. Therefore, there exists an operator Φ: $\mathfrak{D}\to\mathscr{Z}$ such that

$$D\mathscr{I}(z; h) = (\mathfrak{N}z, h) = [\Phi z, h].$$

Moreover, from (1.6) it follows that

$$|[\Phi(z+h)-\Phi z, h]| \leqslant \rho|h|^2, \quad z, h\in\mathfrak{D},$$

where

$$\rho = \max\{\max_{|h|\leqslant 1}|A(h,h)|, \ \max_{|h|\leqslant 1}|B(h,h)|\}. \tag{1.8}$$

Since Φ is a potential operator, the last inequality, by virtue of the results of [13], implies that the estimate

$$|\Phi(z+h)-\Phi z| \leqslant \rho|h|, \quad z, h\in\mathfrak{D},$$

holds. From this it is clear that the operator Φ, and therefore the functional $\mathscr{I}$, can be extended by continuity to the whole $\mathscr{Z}$. Moreover, the extension of Φ is the gradient of the extended functional. The estimate (1.7) follows from (1.6) and from the fact that $\mathfrak{D}$ is dense in $\mathscr{Z}$. The lemma is proved.

It is natural to define a *generalized solution* of the equation

$$\mathfrak{N}z = 0, \quad z\in\mathfrak{D}, \tag{1.9}$$

as a solution $\varphi\in\mathscr{Z}$ of the equation

$$\Phi z = 0, \quad z\in\mathscr{Z}. \tag{1.10}$$

In the following assertion we formulate conditions which are sufficient to guarantee that generalized solutions are classical solutions.

LEMMA 1.2. Let $\mathfrak{R}:\mathfrak{D}\to\mathfrak{H}$ satisfy the conditions of Lemma 1.1 and $\mathfrak{R}=-\mathfrak{L}+\mathcal{F}$, where $\mathfrak{L}:\mathfrak{D}\to\mathfrak{H}$ is a linear self-adjoint positive-definite operator whose energy space has the same topology and the same domain of elements as the space $\mathcal{Z}$; the operator $\mathcal{F}:\mathcal{Z}\to\mathfrak{H}$ is nonlinear and semicontinuous.

Then the generalized solution of Eq. (1.9) belongs to $\mathfrak{D}$ and $\mathfrak{R}\varphi=0$.

Proof. We define the operator $\Phi:\mathcal{Z}\to\mathcal{Z}$ as in Lemma 1.1. By virtue of the representation for $\mathfrak{R}$ we have for $z, h\in\mathfrak{D}$ that

$$[\Phi z, h]=-(\mathfrak{L}z, h)+(\mathcal{F}z, h).$$

Denote by $[\ ,\ ]_{\mathfrak{L}}$ the energy scalar product generated by the operator $\mathfrak{L}$. Then from the previous equality we obtain that

$$[\Phi z, h]=-[z,h]_{\mathfrak{L}}+(\mathcal{F}z,h).$$

Since the energy space of the operator $\mathfrak{L}$ has the same topology and the same domain of elements as the space $\mathcal{Z}$, and since the operator $\mathcal{F}$ is semicontinuous, the last equality holds for all $z\in\mathcal{Z}$. In particular for the generalized solution φ and $h\in\mathfrak{D}$ we have that $(\varphi, \mathfrak{L}h)=(\mathcal{F}\varphi, h)$. Thus, $\varphi\in\mathfrak{D}(\mathfrak{L}^*)$ and $\mathfrak{L}^*\varphi=\mathcal{F}\varphi$. Since $\mathfrak{D}(\mathfrak{L}^*)=\mathfrak{D}$ and $\mathfrak{L}^*=\mathfrak{L}$, $\mathfrak{L}\varphi=\mathcal{F}\varphi$. The lemma is proved.

1.3. Regularity Classes. Everywhere below we shall assume that $\mathcal{Z}$ is imbedded compactly in $\mathfrak{H}$. By $\mathcal{E}(\mathcal{Z})$ we denote a class of linear self-adjoint semibounded from above operators $\mathfrak{L}$ in $\mathfrak{H}$, which are defined on the domain $\mathfrak{D}(\mathfrak{L})\subseteq\mathcal{Z}$, and for which $\mathcal{Z}$ has the same topology and the same domain of elements as the energy space of the operator $lI-\mathfrak{L}$ for sufficient ciently large $l>0$. By Rellich's theorem, see [14, p. 217], the spectrum of the operator $lI-\mathfrak{L}$, and thus the spectrum of the operator $\mathfrak{L}$, is discrete. Moreover, the system of eigenvectors of the operator $\mathfrak{L}$ is complete both in the space $\mathfrak{H}$ and in the space $\mathcal{Z}$.

By $[\ ,\ ]_{\mathfrak{L}}$ we denote the energy scalar product in $\mathcal{Z}$, generated by the operator $lI-\mathfrak{L}$. We introduce a bilinear continuous on $\mathcal{Z}$ functional

$$L(z,h)=l(z,h)-[z,h]_{\mathfrak{L}},\quad z,h\in\mathcal{Z}. \tag{1.11}$$

Note that

$$L(z, h)=(\mathfrak{L}z, h),\ z\in\mathfrak{D}(\mathfrak{L}),\ h\in\mathcal{Z}. \tag{1.12}$$

Consider the operators $\mathfrak{A}, \mathfrak{B}\in\mathcal{E}(\mathcal{Z})$. We write $\mathfrak{A}\leqslant\mathfrak{B}$, if $A(z, z)\leqslant B(z, z)$ for all $z\in\mathcal{Z}$, where the functionals A, B are constructed according to (1.11) from the operators $\mathfrak{A}, \mathfrak{B}$, respectively.

LEMMA 1.3. If $\lambda_1(\mathfrak{L})\geqslant\lambda_2(\mathfrak{L})\geqslant\ldots$ is a complete collection of eigenvalues of the operator $\mathfrak{L}\in\mathcal{E}(\mathcal{Z})$, and if $\varphi_1, \varphi_2, \ldots$ is the corresponding orthonormal (in $\mathfrak{H}$) system of eigenvectors, then the following representation holds:

$$L(z,z)=\sum_{j=1}^{\infty}\lambda_j(\mathfrak{L})(z,\varphi_j)^2,\quad z\in\mathcal{Z}. \tag{1.13}$$

Proof (is in [12, p. 149]. For a definite operator $\mathfrak{L}$ the representation (1.13) is given in [14, p. 221).

An operator $\mathfrak{L}$ is called *regular* if it has a continuous inverse in $\mathfrak{H}$. Therefore, the operator $\mathfrak{L}\in\mathcal{E}(\mathcal{Z})$ is regular precisely when none of its eigenvalues vanishes. We denote the set of all regular operators $\mathfrak{L}\in\mathcal{E}(\mathcal{Z})$ by $\mathfrak{G}(\mathcal{Z})$. The *regularity index* $\operatorname{ind}\mathfrak{L}$ of the operator $\mathfrak{L}\in\mathfrak{G}(\mathcal{Z})$ is defined as a sum of its positive eigenvalues. Since the operator $\mathfrak{L}$ is semibounded from above, $\operatorname{ind}\mathfrak{L}$ is a finite number. We denote the set of all operators from $\mathfrak{G}(\mathcal{Z})$ with the regularity index k by $\mathfrak{G}_k(\mathcal{Z})$ (k = 0, 1, 2,...). It is clear that the classes $\mathfrak{G}_k(\mathcal{Z})$ are not empty for every k = 0, 1, 2,...; they are mutually noninteresting and

$$\mathfrak{G}(\mathcal{Z})=\bigcup_{k=0}^{\infty}\mathfrak{G}_k(\mathcal{Z}).$$

Thus, if an operator $\mathfrak{L}$ belongs to a class $\mathfrak{G}_k(\mathcal{Z})$, it means that its eigenvalues satisfy the following inequalities:

$$\lambda_1(\mathfrak{L})\geqslant\lambda_2(\mathfrak{L})\geqslant\ldots\geqslant\lambda_k(\mathfrak{L})>0>\lambda_{k+1}(\mathfrak{L})\geqslant\ldots$$

The regularity classes for self-adjoint completely continuous operators were first introduced by one of the authors in [15].

LEMMA 1.4. Let operators $\mathfrak{A}, \mathfrak{B}, \mathfrak{S} \in \mathscr{E}(\mathscr{Z})$ and $\mathfrak{A} \leqslant \mathfrak{S} \leqslant \mathfrak{B}$. Then their eigenvalues, numbered in nonascending order, satisfy the inequalities

$$\lambda_j(\mathfrak{A}) \leqslant \lambda_j(\mathfrak{S}) \leqslant \lambda_j(\mathfrak{B}), \quad j = 1, 2, \ldots. \tag{1.14}$$

Moreover, if $\operatorname{ind} \mathfrak{A} = \operatorname{ind} \mathfrak{B} = k$, then $\operatorname{ind} \mathfrak{S} = k$, and

$$\|\mathfrak{S}^{-1}\| \leqslant 1/\min\{\lambda_k(\mathfrak{A}), \ -\lambda_{k+1}(\mathfrak{B})\} \tag{1.15}$$

and

$$\mathfrak{B}^{-1} \leqslant \mathfrak{S}^{-1} \leqslant \mathfrak{A}^{-1}. \tag{1.16}$$

Note that relations (1.14) and (1.15) follow directly from the Courant minimax principle. The proof of (1.16) is less trivial and is given in [12, p. 154].

LEMMA 1.5. Let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}_k(\mathscr{Z})$ and $\mathfrak{A} \leqslant \mathfrak{B}$. Let $\varphi_1, \varphi_2, \ldots, \varphi_n, \ldots$ and $\psi_1, \psi_2, \ldots, \psi_n, \ldots$ be complete orthonormal in $\mathfrak{H}$ systems of eigenvectors of the operators $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Then the space $\mathscr{Z}$ can be represented as $\mathscr{Z} = X \oplus Y$, where X is a linear span of the vectors $\varphi_1, \ldots, \varphi_k$, and Y is a set of all $z \in \mathscr{Z}$, such that $(z, \psi_j) = 0$ for $j = 1, 2, \ldots, k$.

Proof. We show first that

$$\det\{(\varphi_n, \psi_m)\}_{n,m=1}^{k} \neq 0.$$

Let us assume that the opposite is true. Then there exists a vector $h = \sum_{j=1}^{k} a_j \varphi_j \neq 0$ such that $(h, \psi_j) = 0$, $j = 1, \ldots, k$. Since $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}_k$ and $\mathfrak{A} \leqslant \mathfrak{B}$, then by Lemma 1.3 we have that

$$0 < \sum_{j=1}^{k} \lambda_j(\mathfrak{A})(h, \varphi_j)^2 = A(h, h) \leqslant B(h, h) = \sum_{j=k+1}^{\infty} \lambda_j(\mathfrak{B})(h, \psi_j)^2 \leqslant 0.$$

The contradiction obtained proves that the determinant above does not vanish. Therefore, for any $z \in \mathscr{Z}$ the system

$$\left(z - \sum_{j=1}^{k} a_j \varphi_j, \psi_m\right) = 0, \quad m = 1, 2, \ldots, k,$$

has the unique solution $a_1^*, a_2^*, \ldots, a_k^*$. Putting $x = \sum_{j=1}^{k} a_j^* \varphi_j \in X$ and $y = z - x \in Y$, we obtain a decomposition

$$\mathscr{Z} = X \oplus Y.$$

THEOREM 1.2. Let $\mathscr{I}: \mathfrak{D} \to \mathbf{R}$ be a differentiable functional, let $\mathfrak{R} = \operatorname{grad} \mathscr{I}$, and let the condition (1.6) hold, where A and B correspond, according to (1.11), to the operators $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Moreover, let $\operatorname{ind} \mathfrak{A} = \operatorname{ind} \mathfrak{B} = k$. Then Eq. (1.9) has the unique generalized solution φ and the estimate

$$|\varphi| \leqslant |\Phi(0)|/\min(\alpha, \beta) \tag{1.17}$$

holds, where the constants α and β are determined by Eqs. (1.22) and (1.23); if $\varphi, 0 \in \mathfrak{D}(\mathfrak{R})$, then

$$\|\varphi\| \leqslant \|\mathfrak{R}(0)\|/\gamma, \quad \gamma = \min\{\lambda_k(\mathfrak{A}), \ -\lambda_{k+1}(\mathfrak{B})\}. \tag{1.18}$$

Proof. By Lemma 1.1 the functional $\mathscr{I}$ can be extended to the functional differentiable on the whole spece $\mathscr{Z}$. We define the spaces X and Y as in Lemma 1.5. Then $\mathscr{Z} = X \oplus Y$. Now in order to prove the existence of the solution and to establish estimate (1.17) it is sufficient, by virtue of Theorem 1.1, to prove that inequalities (1.2) and (1.3) hold. From the representation (1.13) we obtain immediately that

$$A(x, x) = \sum_{j=1}^{k} \lambda_j(\mathfrak{A})(x, \varphi_j)^2 \geqslant \lambda_k(\mathfrak{A})\|x\|^2, \quad x \in X, \tag{1.19}$$

$$B(y, y) = \sum_{j=k+1}^{\infty} \lambda_j(\mathfrak{B})(y, \psi_j)^2 \leqslant \lambda_{k+1}(\mathfrak{B})\|y\|^2, \ y \in Y. \tag{1.20}$$

Recall that $\mathfrak{A}, \mathfrak{B} \in \mathscr{E}(\mathscr{Z})$, i.e., $\mathscr{Z}$ has the same domain of elements and the same topology as the energy space of the operators $lI-\mathfrak{A}$ and $lI-\mathfrak{B}$ (l is sufficiently large). Hence the norm $|\cdot|$ in $\mathscr{Z}$ and energy norms $|\cdot|_{\mathfrak{A}}$ and $|\cdot|_{\mathfrak{B}}$ are equivalent, i.e., there exist constants $\alpha_1, \beta_1 > 0$ such that

$$|z|_{\mathfrak{A}} \geqslant \alpha_1 |z|, \quad |z|_{\mathfrak{B}} \leqslant \beta_1 |z|, \quad z \in \mathscr{Z}. \tag{1.21}$$

By (1.11) and (1.19) we have that

$$A(x, x) = l\|x\|^2 - |x|_{\mathfrak{A}}^2 \geqslant \lambda_k(\mathfrak{A})\|x\|^2.$$

Hence

$$[l - \lambda_k(\mathfrak{A})]\|x\|^2 \geqslant |x|_{\mathfrak{A}}^2,$$

and taking into account (1.21) we obtain the estimate

$$\|x\|^2 \geqslant \alpha_1^2/[l - \lambda_k(\mathfrak{A})].$$

Therefore, the estimate (1.2) is established with

$$\alpha = \lambda_k(\mathfrak{A})\,\alpha_1^2/[l - \lambda_k(\mathfrak{A})]. \tag{1.22}$$

The estimate (1.3) can be proved in the same manner. We can put

$$\beta = \beta_1^2 |\lambda_{k+1}(\mathfrak{B})|/[l - \lambda_{k+1}(\mathfrak{B})]. \tag{1.23}$$

In order to prove the estimate (1.18) we introduce linear manifolds $\mathscr{X} = X$ and $\mathscr{Y} = \{y \in \mathfrak{D} : (y, \psi_j) = 0,\ j = 1, \ldots, k\}$. As in the proof of Lemma 1.5 we can show that $\mathfrak{D} = \mathscr{X} \oplus \mathscr{Y}$. Let $\mathscr{P} : \mathfrak{D} \to \mathscr{X}$ be the projection of $\mathfrak{D}$ on $\mathscr{X}$ parallel to $\mathscr{Y}$ and let $\mathcal{Q} : \mathfrak{D} \to \mathscr{Y}$ be the complementary projection. By the arguments similar to those from [12, p. 111] we can obtain from (1.6) that

$$(\mathfrak{R}z - \mathfrak{R}h,\ (\mathscr{P} - \mathcal{Q})(z - h)) \geqslant \gamma(\|\mathscr{P}(z - h)\|^2 + \|\mathcal{Q}(z - h)\|^2).$$

Hence

$$\|\mathfrak{R}z - \mathfrak{R}h\| \geqslant \frac{\|\mathscr{P}(z-h)\|^2 + \|\mathcal{Q}(z-h)\|^2}{\|(\mathscr{P} - \mathcal{Q})(z-h)\| \cdot \|(\mathscr{P} + \mathcal{Q})(z-h)\|}\|z - h\| \cdot \gamma \geqslant \gamma\|z - h\|.$$

Putting $z = \varphi$, $h = 0$ we obtain the estimate (1.18).

2. Periodic Boundary-Value Problem for the Second-Order Equations

2.1. Hill's Differential Operators. We apply the results of the previous section to the following boundary-value problem:

$$\frac{d}{dt}\left(P(t)\frac{dz}{dt}\right) + Q(t)z = f(t), \quad 0 \leqslant t \leqslant \omega, \tag{2.1}$$

$$z(0) = z(\omega), \quad \dot{z}(0) = \dot{z}(\omega). \tag{2.2}$$

Let H and Z be Hilbert spaces with the scalar products $\langle\,,\,\rangle$, $\langle\,,\,\rangle_1$ and with the norms $|\cdot|$, $|\cdot|_1$, respectively. We assume that the space Z is dense and compactly imbedded in H.

We investigate the problem (2.1), (2.2) under the following assumptions:

a) for every $t \in [0, \omega]$ $P(t) : H \to H$ is a linear bounded self-adjoint operator such that $P(0) = P(\omega)$; the generator-valued function P(t) is strongly continuously differentiable and

$$\langle P(t)h, h\rangle \geqslant p_0|h|^2, \quad p_0 > 0, \quad h \in H; \tag{2.3}$$

b) for every $t \in [0, \omega]$ $Q(t) : D(Q) \to H$ is a linear self-adjoint operator, which for every t is defined on the domain D(Q); we assume that D(Q) is dense in H, that the operator-valued function Q(t) is strongly continuous and semibounded from above

$$\langle Q(t)h, h\rangle \leqslant a|h|^2, \quad h \in D(Q); \tag{2.4}$$

that the energy space of the operator $lI - Q(t)$ ($l > a$) has the same domain of elements as Z and that there exist constants $c_1, c_2 > 0$ such that

$$c_1|z|_1^2 \leqslant |z|_Q^2 \leqslant c_2|z|_1^2, \tag{2.5}$$

where by $|\cdot|_Q$ we denote the energy norm generated by the operator $lI - Q(t)$;

c) f(·) is an element of the Hilbert space $\mathfrak{H}=L_2([0, \omega]; H)$, in which the scalar product (,) and the norm $\|\cdot\|$ are defined in the usual way.

The solution of the problem (2.1), (2.2), by our definition, will be a function $\varphi:[0, \omega] \to D(Q)$ such that $(Q\varphi)(\cdot)\in\mathfrak{H}$, the functions $\varphi(\cdot)$, and $(P\dot{\varphi})\cdot(\cdot)$ are absolutely continuous, $(P\dot{\varphi})(\cdot)\in\mathfrak{H}$, $\varphi(t)$ satisfies Eq. (2.1) for almost all $t\in[0, \omega]$.

Remark 2.1. The problem of finding periodic in t solutions of boundary-value problems for partial differential equations can be reduced to the problem (2.1), (2.2). As an example of the operator P(t) we consider the multiplication operator. As an operator Q(t), $t\in[0, \omega]$ we consider the operator which is defined on the Hilbert space $H = L_2(\Omega)$ (Ω is a bounded domain in R^n with a sufficiently smooth boundary) and which is generated by a symmetric differential expression

$$Q(t)z=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ji}(t, x)\frac{\partial z}{\partial x_i}\right)+b(t, x)z$$

where the coefficients are sufficiently smooth and ω-periodic in t. Let the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ji}(t, x)\xi_j\xi_i\geqslant\delta\sum_{i=1}^{n}\xi_i^2,\quad t\in[0, \omega],\ x\in\Omega$$

hold for some $\delta > 0$. If we put $D(Q) = \overset{\circ}{W}{}_2^2(\Omega)$, then for every $t\in[0, \omega]$ the operator Q(t) is self-adjoint and semibounded from above. Note that in this case $Z = \overset{\circ}{W}{}_2^1(\Omega)$.

Consider in $\mathfrak{H}$ the differential operator $\mathfrak{L}$, which is defined as follows:

$$\mathfrak{L}z=\frac{d}{dt}\left(P(t)\frac{dz}{dt}\right)+Q(t)z,\quad z\in\mathfrak{D}(\mathfrak{L}), \tag{2.6}$$

where

$$\mathfrak{D}(\mathfrak{L})=\mathfrak{D}(P, Q)=\left\{z\in\mathfrak{H}\ \middle|\ \begin{array}{l} z\in D(Q),\quad Qz\in\mathfrak{H},\\ z,\ P\dot{z}\text{ — is absolutely continuous},\ (P\dot{z})^{\cdot}\in\mathfrak{H},\\ z(0)=z(\omega),\ \dot{z}(0)=\dot{z}(\omega). \end{array}\right\} \tag{2.7}$$

The operator $\mathfrak{L}:\mathfrak{D}(\mathfrak{L})\to\mathfrak{H}$ is called *Hill's operator* [generated by the pair of coefficients {P(t), Q(t)}]. Now the solvability of the problem (2.1), (2.2) is equivalent to the solvability of the equation

$$\mathfrak{L}z=f. \tag{2.8}$$

LEMMA 2.1. The domain $\mathfrak{D}(\mathfrak{L})$ is dense in $\mathfrak{H}$, the operator $\mathfrak{L}:\mathfrak{D}(\mathfrak{L})\to\mathfrak{H}$ is symmetrical and semibounded from above.

Proof. In order to show the density of $\mathfrak{D}(\mathfrak{L})$ in $\mathfrak{H}$ we choose a complete in H system of elements $\{u_\alpha,\ \alpha\in A\}$ from D(Q). Then the system of functions

$$\{\cos\varkappa kt u_\alpha,\ \sin\varkappa kt u_\alpha;\ \varkappa=2\pi/\omega,\ k=0, 1, \ldots,\ \alpha\in A\} \tag{2.9}$$

is complete in $\mathfrak{H}$, and therefore each $z\in\mathfrak{H}$ has the Fourier expansion

$$z(t)=\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}[\alpha_{km}\cos\varkappa kt+\beta_{km}\sin\varkappa kt]\,u_m,$$

where $\alpha_{km}=(z, \cos\varkappa kt u_m)$, $\beta_{km}=(z, \sin\varkappa kt u_m)$.

We put

$$z_N(t)=\sum_{k=0}^{N}\sum_{m=1}^{N}[\alpha_{km}\cos\varkappa kt+\beta_{km}\sin\varkappa kt]\,u_m.$$

It is clear that $z_N\in\mathfrak{D}(\mathfrak{L})$ for every fixed N. Moreover,

$$\|z_N-z\|\xrightarrow[N\to\infty]{}0.$$

This proves the completeness of $\mathfrak{D}(\mathfrak{L})$ in $\mathfrak{H}$.

For $z, h \in \mathfrak{D}(\mathfrak{L})$ we have that

$$(\mathfrak{L}z, h) = \int_0^\omega \{\langle (P\dot{z})^{\cdot}(t),\ h(t)\rangle + \langle Q(t)z(t),\ h(t)\rangle\}\,dt.$$

We integrate by parts in the first integral and obtain

$$(\mathfrak{L}z, h) = \int_0^\omega \{-\langle P(t)\dot{z}(t),\ \dot{h}(t)\rangle + \langle Q(t)z(t),\ h(t)\rangle\}\,dt.$$

Hence, taking into account that the generators P(t) and Q(t) are self-adjoint, we obtain that the operator $\mathfrak{L}$ is symmetrical and from (2.3), (2.4) we obtain the estimate

$$(\mathfrak{L}z,\ z) \leqslant -p_0\|\dot{z}\|^2 + a\|z\|^2 \leqslant a\|z\|^2. \tag{2.10}$$

The lemma is proved.

We fix some $l \geqslant a$ and denote by $H_{\mathfrak{L}}$ the energy space of the operator $lI - \mathfrak{L}$. Note that for $z, h \in \mathfrak{D}(\mathfrak{L})$ the energy scalar product and the energy norm are

$$[z, h]_{\mathfrak{L}} = -(P\dot{z}, \dot{h}) + ((lI - Q)z, h), \tag{2.11}$$

$$|z|^2_{\mathfrak{L}} = \int_0^\omega \{-\langle P(t)\dot{z}(t),\ \dot{z}(t)\rangle + |z(t)|^2_Q\}\,dt. \tag{2.12}$$

Actually these formulas are true for every element from $H_{\mathfrak{L}}$.

We introduce the space $L_2([0, \omega]; Z)$ with the scalar product

$$(z, h)_1 = \int_0^\omega \langle z(t), h(t)\rangle_1 dt$$

and with the norm $\|z\|_1 = (z, z)_1^{1/2}$. We also introduce the space $W_2^1([0, \omega]; H; \omega)$, which consists of all functions z from the Sobolev space $W_2^1([0, \omega]; H)$, such that $z(0) = z(\omega)$. In this space the norm is defined as usual:

$$\|z\|^2_{W_2^1} = \|z\|^2 + \|\dot{z}\|^2.$$

We consider now the space

$$\mathscr{Z} = L_2([0, \omega], Z) \cap W_2^1([0, \omega]; H; \omega)$$

with the scalar product

$$[z, h] = (z, h)_1 + (\dot{z}, \dot{h})$$

and with the norm

$$|z|^2 = \|z\|_1^2 + \|\dot{z}\|^2.$$

It is clear that $\mathscr{Z}$ is a Hilbert space and for $z \in \mathfrak{D}(\mathfrak{L})$ the estimate

$$c_3|z| \leqslant |z|_{\mathfrak{L}} \leqslant c_4|z| \tag{2.13}$$

holds.

LEMMA 2.2. The space $H_{\mathfrak{L}}$ has the same domain of elements and the same topology as the space $\mathscr{Z}$.

Proof. Let $z \in H_{\mathfrak{L}}$. Then there exists a sequence $\{z_n\}$ of elements from $\mathfrak{D}(\mathfrak{L})$ such that

$$|z_n - z_m|_{\mathfrak{L}} \xrightarrow[n,m\to\infty]{} 0, \quad \|z_n - z\| \xrightarrow[n\to\infty]{} 0. \tag{2.14}$$

Since $z_n - z_m \in \mathfrak{D}(\mathfrak{L})$, it follows from (2.12) that the sequence $\{\dot{z}_n\}$ is a Cauchy sequence in $\mathfrak{H}$. Therefore, there exists $w \in \mathfrak{H}$ such that $\|\dot{z}_n - w\| \to 0$ as $n \to \infty$. From the second relation in (2.14) it follows that $w = \dot{z} \in \mathfrak{H}$. Let us show that $z(0) = z(\omega)$. The continuity of the imbedding of $W_2^1([0, \omega]; H)$ in the space $C([0, \omega]; H)$ of continuous functions $z:[0, \omega] \to H$ with the uniform norm implies that the following estimate holds:

$$\max_{0\leqslant t\leqslant\omega} |z(t)| \leqslant c(\|z\| + \|\dot{z}\|). \tag{2.15}$$

Therefore, taking into account that $z_n(0) = z_n(\omega)$, we obtain

$$|z(0) - z(\omega)| \leqslant |z(0) - z_n(0)| + |z_n(\omega) - z(\omega)| \leqslant c(\|z - z_n\| + \|\dot{z} - \dot{z}_n\|) \xrightarrow[n\to\infty]{} 0$$

Hence $z \in W_2^1([0,\omega]; H; \omega)$.

From (2.14), (2.12), and (2.5) it follows that the sequence $\{z_n\}$ is a Cauchy sequence in $L_2([0, \omega]; Z)$. Since this space is imbedded in $\mathfrak{H}$ and $\|z_n - z\| \to 0$ as $n \to \infty$ we obtain that $z \in L_2([0, \omega]; Z)$. Thus we have proved that $H_{\mathfrak{L}} \subseteq \mathscr{Z}$. By taking limits in (2.12) we can easily prove that the formula (2.12), and therefore (2.11), holds for every $z, h \in H_{\mathfrak{L}}$.

Let us prove that $\mathscr{Z} \subseteq H_{\mathfrak{L}}$. Let $z \in \mathscr{Z}$. We construct a sequence $\{z_N\}$, $z_N \in \mathfrak{D}(\mathfrak{L})$, with the properties (2.14). For this we choose a system of type (2.9) which is complete in $\mathfrak{H}$, where $u_\alpha \in D(Q)$ and the following condition holds:

$$\langle Q(0)u_m, u_n\rangle = \delta_{mn}, \quad \delta_{mn} - \text{ is the Kronecker delta symbol.} \tag{2.16}$$

We consider the Fourier expansion for the function $\dot{z} \in \mathfrak{H}$:

$$\dot{z}(t) = \sum_{k,m=1}^{\infty} [a_{km}\cos \varkappa kt + b_{km}\sin \varkappa kt]\, u_m, \tag{2.17}$$

where

$$a_{km} = (\dot{z}, \cos \varkappa kt u_m), \quad b_{km} = (\dot{z}, \sin \varkappa kt u_m).$$

Note that in (2.17) we sum over $k \neq 0$, since $a_{0m} = 0$ by the condition that $z(0) = z(\omega)$. We integrate every term in (2.17) and obtain the expansion

$$z(t) = z(0) + \sum_{k,m=1}^{\infty} \frac{1}{\varkappa k}[a_{km}\sin \varkappa kt + b_{km}(1 - \cos \varkappa kt)]\, u_m.$$

Since $D(Q)$ is dense in H, for every $\varepsilon > 0$ we choose such $\xi \in D(Q)$ that $|z(0) - \xi| < \varepsilon$. We consider the function

$$z_N(t) = \xi + \sum_{k,m=1}^{N} \frac{1}{\varkappa k}[a_{km}\sin \varkappa kt + b_{km}(1 - \cos \varkappa kt)]\, u_m.$$

It is not difficult to see that $z_N \in \mathfrak{D}(\mathfrak{L})$ and that $\|z_N - z\| \to 0$ as $n \to \infty$. Let us show that the sequence $\{z_N\}$ is a Cauchy sequence $H_{\mathfrak{L}}$. in Firstly,

$$\int_0^\omega \langle P(t)(\dot{z}_N(t) - \dot{z}_M(t)),\ \dot{z}_N(t) - \dot{z}_M(t)\rangle\, dt \xrightarrow[N,M\to\infty]{} 0$$

and

$$\|z_N - z_M\| \xrightarrow[N,M\to\infty]{} 0.$$

Secondly, taking into account that the operator-valued function $[lI - Q(t)]^{1/2} \cdot [lI - Q(0)]^{-1/2}$ is bounded and (2.16) (for $N > M$)

$$\int_0^\omega |z_N(t) - z_M(t)|^2 dt \leqslant \max_{0<t\leqslant\omega} \left|[lI - Q(t)]^{1/2} \cdot [lI - Q(0)]^{-1/2}\right|^2 \cdot$$

$$\cdot \int_0^\omega \langle [lI - Q(0)](z_N(t) - z_M(t)),\ z_N(t) - z_M(t)\rangle\, dt \leqslant$$

$$\leqslant d_1 \Big\{ l\|z_N - z_M\|^2 + \sum_{k,m=M}^{N} \sum_{l,n=M}^{N} \frac{1}{\varkappa k}\frac{1}{\varkappa l} \int_0^\omega [a_{km}\sin \varkappa kt +$$

$$+ b_{km}(1 - \cos \varkappa kt)][a_{ln}\sin \varkappa lt + b_{ln}(1 - \cos \varkappa lt)]\, dt \cdot \langle Q(0)u_m, u_n\rangle \Big\} \leqslant$$

$$\leqslant d_2 \Big\{ \|z_N - z_M\|^2 + \sum_{k,m=M}^{N} \frac{1}{\varkappa^2 k^2}(a_{km}^2 + b_{kM}^2) \Big\} \xrightarrow[N,M\to\infty]{} 0$$

by Bessel's inequality. Therefore, we have proved that the sequence $\{z_N\}$ satisfies (2.14) and hence $z \in H_{\mathfrak{L}}$.

Thus, $H_{\mathfrak{L}}$ has the same domain of elements as $\mathscr{Z}$. The equivalence of the norms in these spaces follows from (2.13) and from the fact that $\mathfrak{D}(\mathfrak{L})$ is dense in $H_{\mathfrak{L}} = \mathscr{Z}$.

LEMMA 2.3. The space $\mathscr{Z}$ is compactly imbedded in $\mathfrak{H}$.

Proof. Let $K = \{z \in \mathscr{Z} : |z| \leqslant r\}$. We have to prove that this is totally bounded in $\mathfrak{H}$. Moreover, we shall prove that this set is totally bounded in $C([0, \omega]; H)$. It is not difficult to see that firstly the set K is uniformly bounded, since by (2.15)

$$\max_{0 \leqslant t \leqslant \omega} |z(t)| \leqslant c|z| \leqslant cr.$$

Secondly, this family is equicontinuous since

$$|z(t) - z(s)| = \left| \int_s^t \dot{z}(\tau)\, d\tau \right| \leqslant \|\dot{z}\| \cdot |t - s|^{1/2} \leqslant r\, |t - s|^{1/2}.$$

Finally, the set $K_t = \{z(t) : z \in K\}$ of values of the functions from K for every $t \in [0, \omega]$ is totally bounded in H. In order to prove this fact we define the functions from this family on the whole axis by continuing them ω-periodically, and we construct for them Steklov's mean functions

$$z_\varepsilon(t) = \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} z(\tau)\, d\tau.$$

We have that

$$|z_\varepsilon(t)|_1 \leqslant \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} |z(\tau)|_1 d\tau \leqslant (2\varepsilon)^{-1/2} \|z\|_1 \leqslant cr\,(2\varepsilon)^{-1/2}.$$

Since Z is compactly imbedded in H, the set $K_{t,\varepsilon} = \{z_\varepsilon(t) : z \in K\}$ for every t and $\varepsilon > 0$ is totally bounded in H. From the relation

$$|z_\varepsilon(t) - z(t)| \leqslant \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} |z(\tau) - z(t)|\, d\tau \leqslant r\,(2\varepsilon)^{-1/2} \int_{t-\varepsilon}^{t+\varepsilon} |\tau - t|^{1/2} d\tau \leqslant 2r3^{-1}\varepsilon^{1/2} \underset{\varepsilon \to 0}{\longrightarrow} 0$$

it follows, by Frechet's theorem, that the set H is totally bounded in K_t. By the Arzela–Ascoli theorem the set K is totally bounded in $C([0, \omega]; H)$, and therefore, in $\mathfrak{H}$. The lemma is proved.

Remark 2.2. If the operator-valued function Q(t) is strongly differentiable in D(Q) then Hill's operator has a regular point and hence it is self-adjoint. For P(t) = I this fact was established in [5].

We denote by $\tilde{\mathfrak{L}}$ the Friedrichs self-adjoint extension of the operator $\mathfrak{L}$. As is known the energy spaces of the operators $lI - \mathfrak{L}$ and $lI - \tilde{\mathfrak{L}}$ coincide. Therefore, the self-adjoint extension of Hill's operators, generated, according to (2.6), (2.7), by coefficients {P(t), Q(t)} which satisfy the conditions a), b), belong to the class $\mathscr{E}(\mathscr{Z})$. This class was defined in the previous section. Hence these operators can be subdivided into the regularity classes. In particular if P(t) = I, and if Q(t) = Q does not depend on t, has a discrete spectrum and $\nu_1, \nu_2, \ldots, \nu_s$ are the positive eigenvalues of the operator Q, then (see [5]) the operator $\mathfrak{L}$ is regular precisely when

$$\nu_j \neq (2\pi n/\omega)^2, \quad j = 1, 2, \ldots, s; \quad n = 0, 1, \ldots \tag{2.18}$$

Moreover, its spectrum coincides with the set

$$\sigma(\mathfrak{L}) = \{\mu \in \mathbf{R} : \mu = \nu^2 - (2\pi n/\omega)^2, \ \nu \in \sigma(Q), \ n = 0, 1, \ldots\}.$$

If condition (2.18) is satisfied, then

$$\operatorname{ind} \mathfrak{L} = s + 2 \sum_{j=1}^{s} \left[\frac{\omega}{2\pi} \sqrt{\nu_j} \right], \tag{2.19}$$

where by [·] we denote the integer part of a number.

Remark 2.3. If Q(t) is a negative-definite operator-valued function, then by (2.10), ind $\mathfrak{L} = 0$.

2.2. Solvability Theorems. We consider a periodic boundary-value problem for the nonlinear second-order equation

$$\frac{d}{dt}\left(P(t)\frac{dz}{dt}\right) + F(t, z) = 0, \tag{2.20}$$

$$z(0) = z(\omega), \quad \dot{z}(0) = \dot{z}(\omega), \tag{2.21}$$

where as before P(t) possesses the property a), and F:[0, ω] × D → H (D is a linear set which is dense in H), and F(t, z) for every $z \in D$ is an element from the space $\mathfrak{H}$.

We define the solution of the problem (2.20), (2.21) as a function $\varphi:[0, \omega] \to D$ such that $\varphi(\cdot)$ is an element of the space $\mathscr{Z}$, the functions $(P\dot{\varphi})^{\cdot}(\cdot)$, $F(t, \varphi(\cdot))$ belong to the space $\mathfrak{H}$, $\varphi(t)$ satisfies the conditions (2.21), and for almost all $t \in [0, \omega]$ it satisfies Eq. (2.20).

The problem (2.20), (2.21) generates the operator in $\mathfrak{H}$

$$\mathfrak{N}z = \frac{d}{dt}\left(P(t)\frac{dz}{dt}\right) + F(t, z)$$

which is defined on the domain

$$\mathfrak{D}(\mathfrak{N}) = \left\{ z \in \mathscr{Z} \,\middle|\, \begin{array}{l} Pz - \text{is absolutely continuous } (P\dot{z}) \in \mathfrak{H}, \\ F(t, z) \in \mathfrak{H}, \quad \dot{z}(0) = \dot{z}(\omega). \end{array} \right\}$$

We assume that the set $\mathfrak{D}(\mathfrak{N})$ is linear and dense in $\mathscr{Z}$. For example, this is true if F(t, z) = Q(t)z + f(t, z), f is a bounded in $\mathscr{Z}$ operator and Q satisfies the condition b).

Now the solvability of the problem (2.20), (2.21) is equivalent to the solvability of the equation

$$\mathfrak{N}z = 0. \tag{2.22}$$

Recall that if $\mathfrak{N}$ is a potential operator, such that $\mathscr{I} = \text{pot}\,\mathfrak{N}:\mathfrak{D}(\mathfrak{N}) \to \mathbf{R}$ can be extended to the whole space $\mathscr{Z}$ without the loss of differentiability, then (by definition) a stationary point of the functional $\mathscr{I}:\mathscr{Z} \to \mathbf{R}$, i.e., the solution of the equation

$$D\mathscr{I}(z;\ h) = 0 \quad \forall h \in \mathscr{Z}, \tag{2.23}$$

is called a generalized solution of Eq. (2.22).

THEOREM 2.1. Assume that there exists a function g:[0, ω] × D → **R**, such that it is continuous with respect to t, differentiable with respect to z, and for every z, $h \in \mathfrak{D}(\mathfrak{N})$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{g(t, z + \varepsilon h) - g(t, z)\} = \langle F(t, z), h\rangle. \tag{2.24}$$

Assume that there exist two pairs of coefficients $\{P_A(t), A(t)\}$, $\{P_B(t), B(t)\}$, which generate Hill's operators $\mathfrak{A}$ and $\mathfrak{B}$, respectively, such that $\mathfrak{A}, \mathfrak{B} \in \mathfrak{G}_k$ for some k > 0 and the inequalities

$$\langle P_B(t)z,\ z\rangle \leqslant \langle P(t)z,\ z\rangle \leqslant \langle P_A(t)z,\ z\rangle,\ z \in H, \tag{2.25}$$

$$\langle A(t)h,\ h\rangle \leqslant \langle F(t,\ z+h) - F(t,\ z),\ h\rangle \leqslant \langle B(t)h,\ h\rangle,\ z,\ h \in D. \tag{2.26}$$

hold.

Then Eq. (2.22) has the unique generalized solution φ and the following estimate holds:

$$|\varphi| \leqslant \gamma_1 \|F(t,\ 0)\| \tag{2.27}$$

with some constant $\gamma_1 > 0$.

If $\varphi \in \mathfrak{D}(\mathfrak{N})$, then the estimate

$$\|\varphi\| \leqslant \|F(t,\ 0)\| / \min\{\lambda_k(\mathfrak{A}),\ -\lambda_{k+1}(\mathfrak{B})\} \tag{2.28}$$

holds.

Proof. We consider the functional

$$\mathcal{I}(z)=\int_0^{\omega}\left\{-\frac{1}{2}\langle P(t)\dot z(t),\ \dot z(t)\rangle+g(t,z(t))\right\}dt,\quad z\in\mathfrak{D}(\mathfrak{R}). \tag{2.29}$$

Let us prove that it is differentiable and for z, $h\in\mathfrak{D}(\mathfrak{R})$:

$$D\mathcal{I}(z;h)=\int_0^{\omega}\{-\langle P(t)\dot z(t),\ \dot h(t)\rangle+\langle F(t,z(t)),\ h(t)\rangle\}\,dt. \tag{2.30}$$

We first note that it follows from (2.26) that

$$|\langle F(t,z+h)-F(t,z),h\rangle|\leqslant\rho|h|_1^2,\quad z,h\in D, \tag{2.31}$$

with some constant $\rho > 0$. Therefore, the function $\langle F(t, z(t) + \theta h(t)), h(t)\rangle$ is continuous in θ, and hence, the following representation holds:

$$\frac{1}{\varepsilon}\{g(t,z(t)+\varepsilon h(t))-g(t,z(t))\}=\int_0^1\langle F(t,z(t)+\theta\varepsilon h(t)),\ h(t)\rangle\,dt. \tag{2.32}$$

Moreover, by (2.26) we have that

$$|\langle F(t,z(t)+\varepsilon\theta h(t)),\ h(t)\rangle|\leqslant\varepsilon\theta\rho|h(t)|_1^2+|F(t,z(t))|\cdot|h(t)|$$

Hence for $0<\varepsilon, 0\leqslant\theta\leqslant 1$ the integrand in (2.32) has a majorant with respect to ε, which is integrable with respect to t and θ. Therefore

$$\lim_{\varepsilon\to 0}\int_0^{\omega}\frac{1}{\varepsilon}\{g(t,z(t)+\varepsilon h(t))-g(t,z(t))\}\,dt=\int_0^{\omega}\langle F(t,z(t)),\ h(t)\rangle\,dt.$$

From this it is easy to obtain the formula (2.30) and the relation

$$D\mathcal{I}(z;\ h)=(\mathfrak{R}z,\ h),\quad z,\ h\in\mathfrak{D}(\mathfrak{R}). \tag{2.33}$$

We consider the functionals $A, B:\mathcal{Z}\times\mathcal{Z}\to\mathbf{R}$, which correspond, according to (1.11), to the operators $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Taking into account (2.11) we can write that

$$A(z,z)=\int_0^{\omega}\{-\langle P_A(t)\dot z(t),\ \dot z(t)\rangle+\langle A(t)z(t),\ z(t)\rangle\}\,dt,$$

$$B(z,z)=\int_0^{\omega}\{-\langle P_B(t)\dot z(t),\ \dot z(t)\rangle+\langle B(t)z(t),z(t)\rangle\}\,dt.$$

By (2.25) and (2.26) the functional $\mathcal{I}$ satisfies the conditions of Lemma 1.2. Therefore, it can be extended to the whole space $\mathcal{Z}$. The extended operator is differentiable and the inequality (1.7) holds. By Theorem 1.2 Eq. (2.22) has the unique generalized solution. The estimate (1.17) holds for this solution. By (2.30)

$$|D\mathcal{I}(0;\ h)|\leqslant\|F(t,0)\|\cdot\|h\|\leqslant c\|F(t,0)\|\cdot|h|,$$

where c is the norm of imbedding of $\mathcal{Z}$ in $\mathfrak{H}$, hence the estimate (2.27) holds. If $\varphi\in\mathfrak{D}(\mathfrak{R})$, then (2.28) follows directly from (1.18). The theorem is proved.

COROLLARY 2.1. Let constants $0 < m < M$ exist and let constant operators A and B with properties b) exist such that

$$m|z|^2\leqslant\langle P(t)z,\ z\rangle\leqslant M|z|^2,$$
$$\langle Ah,\ h\rangle\leqslant\langle F(t,\ z+h)-F(t,\ z),\ h\rangle\leqslant(Bh,\ h);$$

where F is a potential operator, i.e., Eq. (2.24) holds. Let $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_l>0$ and $\mu_1\geqslant\mu_2\geqslant\ldots\geqslant\mu_l$ be all positive eigenvalues of the operators A and B, respectively. If for every $j = 1,\ldots,l$ we can find an integer $n_j\geqslant 0$ such that

$$\left(\frac{2\pi}{\omega}n_j\right)^2<\frac{\lambda_j}{M}\leqslant\frac{\mu_j}{m}<\left(\frac{2\pi}{\omega}(n_j+1)\right)^2,\quad j=1,\ldots,l, \tag{2.34}$$

then Eq. (2.22) has the unique generalized solution.

Proof. Inequalities (2.34) guarantee that Hill's operators $\mathfrak{A}$ and $\mathfrak{B}$, generated by the coefficients {mI, A} and {M·I, B}, respectively, belong to the same regularity class.

Note that Corollary 2.1 contains as a particular case the results of [6-11] where the problem (2.20)-(2.21) is investigated in the finite dimensional case.

COROLLARY 2.2. Let F(t, z) = Q(t)z + f(t, z), Q(t):D → H, satisfy the conditions b), the operator-valued function Q(t) is strongly continuously differentiable on D, and f:[0, ω] × H → H is such that the composition operator (fz)(t) = f(t, z(t)) is continuous in $\mathfrak{H}$. Moreover, let the conditions of Theorem 2.1 hold.

Then the generalized solution of Eq. (2.22) $\varphi \in \mathfrak{D}(\mathfrak{N})$.

Proof. Note that the operator $\mathfrak{N}$ can be represented as $\mathfrak{N}=\mathfrak{L}_0+\mathscr{F}$, where $\mathfrak{L}_0$ is Hill's operator, generated by the coefficients $\{P(t),\ Q(t)-a\cdot I\}$, and $(\mathscr{F}z)(t)=f(t,\ z(t))+az(t)$, where the constant a is determined from condition (2.24). By virtue of Remarks 2.2 and 2.3 the operator $\mathfrak{L}_0$ is self-adjoint and negative-definite. The corollary follows from Lemma 1.2.

Consider a linear problem (2.1), (2.2) for $f \in \mathfrak{H}$, or, which is the same, the equation

$$\mathfrak{L}z = f. \tag{2.35}$$

Note that in the linear case the generalized solution of Eq. (2.35) φ, and only it, satisfies the identity

$$L(\varphi,\ h) = (f,\ h) \quad \forall h \in \mathscr{Z}. \tag{2.36}$$

The functionals L(z, h) and $\tilde{L}$(z, h), generated by the operators $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$, coincide; therefore, the usual solution of the equation

$$\tilde{\mathfrak{L}}z = f \tag{2.37}$$

is the generalized solution of Eq. (2.35).

THEOREM 2.2. Let the pairs $\{P_A(t),\ A(t)\}$ and $\{P_B(t),\ B(t)\}$ be such that the inequalities (2.25) hold and for $h \in D(Q)$

$$\langle A(t)h,\ h\rangle \leqslant \langle Q(t)h,\ h\rangle \leqslant \langle B(t)h,\ h\rangle, \tag{2.38}$$

and let the operators $\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}}$ belong to the same regularity class $\mathfrak{G}_k$.

Then Eq. (2.35) has the unique generalized solution and the following estimates hold:

$$|\varphi| \leqslant \gamma_1 \|f\|, \tag{2.39}$$

$$\|\varphi\| \leqslant \|f\|/\min\{\lambda_k(\tilde{\mathfrak{A}}),\ -\lambda_{k+1}(\tilde{\mathfrak{B}})\}. \tag{2.40}$$

Proof. The existence of the solution and the estimate (2.39) follows from Theorem 2.1. The estimate (2.40) follows from the representation

$$\varphi = \tilde{\mathfrak{L}}^{-1} f$$

and from the estimate (1.14).

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AN OPTIMAL CONTROL PROBLEM FOR AN ELLIPTICAL SYSTEM WITH POWER NONLINEARITY

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UDC 517.977.56

Lions and others [1, 2] considered the method of solving linear problems of optimal control using the necessary conditions for optimality in the form of variational inequalities. The direct extension of these results to nonlinear systems is difficult, connected with the necessity to prove the differentiability of the function of the state of the control system.

Another method of obtaining optimality conditions was suggested in [3], connected with the construction of the so-called "quasiconjugate system." Moreover, to obtain optimality conditions it is sufficient that the function of the state of control be weakly continuous. Thus, we are able to extend significantly the class of optimization problems which are soluble using variational inequalities. These results were developed further in [4].

In this article the method developed in [3, 4] is applied to the solution of an optimal control problem for an elliptical system with power nonlinearity.

1. Statement of the Problem

In the open bounded region Ω of the space R^2, with sufficiently smooth boundary $\partial\Omega$, consider the control process described by the equations

$$-\Delta y(v)+[y(v)]^3=f+v \quad x\in\Omega, \tag{1.1}$$

$$y(v)=0 \quad x\in\partial\Omega, \tag{1.2}$$

where Δ is the Laplace operator, v is a control defined on the set

$$U=\{v|v\in L^2(\Omega),\quad v(x)\in K \text{ a.e.}\},$$

K is a convex closed set, y(v) is the function of the state of the system corresponding to the control v, and $f\in L^{4/3}(\Omega)$ is a known function.

Define the functional space $Y=H_0^1(\Omega)\cap L^4(\Omega)$. Clearly, the space Y with the norm $\|\cdot\|_Y=\|\cdot\|+\|\cdot\|_{L^4\Omega}$, where $\|\cdot\|$ is the norm in $H_0^1(\Omega)$, is Banach.

Definition 1. The function y(v) is called a generalized solution of problem (1.1), (1.2), corresponding to the control $v\in U$, if $y(v)\in Y$ and we have the equation

$$a[y(v),\lambda]+\langle[y(v)]^3,\lambda\rangle=\langle f+v,\lambda\rangle \quad \forall\lambda\in Y,$$

Translated from Sibirskii Matematicheskii Zhurnal, Vol. 25, No. 1, pp. 120-125, January-February, 1984. Original article submitted January 22, 1982.

 0037-4466/84/2501-0100$8.50