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STRONGLY HOMOGENEOUS TORSION-FREE ABELIAN GROUPS

P. A. Krylov

UDC 512.541

A torsion-free Abelian group G is called *strongly homogeneous* if the automorphism group of G acts transitively on the set of all pure subgroups of rank 1, i.e., if A and B are any pure subgroups of G of rank 1, then $\alpha A = B$ for some $\alpha \in \mathrm{Aut}\, G$.

Strongly homogeneous groups form an important and interesting class of groups. Closely connected with these groups are the strongly homogeneous torsion-free rings. An associative ring R with unity is called *strongly homogeneous* if each element is an integral multiple of some element that is invertible in R. The additive group R^+ of a strongly homogeneous torsion-free ring R is strongly homogeneous. Indeed, if A and B are pure subgroups of R^+ of rank 1, then there exist invertible elements $u \in A$ and $v \in B$. Left multiplication of R by the element $w = vu^{-1}$ is an automorphism of the group R^+ and $wA = B$.

In certain special cases, strongly homogeneous torsion-free groups of finite rank were described in [1-3]. Arnold [4] completed the description of strongly homogeneous groups of finite rank. In the present paper we study strongly homogeneous torsion-free groups of arbitrary rank and their endomorphism rings. We prove that a strongly homogeneous torsion-free

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V. V. Kuibyshev Tomsk State University. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 24, No. 2, pp. 77-84, March-April, 1983. Original article submitted March 23, 1981.

group G is isomorphic to a tensor product of a module over a strongly homogeneous ring, all of whose submodules of countable rank are free, by a torsion-free group of rank 1 (Theorem 1). It follows that a countable torsion-free group G is strongly homogeneous if and only if $G \cong F \otimes_Z A$, where F is a finitely or countably generated free module over a countable strongly homogeneous ring and A is a group of rank 1 (Corollary 2). Since a torsion-free group of finite rank is countable, this corollary implies Arnold's theorem [4, Theorem 1]. We then prove (Theorem 2, Corollary 4) that a strongly homogeneous torsion-free group is essentially determined by its endomorphism ring in the class of all strongly homogeneous groups.

All groups considered in this paper are Abelian. The group terminology and notation are taken from [5, 6]. If G is a torsion-free group and p is a prime, then the *p-height* $h_p(a)$ of an element $a \in G$ is the largest integer k for which the equation $a = p^k x$ is solvable. If there exists no such x, then $h_p(a) = \infty$. If $p_1, p_2, \ldots$ is the sequence of all primes, then $\chi(a) = \{h_{p_1}(a), h_{p_2}(a), \ldots\}$ is the *characteristic* of a. If $b \in G$ and $h_{p_i}(a) \leqslant h_{p_i}(b)$ (1, 2,...), then we write $\chi(a) \leqslant \chi(b)$. A class of equivalent characteristics is called a *type* [6, Sec. 85]. A torsion-free group of rank 1 is completely determined by its type. A group G in which all nonzero elements have the same type is called *homogeneous*. This common type is called the *type of the group* G and is denoted by t(G). A strongly homogeneous group is homogeneous. A type is called *idempotent* if it contains a characteristic consisting of 0 and ∞. Let E(G) denote the endomorphism ring of the group G, Z the ring of integers, and Q the field of rational numbers. If A is a torsion-free group and $g \otimes a \in G \otimes_Z A$, $g \in G$, $a \in A$, then $h_p(g \otimes a) = h_p(g) + h_p(a)$ (see [5, Sec. 60, Exercise 9]). If A has rank 1, i.e., is isomorphic to a subgroup of Q, then each element of $G \otimes_Z A$ can be written in the form $g \otimes a$ for certain $g \in G$ and $a \in A$.

We begin our exposition of the results with the following

Remark 1. A strongly homogeneous torsion-free group G has the following property. If $0 \neq a, b \in G$ and $\chi(a) \leqslant \chi(b)$ ($\chi(a) = \chi(b)$), then $\alpha a = b$ for some $\alpha \in E(G)$ ($\alpha \in \mathrm{Aut}\, G$). Let A and B be the pure subgroups of G generated by a and b, respectively. Then $\beta A = B$, $\beta \in \mathrm{Aut}\, G$. Since $\beta a, b \in B$ and B has rank 1, we have $n(\beta a) = mb$ for certain natural numbers n and m. We may assume (n, m) = 1. An automorphism preserves characteristics of elements. Therefore, $\chi(\beta a) \leqslant \chi(b)$, hence mG = G. Consequently, multiplication by n/m is an endomorphism of G, and $(n\beta/m)a = b$. If $\chi(a) = \chi(b)$, then also nG = G, $n \cdot 1/m$ is an automorphism of G, and $(n\beta/m)a = b$. In particular, if G has idempotent type, then each pure subgroup contains a generator of the group G as an E(G)-module. Indeed, if A is pure in G, then A contains an element $a \neq 0$ whose characteristic consists of 0 and ∞. Then $\chi(a) \leqslant \chi(b)$ for any $b \in G$ and $\alpha a = b$, $\alpha \in E(G)$, i.e., a is a generator of the E(G)-module G.

LEMMA 1. 1) Suppose G is a strongly homogeneous torsion-free group. Then for any element g, $0 \neq g \in G$, the orbit $O(g) = \{f \in G \mid f = \alpha g, \alpha \in E(G)\} = F$ is a strongly homogeneous group of idempotent type and $G \cong F \otimes_Z A$, where A is a torsion-free group of rank 1 and of type t(G) and $E(G) \cong E(F)$.

2) Suppose F is a strongly homogeneous torsion-free group, A is a torsion-free group of rank 1, and, for each prime p, if pA = A, then pF = F. Then $G = F \otimes_Z A$ is a strongly homogeneous group and the mapping $\alpha \to \alpha \otimes 1$, $\alpha \in E(F)$, is an isomorphism of the rings E(F) and E(G).

Proof. 1) Consider any pure subgroups X' and Y' in F of rank 1. Let X and Y be the pure subgroups of G generated by X' and Y', respectively. Then $X \cap F = X'$ and $Y \cap F = Y'$. If $\alpha \in \mathrm{Aut}\, G$ and $\alpha X = Y$, then obviously $\alpha' = \alpha|_F$ (the restriction of α to F) is an automorphism of F and $\alpha' X' = Y'$. Consequently, F is strongly homogeneous. By construction, the element g is a generator of the E(F)-module F. Since an endomorphism does not lower characteristics of elements, we have $\chi(g) \leqslant \chi(f)$ for all $f \in F$ (characteristics in F). This is possible only if $\chi(g)$ consists of 0 and ∞, i.e., g, hence also F, has idempotent type.

Suppose A is a subgroup of Q containing unity and of type t(G). We will show that $G \cong F \otimes_Z A$. Note first that if B is a torsion-free group of rank 1 and of type t(F), then t(B) = t(F) ⩽ t(G) = t(A). Therefore, since t(B) is an idempotent type, $B \otimes_Z A \cong A$ [6, Proposition 85.3]. The mapping $F \times A \to G$ defined by $\left(b, \frac{n}{m}\right) \to \frac{n}{m} b$, $b \in F$, $\frac{n}{m} \in A$, is bilinear. The element nb/m exists in G. This follows from the existence of an isomorphism $B \otimes_Z A \cong A$, where B is the pure subgroup of F generated by b. Consequently, there exists a homomorphism

$\psi: F\otimes_Z A\to G$ such that $\psi\left(b\otimes\frac{n}{m}\right)=\frac{n}{m}b$ [7, Sec. 5.1, Proposition 2]. Clearly, ψ is a monomorphism. Suppose $a\in G$, C is the pure subgroup of G generated by a, and $B=C\cap F$. Then $B\otimes_Z A\cong A$. This implies the existence of a natural number m such that $ma\in B$ and $\frac{1}{m}\in A$. Now $ma\otimes\frac{1}{m}\in F\otimes_Z A$, $\psi\left(ma\otimes\frac{1}{m}\right)=a$, and ψ is an isomorphism. The isomorphism E(G) ≅ E(F) follows from 2).

2) Let X and Y be pure subgroups of $G=F\otimes_Z A$ of rank 1. Choose $0\neq(x\otimes a)\in X$ and $y\otimes b\in Y$ (x, $y\in F$; a, $b\in A$) so that $\chi(x\otimes a)=\chi(y\otimes b)$. Also, choose natural numbers n and m so that $na = mb$. Then $nm(x\otimes a)=mx\otimes na$ and $nm(y\otimes b)=ny\otimes mb$. Since $\chi(nm(x\otimes a))=\chi(nm(y\otimes b))$, we have $\chi(mx)=\chi(ny)$. Thus, X and Y contain elements $x\otimes a$ and $y\otimes a$, (changing the notation) such that $\chi(x)=\chi(y)$. Therefore, $\alpha x = y$ for some $\alpha\in \mathrm{Aut} F$ (Remark 1). Then $\alpha\otimes 1\in \mathrm{Aut}\, G$ and $(\alpha\otimes 1)(x\otimes a)=y\otimes a$. Therefore, $(\alpha\otimes 1)X=Y$ and G is strongly homogeneous.

The mapping $\alpha\to\alpha\otimes 1$, $\alpha\in E(F)$, is a ring homomorphism E(F) → E(G). If $x\otimes a=y\otimes a$, x, $y\in F$, $a\in A$, then the existence of a natural isomorphism $F\otimes_Z\langle a\rangle\cong F$ implies x = y. Thus, the above mapping is a monomorphism. Suppose $\gamma\in E(G)$. We will show that γ is induced by some $\alpha\in E(F)$, i.e., $\gamma=\alpha\otimes 1$. Suppose $x\in F$, $a\in A$, and $\gamma(x\otimes a)=z\otimes b$, $z\in F$, $b\in A$. The elements a and b have equivalent characteristics. Consequently, there are only finitely many primes p for which $h_p(a) > h_p(b)$, and in this case we always have $h_p(a) < \infty$. Let p be one of these primes. Write $h_p(a) = h_p(b) + k$. Since $h_p(x\otimes a)=h_p(x)+h_p(a)$, $h_p(z\otimes b)=h_p(z)+h_p(b)$, and $h_p(x\otimes a)\leqslant h_p(z\otimes b)$, it follows that $h_p(z)=h_p(x)+l$, where $l\geqslant k$ (l is a natural number or $l=\infty$). Then $z=p^k z'$, $z'\in F$, and $z\otimes b=z'\otimes p^k b$. Here $h_p(a) = h_p(p^k b)$ and, in addition, $h_q(b) = h_q(p^k b)$ for all primes $q\neq p$. There exist elements $z_1\in F$ and $b_1\in A$ such that $z\otimes b=z_1\otimes b_1$ and $\chi(a)\leqslant\chi(b_1)$. The last inequality implies $na = mb_1$ for certain natural numbers n and m, where (n, m) = 1 and mA = A. In view of 2), we also have mF = F. Therefore, $z_1=mz_2$, $z_2\in F$. We now have $z_1\otimes b_1=mz_2\otimes b_1=z_2\otimes mb_1=nz_2\otimes a$. Put $y = nz_2$. We have shown that for any $x\otimes a\in F\otimes_Z A$ its image satisfies $\gamma(x\otimes a)=y\otimes a$ for a unique $y\in F$. Uniqueness follows from the existence of an isomorphism $F\otimes_Z\langle a\rangle\cong F$. The mapping $\alpha: x\to y$, $x\in F$, is an endomorphism of F and $\gamma=\alpha\otimes 1$. The lemma is proved.

A ring R is called an *E-ring* if the left regular representation of R is an isomorphism, i.e., each endomorphism of the group R^+ is equal to left multiplication of the ring R by some element of R. An E-ring R is commutative [8, Corollary 1.3].

Remark 2. All ideals of a strongly homogeneous ring R are exhausted by the principal ideals (n·1) = nR, n = 0, 1, 2,... (see [9, Corollary 1]). Therefore, a strongly homogeneous E-ring is a principal ideal ring.

THEOREM 1. Suppose G is a strongly homogeneous torsion-free group. Then the center C of the ring E(G) is a strongly homogeneous ring and $G\cong F\otimes_Z A$, where F is a C-module in which all submodules of countable rank are free and A is a torsion-free group of rank 1 and of type t(G).

Proof. 1) Suppose F and A are the same as in the lemma. Then $G\cong F\otimes_Z A$, $E(G)\cong E(F)$, and F is a strongly homogeneous group of idempotent type. It suffices to prove that the center C of the ring E(F) is a strongly homogeneous ring and that all submodules of countable rank of the C-module F are free. We will assume that G itself has idempotent type.

2) Put $R=E(G)$, $S=R\otimes_Z Q$, and $V=G\otimes_Z Q$. Then V is a left S-module under the operation $(\alpha\otimes r)(a\otimes\varphi)=\alpha a\otimes r\varphi$, $\alpha\in R$, $a\in G$, and r, $\varphi\in Q$. For a fixed element $\alpha\otimes r$ its action on V is a linear transformation of the Q-space V. We will assume that S is contained in the ring L(V) of all linear transformations of the Q-space V. We identify the ring R with its image under the canonical monomorphism $R\to R\otimes_Z Q$. Then $S=\{\alpha\in L(V)\mid n\alpha\in R$ for some natural number n$\}$. We will prove that the S-module V is irreducible. Suppose $a\otimes r$, $b\otimes\varphi\in V$, where a, $b\in G$; r, $\varphi\in Q$, and $a\otimes r\neq 0$. If A and B are the pure subgroups of G generated by a and b, respectively, then $\alpha A = B$ for some $\alpha\in R$. Since αa, $b\in B$, it follows that $n(\alpha a) = mb$ for certain natural numbers n and m. Consider the element $\alpha\otimes\frac{n\varphi r^{-1}}{m}$ of S. We have $\left(\alpha\otimes\frac{n\varphi r^{-1}}{m}\right)(a\otimes r)=\alpha a\otimes\frac{n\varphi}{m}=n(\alpha a)\otimes\frac{\varphi}{m}=mb\otimes\frac{\varphi}{m}=b\otimes\varphi$. Thus, V is generated as an S-module by any nonzero element. This means that V is an irreducible module.

3) Thus, V is a faithful irreducible S-module. By Schur's lemma, $D = \mathrm{End}_S V$ is a division ring and, by the density theorem for irreducible modules [10, Chap. 2], S is a dense ring of linear transformations of V over D. Note that D consists of those linear transformations in L(V) that commute with all elements of S. This follows from the fact that each endomorphism of the group V is a linear transformation of the Q-space V.

4) We will show that $D = C \otimes_Z Q$. This equality is very important for what follows (we identify $C \otimes_Z Q$ with its image under the canonical monomorphism $C \otimes_Z Q \to R \otimes_Z Q$; we do the same thing in analogous situations). The center of the ring S is $C \otimes_Z Q$, hence $C \otimes_Z Q \subseteq D$. Suppose $d \in D$. Choose some generator a of the R-module G (Remark 1). Since $da \in V$, we have $n(da) \in G$ for some natural number n. Now for any $\alpha \in R$ we have $nd(\alpha a) = \alpha(n(da)) \in G$. But αa ranges over the whole group G as α ranges over R, hence $nd(G) \leqslant G$, $nd \in R$, and $d \in S$. The element d commutes with all elements of S; hence d lies in the center of S, i.e., $d \in C \otimes_Z Q$ and $D = C \otimes_Z Q$.

5) Fix a generator a of the R-module G. Let $H = Ca = \{ca \mid c \in C\}$ be the C-submodule of G generated by a, and let $W = Da = \{da \mid d \in D\}$ be the D-subspace of V generated by a. Then $H = W \cap G$. Suppose $g \in W \cap G$ and $g = da$, $d \in D$. For any $\alpha \in R$ we have $d(\alpha a) = \alpha(da) = \alpha g \in G$. Since a is a generator, it follows that $dG \subseteq G$ and $d \in R$. Thus, $d \in D \cap R = C$ and $g = da \in H$. This shows that $W \cap G \subseteq H$.. The reverse inclusion is obvious.

If $c \in C$ and $ca = 0$, then $c(\alpha a) = \alpha(ca) = 0$ for any $\alpha \in R$. Consequently, $cG = 0$ and $c = 0$. Therefore, the mapping $c \to ca$, $c \in C$, is an isomorphism of the C-modules C and H.

6) We will show that the ring C is strongly homogeneous. Since C is contained in D and D is a division ring, it follows that each nonzero endomorphism of G contained in C is a monomorphism. Fix a pure subgroup B of rank 1 in G. Suppose $0 \neq \alpha \in C$, and let E be the pure subgroup of G generated by the image αB. Since G is a homogeneous group and α is a monomorphism, we have $\alpha B \cong B \cong E$. Since the rank of E is 1, it follows that $\alpha B = nE$ for some natural number n. We shall show that $\alpha G = nG$. Suppose X is any pure subgroup of G of rank 1. Choose endomorphisms $\varphi, \psi \in E(G)$, for which $\varphi B = X$ and $\psi E = X$. We have $\alpha X = \alpha(\varphi B) = \varphi(\alpha B) = \varphi(nE) \subseteq nG$. Since X was chosen arbitrarily, $\alpha G \subseteq nG$. Also, $nX = n(\psi E) = \psi(\alpha B) = \alpha(\psi B) \subseteq \alpha G$. Therefore, $nG \subseteq \alpha G$, hence $\alpha G = nG$. We now define an automorphism β of G as follows. If $b \in G$, then $\beta a = nb$ for a unique $a \in G$. Put $\beta a = b$. Since $\alpha G = nG$ and α is a monomorphism, it is clear that β is an automorphism of G and $\alpha = n\beta$. Moreover, $\beta \in C$ and the strong homogeneity of the ring C is established.

7) We will show that the C-submodules of G of finite C-rank are free. Since C is a principal ideal ring (Remark 2), submodules of free C-modules are free. Also, if M is a C-submodule of G, there exists a C-pure submodule M* of G generated by M. Therefore, it suffices to prove that C-pure submodules of G of finite C-rank are free. Suppose first that M is a C-pure submodule of G of C-rank 1. Put $W_1 = M \otimes_Z Q$. Since $D = C \otimes_Z Q$ acts naturally on $V = G \otimes_Z Q$, , it follows that W_1 is a D-subspace of V. If $W_1 = W' \oplus W''$, where W' and W'' are D-subspaces, then $W' \cap M$ and $W'' \cap M$ are disjoint C-submodules of M, which is impossible inasmuch as the C-rank of M is 1. Consequently, $\dim_D W_1 = 1$. Note also that $M = W_1 \cap G$. If $g \in W_1 \cap G$, then $g \in M \otimes_Z Q$, hence $ng \in M$ for some natural number n. Therefore, since M is pure, we have $g \in M$ and $M = W_1 \cap G$. Choose in M a generator b of the R-module G, and suppose $\alpha, \beta \in R$, are such that $\alpha a = b$ and $\beta b = a$. Since α and β are linear transformations of the D-space V, it follows that W and W_1 are D-subspaces of dimension 1, hence $\alpha W = W_1$ and $\beta W_1 = W$. Therefore $\alpha H \subseteq W_1 \cap G = M$ and $\beta M \subseteq W \cap G = H$ [a, W, and H are the same as in 5)]. The equalities $(\beta\alpha)a = a$ and $(\alpha\beta)b = b$ show that $(\beta\alpha)|_W = 1_W$ and $(\alpha\beta)|_{W_1} = 1_{W_1}$. Let α' (resp., β') be the restriction of the endomorphism α (resp., β) to the subgroup H (resp., M). Then we can write $\beta'\alpha' = 1_H$ and $\alpha'\beta' = 1_M$. Therefore, α' is an isomorphism of H onto M. Since $\alpha \in R$, this isomorphism is also a C-module isomorphism. Thus, $M \cong H \cong C$.

Assume that all C-submodules of G of C-rank less than k are free, and suppose M is a C-pure submodule of G of C-rank k. Put $W_1 = M \otimes_Z Q$. Then, as in the beginning of this section, $M = W_1 \cap G$ and W_1 is a D-subspace of V of dimension k. We can write $W_1 = W' \oplus W''$, where $\dim_D W' = 1$, $\dim_D W'' = k - 1$. Then $M_1 = W' \cap G$ and $M_2 = W'' \cap G$ are C-submodules of M of rank 1 and k − 1, respectively. By our assumption, these submodules are free. Since the ring S is dense in $\mathrm{End}_D V$ (Sec. 3), there exists $\alpha \in S$, acting identically on W' and such that $\alpha W'' = 0$. Choose a natural number n such that $n\alpha \in R$; we will show that $nM \subseteq M_1 \oplus M_2$. Since the C-modules M and nM are isomorphic, the proof will be complete, inasmuch as M will be isomorphic to a submodule of a free C-module. Suppose $x \in M$. Then $nx = \alpha(nx) + (nx - \alpha(nx))$.

Here $\alpha(nx) \in W'$ and $\alpha(nx) = (n\alpha)x \in G$. Thus, $\alpha(nx) \in M_1$. Then $nx - \alpha(nx) \in W''$, $nx - (n\alpha)x \in$ G, hence $nx - \alpha(nx) \in M_2$. Thus, $nx \in M_1 \oplus M_2$ and $nM \subseteq M_1 \oplus M_2$.

8) Suppose M is a C-submodule of G of countable C-rank. All submodules of M of finite rank are free. Since C is a principal ideal ring, we have for C-modules an analogue of the well-known criterion of freeness of a countable Abelian group [5, Theorem 19.1]. Therefore, M is a free C-module. The theorem is proved.

Suppose R is a commutative integral domain whose quotient field is an algebraic number field. In [4, Proposition 5] conditions were given for R to be strongly homogeneous. We denote by $\sum_{\mathfrak{M}}^{\oplus} H$ the direct sum of $\mathfrak{M}$ groups isomorphic to H ($\mathfrak{M}$ is some cardinal number).

COROLLARY 1. If G is a strongly homogeneous torsion-free group of finite or countable rank over the center C of its endomorphism ring, then $G \cong F \otimes_Z A$, where F is a free C-module and A is a group of rank 1. Thus, $G \cong \sum_{\mathfrak{M}} \oplus (C \otimes_Z A)$, $\mathfrak{M} \leqslant \aleph_0$. If, in addition, G has idempotent type, then G is a free C-module.

COROLLARY 2. The following properties of a countable torsion-free group G are equivalent:

1) G is strongly homogeneous;

2) $G \cong F \otimes_Z A$, where F is a finitely or countably generated free module over some countable strongly homogeneous torsion-free E-ring T, A is a group of rank 1, and if p is a prime and pA = A, then pF = F; in this case the center of the ring E(G) is isomorphic to T;

3) $G \cong \sum_{\mathfrak{M}}^{\oplus} H\ (\mathfrak{M} \leqslant \aleph_0)$, where H is an indecomposable strongly homogeneous group.

Proof. 1) ⇒ 2). If C is the center of the ring E(G), then, by Theorem 1, $G \cong F \otimes_Z A$, where F and A are the same as in 2), and C is a strongly homogeneous ring. We need only show that C is an E-ring. Since $F \cong \sum_{\mathfrak{M}}^{\oplus} C$, we have $G \cong \sum_{\mathfrak{M}}^{\oplus} (C \otimes_Z A)$. Identify G with $\sum_{\mathfrak{M}}^{\oplus} (C \otimes_Z A)$. Then in the notation of the proof of Theorem 1 we have $V = G \otimes_Z Q = \sum_{\mathfrak{M}}^{\oplus} (C \otimes_Z Q) = \sum_{\mathfrak{M}}^{\oplus} D$, where D = $\mathrm{End}_S V$ (Secs. 3 and 4 of the proof). Each endomorphism of the group G is a linear transformation of the D-space V. Then it is clear that the endomorphism ring $E(C \otimes_Z A)$ can be embedded in the ring of linear transformations $\mathrm{End}_D D \cong D$. By Lemma 1, $E(C^+) \cong E(C \otimes_Z A)$. Thus, the ring $E(C^+)$ can be embedded in D and, in particular, is commutative. Therefore, C is an E-ring by virtue of Proposition 1.2 of [8].

2) ⇒ 3). If $F \cong \sum_{\mathfrak{M}}^{\oplus} T$, $\mathfrak{M} \leqslant \aleph_0$, then $G \cong \sum_{\mathfrak{M}}^{\oplus} (T \otimes_Z A)$. Put $H = T \otimes_Z A$. The group T^+ is strongly homogeneous; hence, by Lemma 1, so is H. By the same lemma, $E(H) \cong E(T^+) \cong T$. Consequently, E(H) has no nontrivial idempotents and H is indecomposable.

3) ⇒ 1). Let T be the center of the ring E(H). Since H is countable and indecomposable, it follows from Theorem 1 that $H \cong T \otimes_Z A$, where A is a group of rank 1 and of type t(H). Then $G \cong F \otimes_Z A$, where $F \cong \sum_{\mathfrak{M}}^{\oplus} T$ is a free module over the principal ideal ring T. Each element of F can be embedded in a direct summand isomorphic to T^+. Since the group T^+ is strongly homogeneous, it is clear that F is also. By Lemma 1, G is strongly homogeneous.

It remains to show that in 2) the center C of E(G) is isomorphic to T. We have $G \cong \sum_{\mathfrak{M}}^{\oplus} (T \otimes_Z A)$. The center of E(G) is isomorphic to the center of $E(T \otimes_Z A)$. By Lemma 1, $E(T \otimes_Z A) \cong E(T^+) \cong T$ and $C \cong T$, since the E-ring T is commutative (see the reference preceding Remark 2).

COROLLARY 3. A countable indecomposable torsion-free group G is strongly homogeneous if and only if $G \cong R \otimes_Z A$, where R is a countable strongly homogeneous E-ring, A is a group of rank 1, and if pA = A, then pR = R; in this case, $E(G) \cong R$.

We now consider the endomorphism rings of strongly homogeneous groups. If M is a module and R its endomorphism ring, then by taking in R the annihilators of the finite subsets of M as a neighborhood base of zero, we obtain the finite topology on R. It is known that R is a complete Hausdorff topological ring in the finite topology [6, Theorem 107.1]; the proof is analogous to modules. We regard endomorphism rings as topological with respect to the finite topology.

THEOREM 2. If G and H are strongly homogeneous torsion-free groups whose endomorphism rings are topologically isomorphic, then $G\otimes_Z B\cong H\otimes_Z A$, where B and A are torsion-free groups of rank 1 and of types t(H) and t(G), respectively. More precisely, if $G\cong F_1\otimes_Z A$ and $H\cong F_2\otimes_Z B$ are written as in Lemma 1, where F_1 and F_2 are strongly homogeneous groups of idempotent types, then the rings $E(F_1)$ and $E(F_2)$ are topologically isomorphic and any topological isomorphism $\psi: E(F_1)\to E(F_2)$ is induced by some group isomorphism $\varphi: F_1\to F_2$, i.e., $\psi:\eta\to\varphi\eta\varphi^{-1}$, $\eta\in E(F_1)$.

Proof. Once the second assertion has been proved, $F_1\cong F_2$ and $G\otimes_Z B\cong (F_1\otimes_Z A)\otimes_Z B\cong (F_2\otimes_Z B)\otimes_Z A\cong H\otimes_Z A$. So let us prove the second assertion. By Lemma 1, $G\cong F_1\otimes_Z A$ and $H\cong F_2\otimes_Z B$, where F_1 and F_2 are strongly homogeneous groups of idempotent types and A and B are groups of rank 1 of types t(G) and t(H), respectively. The mappings $\alpha\to\alpha\otimes 1$, $\alpha\in E(F_1)$, and $\beta\to\beta\otimes 1$, $\beta\in E(F_2)$ are ring isomorphisms $(EF_1)\to E(G)$ and $E(F_2)\to E(H)$, respectively (Lemma 1). These are obviously topological isomorphisms. Consequently, the rings $E(F_1)$ and $E(F_2)$ are also topologically isomorphic.

Suppose $\psi: E(F_1)\to E(F_2)$ is a topological isomorphism. Put $V_i=F_i\otimes_Z Q$ and $S_i=E(F_i)\otimes_Z Q$ (i = 1, 2). Then V_i is a faithful irreducible S_i-module (Sec. 2) of the proof of Theorem 1. Let $D_i=\mathrm{End}_{S_i}V_i$ and $L_i=\mathrm{End}_{D_i}V_i$ (i = 1, 2). Here D_i is a division ring and S_i is dense in the finite topology of the ring L_i [10, Chap. 2]. As before, we identify $E(F_i)$ with its image under the canonical monomorphism $E(F_i)\to S_i$ (i = 1, 2). Then the finite topology of the ring $E(F_i)$ is the same as the topology induced by the finite topology of the ring L_i. Therefore, $\psi\otimes 1_Q$ is a topological isomorphism of the rings S_1 and S_2, which isomorphism we also denote by ψ. Since S_i is dense in the full ring L_i (i = 1, 2), ψ can be uniquely extended to an isomorphism of L_1 and L_2, and we again denote this isomorphism by ψ. We will write η^* instead of $\psi(\eta)$.

Choose a generator g of the $E(F_1)$-module F_1 (Remark 1). Let W be the D_1-subspace of V_1 generated by g, and let $\pi: V_1\to W$ be a projection. Then $\pi\in L_1$ and $\pi^2=\pi$. Therefore, $(\pi^*)^2=\pi^*$ and $\pi^*: V_2\to\pi^*V_2$ is a projection. Fix in $\pi^*V_2\cap F_2$ some generator h of the $E(F_2)$-module F_2.

We define $\varphi: F_1\to F_2$ as follows. If $a\in F_1$, then $a=\eta g$ for some $\eta\in E(F_1)$. Put $\varphi a=\eta^*h$. We will show that φ is a mapping. If $a=\eta_1 g$, $\eta_1\in E(F_1)$, then $(\eta-\eta_1)g=0$. Consequently, $(\eta-\eta_1)\pi=0$. Therefore, $(\eta^*-\eta_1^*)\pi^*=0$ and $(\eta^*-\eta_1^*)h=(\eta^*-\eta_1^*)\pi^*h=0$.

Obviously, φ is a homomorphism. If $a=\eta g\neq 0$, then $\eta\pi\neq 0$ and $\eta^*\pi^*\neq 0$. Therefore, $\eta^*h\neq 0$, since $\dim_{D_2}\pi^*V_2=1$. Thus, φ is a monomorphism. If $c\in F_2$, we write $c=\theta h$, $\theta\in E(F_2)$. Put $\eta=\psi^{-1}(\theta)$ and $a=\eta g$. Then $\varphi a=\eta^*h=\theta h=c$, i.e., φ is an isomorphism.

It remains to show that φ induces ψ. Suppose $\xi\in E(F_1)$. Write the element $c\in F_2$ as θh for some $\theta\in E(F_2)$. Put $a=\varphi^{-1}c$ and write $a=\eta g$ for some $\eta\in E(F_1)$. Then $c=\eta^*h$. Now $\xi^*c=\xi^*\eta^*h=(\xi\eta)^*h=\varphi((\xi\eta)g)=\varphi\xi(\eta g)=\varphi\xi(\varphi^{-1}c)=(\varphi\xi\varphi^{-1})c$. Therefore, $\psi(\xi)=\xi^*=\varphi\xi\varphi^{-1}$ for any $\xi\in E(F_1)$, and φ induces ψ. The theorem is proved.

COROLLARY 4. If G and H are strongly homogeneous torsion-free groups of idempotent or equal types whose endomorphism rings are topologically isomorphic, then any topological isomorphism between E(G) and E(H) is induced by some group isomorphism between G and H.

Proof. In the case of idempotent types, we may assume $G=F_1$ and $H=F_2$ in Theorem 2. If t(G) = t(H), then $G\cong F_1\otimes_Z A$ and $H\cong F_2\otimes_Z A$ (in the notation of Theorem 2). We identify the left- and right-hand sides in these isomorphisms. Suppose $\psi: E(G)\to E(H)$ is a topological isomorphism. If $\alpha\in E(F_1)$ and $\psi:\alpha\otimes 1\to\beta\otimes 1$, $\beta\in E(F_2)$, then $\psi':\alpha\to\beta$ is a topological isomorphism $E(F_1)\to E(F_2)$ (see the beginning of the proof of Theorem 2). Suppose the isomorphism $\varphi: F_1\to F_2$ induces ψ'. Then for each $\alpha\in E(F_1)$ we have $\psi(\alpha\otimes 1)=\psi'(\alpha)\otimes 1=(\varphi\alpha\varphi^{-1})\otimes 1=(\varphi\otimes 1)(\alpha\otimes 1)(\varphi\otimes 1)$. Consequently, the isomorphism $\varphi\otimes 1$ between G and H induces ψ.

The requirement that the isomorphism ψ in Corollary 4 be continuous is necessary. If G and H are periodic groups, then any isomorphism $\psi: E(G)\to E(H)$ is topological. The same is true if G and H are separable torsion-free groups [6, Sec. 87]. This follows from the fact that the finite topology on E(G) can be defined by taking the left annihilators of primitive idempotents of E(G) as a neighborhood subbasis of zero [6]. Since a homogeneous separable torsion-free group is strongly homogeneous [6, Proposition 87.2], we have:

COROLLARY 5 (see [11, 12]). If G and H are homogeneous separable torsion-free groups whose endomorphism rings are isomorphic, then $G\otimes_Z B\cong H\otimes_Z A$, where B and A are groups of rank 1 of types t(H) and t(G), respectively.

COROLLARY 6. Any topological automorphism of the endomorphism ring of a strongly homogeneous torsion-free group is inner.

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THE EQUALITY PROBLEM AND FREE PRODUCTS OF LIE ALGEBRAS AND OF ASSOCIATIVE ALGEBRAS

G. P. Kukin

UDC 519.48

The class of groups with soluble equality problem is closed under free products. It follows immediately from Shirshov's work [1] that a free product of Lie algebras with recursive basis again has a recursive basis. The analogous statement for associative algebras holds as well.

Let P_i be a class of finitely presented (f.p.) Lie algebras over a field F, in which the problem of linear independence is soluble for an arbitrary set of $c \leqslant i$ elements. The class P_1 consists of f.p. Lie algebras with soluble equality problem, and the class $P=\bigcap_{i=1}^{\infty} P_i$ consists of f.p. Lie algebras with recursive basis. The concept of *recursive basis* was introduced by Bokut' [2].

The inclusions

$$P_1\supseteq P_2\supseteq\ldots\supseteq P\supseteq\Phi,$$

are obvious, Φ being the class of residually finite dimensional algebras. The inclusion $P\supset\Phi$ is strict over every field (see the example in [3, p. 229]). Over a finite field, we

Omsk State University. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 24, No. 2, pp. 85-96, March-April, 1983. Original article submitted December 26, 1980.

0037-4466/83/2402-0221$07.50