ADMISSIBLE RULES FOR LOGICS CONTAINING S4.3

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UDC 517.11

In this article we shall study the admissible rules of deduction in modal logics containing S4.3. We prove that the free algebras in the corresponding varieties of algebras of closures have a finite basis of quasiidentities, and therefore the problem of admissibility of rules is soluble in all extensions of S4.3. As a corollary we obtain a solution to two problems of Porte [1] on the rules of deduction in the Lewis system S5. The necessary notation of definitions is to be found in [2, 3].

We begin with the description of free algebras in eq(λ) for $\lambda \supseteq$S4.3. Let $\mathfrak{B}_j$, $j \in J$ be all the finite subdirectly indecomposable n-generated algebras in the variety eq(λ). The scales $\mathfrak{B}_j^+$ are then of the form $C_1 \uparrow C_2 \uparrow \ldots \uparrow C_k$, where C_j are clots of cardinality not greater than 2^n, and $\uparrow$ is the operation of sequential combination of scales. Let $\mathfrak{B}_j^+ = \langle T_j, R\rangle$, $\mathcal{T}_j = \langle T_j, R, V\rangle$, where $V(p_i) \rightleftharpoons x_i$ and x_i, $i \le n$ are generators of $\mathfrak{B}_j$. In this case we have

$$\forall x \forall y\, (\forall \theta\, (x, y \in \mathcal{T}_i \wedge (x \Vdash_V \theta(p_1, \ldots, p_n) \Leftrightarrow y \Vdash_V \theta(p_1, \ldots, p_n))) \Rightarrow x = y). \tag{1}$$

We introduce an equivalence relation on $\Sigma\mathcal{T}_i$; $x \equiv y \Leftrightarrow$ there exists an isomorphism of models, generated in $\Sigma\mathcal{T}_i$ by x and y, taking x to y. We take the factor-set in $\Sigma\mathcal{T}_i$ with respect to this equivalence, and introduce a relation R on it setting $[x]R[y] \Leftrightarrow \exists x_1 \in [x] \exists y_1 \in [y] (x_1 R y_1)$; moreover we set $[x] \in V(p_i) \Leftrightarrow x \in V(p_i)$, $\mathcal{T} = \langle T, R, V\rangle$. We denote the submodel of the arbitrary model W, generated by the element x, by $\langle x\rangle$. The following properties of this model are easily verified:

$$\begin{gathered} \forall x \in \mathcal{T}_i \left(\langle x\rangle \overset{f}{\cong} \langle [x]\rangle\right), \quad f(z) = [z], \\ (\forall [x] \in T)(\exists i)(\langle [x]\rangle \cong \mathcal{T}_i), \quad (\forall i)(\exists [x] \in T)(\mathcal{T}_i \cong \langle [x]\rangle). \end{gathered} \tag{2}$$

In view of the above, and the fact that λ is finitely approximable [4], we obtain $\varphi(p_1, \ldots, p_n) \in \lambda \Leftrightarrow \mathcal{T} \Vdash \varphi(p_1, \ldots, p_n)$. Therefore the subalgebra $\langle V(p_1), \ldots, V(p_n)\rangle$ of the algebra $\langle T, R\rangle^+$, generated by the elements $V(p_i)$, is a free algebra of rank n in the variety eq(λ), and $V(p_i)$ are its free generators. Then $\mathcal{F}_n(\lambda) \cong \langle V(p_1), \ldots, V(p_n)\rangle$.

The element a in the model $W = \langle W, R, V\rangle$ is called formular, if there exists a formula φ such that $\forall x \in W (x \Vdash_V \varphi \Leftrightarrow x = a)$. The element $a \in W$ has depth 1, if $\forall b (aRb \Rightarrow bRa)$, and $b \in W$ has depth n + 1 if there exists $c \in W$ of depth n such that bRc but not cRb, and for any z, if bRz but not zRb, then z is of depth k, where $k \le n$.

LEMMA 1. Any element $[x] \in \mathcal{T}$ is formular, and there is a finite number of elements of depth n in $\mathcal{T}$.

Proof. Let the depth of [x] be equal to 1, and then by (2) we have $\langle [x]\rangle \cong \mathcal{T}_i$, and let $x_1, \ldots, x_n$ be elements in the clot $\mathcal{T}_i$. Then for $i \ne j$ there exists p_k; $x_i \Vdash_V p_k$ and $\neg(x_j \Vdash_V p_k)$, or the contrary. Introduce the formulas

$$\psi(x_i) \rightleftharpoons \left(\bigwedge_{j \in A} p_j\right) \wedge \left(\bigwedge_{j \notin A} \neg p_j\right),$$

where $A \rightleftharpoons \{j \mid x_i \Vdash_V p_j,\ j \le n\}$. Clearly, $x_j \Vdash_V \psi(x_i) \Leftrightarrow i = j$. Moreover, let $\varphi(x_i) = \psi(x_i) \wedge \Box(\bigvee_j \psi(x_j)) \wedge \left(\bigwedge_j \Box\Diamond\psi(x_j)\right)$.

We show that $\forall x \in \mathcal{T}_j$ it follows from $x \Vdash_V \varphi(x_i)$ that $\langle x\rangle \cong \langle x_i\rangle$. In fact, if C_k is a maximal clot in $\mathcal{T}_j$, attainable from x, then it is easily seen that $C_k \cong \langle x_i\rangle$. Moreover, if xRz $z \Vdash_V \psi(x_j)$ and $t \in C_k$, $t \Vdash_V \psi(x_j)$, then $z \Vdash_V \theta(p_1, \ldots, p_n) \Leftrightarrow t \Vdash_V \theta(p_1, \ldots, p_n)$. Then since $V(p_i)$ are generators of $\mathfrak{B}_j$ and bearing in mind (1), we have z = t. Therefore $\langle x\rangle$ is a clot and $\langle x\rangle \cong C_k$, i.e., $\langle x\rangle \cong \langle x_i\rangle$, which is what we required. Let $[y] \in T$ and let

Krasnoyarsk. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 25, No. 5, pp. 141-145, September-October, 1984. Original article submitted June 15, 1982.

$[y] \Vdash_V \varphi(x_i)$, then in view of (2) we have $\langle [y] \rangle \cong \langle y \rangle$ and $y \Vdash_V \varphi(x_i)$. In view of the above, then $\langle y \rangle \cong \langle x_i \rangle$, and $[y] = [x_i]$. Therefore, $[y] \Vdash_V \varphi(x_i) \Leftrightarrow [y] = [x_i]$. Thus any [x] of depth 1 is formular. It is clear from the form of the distinguishing formulae and from the restriction on the cardinality of the clots (not greater than 2^n), that there is only a finite number of elements of depth 1.

Suppose that we have proved that elements of depth not greater than n are formular and that this class is finite, and then prove these facts for n + 1. Let $[z_1], \ldots, [z_m]$ be all the elements of depth not greater than n, and let $\varphi_1, \ldots, \varphi_n$ be the formulae distinguishing them. Let [x] be an element of depth n + 1, then for some $j_0 [x]R[z_{j_0}]$, where z_{j_0} is an element of depth n. The element [x] appears in the clot C, and let $[x_1], \ldots, [x_k]$ be its elements. As before, we choose formulas $\psi(x_i)$ distinguishing $[x_i]$ in the clot. Set

$$A = \{j \mid \neg([x]\, R\, [z_j])\}, \quad \theta = \bigvee_{h \notin A} \varphi_h \vee \nabla,$$

$$\nabla = \bigvee_i \Big(\psi(x_i) \wedge \Diamond \varphi_{j_0} \wedge \bigwedge_j \Diamond \Big(\psi(x_j) \wedge \Diamond \varphi_{j_0} \wedge \neg\Big(\bigvee_{h \notin A} \varphi_h\Big)\Big)\Big),$$

$$\varepsilon(x_i) = \bigwedge_j \neg\varphi_j \wedge \bigwedge_{j \in A} \neg\Diamond \varphi_j \wedge \Diamond\varphi_{j_0} \wedge \psi(x_i) \wedge \Box\theta.$$

Clearly, $[x_i] \Vdash_V \varepsilon(x_i)$. We show that $\varepsilon(x_i)$ distinguishes $[x_i]$ in T. Let $[y] \Vdash_V \varepsilon(x_i)$, then $[y]R[z_{j_0}]$ and $[y] \neq [z_{j_0}]$. Let C_p be a clot attainable from [y] and immediately preceding the element $[z_{j_0}]$. Then it is easily seen that $C_p \cong C$. Suppose that $[y]R[z]$, $[z]R[z_{j_0}]$ and $[z] \Vdash_V \psi(x_j) \wedge \neg\Big(\bigvee_{h \notin A} \varphi_h\Big)$ and $[t] \in C_p$, $[t] \Vdash_V \psi(x_j) \vee \neg\Big(\bigvee_{h \notin A} \varphi_h\Big)$. It is easily seen that when $[z] \Vdash_V \theta(p_1, \ldots, p_n) \Leftrightarrow [t] \Vdash_V \theta(p_1, \ldots, p_n)$. By (2), $\langle y \rangle \cong \langle [y] \rangle$ and by (1) we have z = t, and therefore [z] = [t]. Thus, the element [z] with the above properties must appear in C_p. Then [y] appears in C_p and $C_p = C$. Therefore, $\langle [y] \rangle \cong \langle [x] \rangle$ and [y] = [x]. In view of the form of the formulas $\varepsilon(x_i)$, we may conclude that there is only a finite number of elements of depth n + 1. The Lemma is proved.

For any scale $V = \langle V, R \rangle$, we denote by $V \oplus 1$ the scale $\langle V \cup \{1\}, R_1 \rangle$, $\forall x, y \in V(xR_1y \Leftrightarrow xRy)$, $\forall x \in V(\neg(xR_11) \wedge \neg(1R_1x))$, $1R_11$.

LEMMA 2. For any j, $(\langle T_j, R \rangle \oplus 1)^+$ is a subalgebra of the algebra $\mathcal{F}_n(\lambda)$.

Proof. Let $\mathcal{T}_j = C_1 \uparrow C_2 \uparrow \cdots \uparrow C_k$, and in view of (2), $\mathcal{T}_j \cong \langle [x] \rangle$. By Lemma 1, any element $[x_i(m)] \in C_m$ is distinguished in $\mathcal{T}$ by some formula $\varphi(i, m)$. Let $\psi(i, m) \rightleftharpoons \varphi(i, m)$, for a) $\overline{C}_m = e > 1, i < e$; $\varphi(1, m) \wedge \neg\Diamond \varphi(1, m-1) \wedge \bigwedge_{j<l} \neg\varphi(i, m)$, for b) $C_m = l > 1, i = l, m > 1$; $\bigwedge_{i<l} \neg\varphi(i\ m) \wedge \Diamond \varphi(1, m)$, for c) $\overline{C}_m = l > 1, i = l$, m = 1; $\Diamond \varphi(1, m)$, for d) $\overline{\overline{C}}_m = 1, m = 1$; $\Diamond \varphi(1, m) \wedge \neg\Diamond \varphi(1, m-1)$, and for e) $\overline{\overline{C}}_m = 1, m > 1$. It is easily seen that $[x] \Vdash_V \psi(i, m) \Leftrightarrow [x] = [x_i(m)]$ for a), $[x] = [x_i(m)] \vee ([x]R[x_i(m)] \wedge \neg([x]R[x_i(m-1)]))$ for b), $[x] = [x_i(m)] \vee ([x]R[x_i(m)] \wedge \forall j([x] \neq [x_j(m)]))$ for c), $[x]R[x_i(m)]$ for d), and $[x]R[x_i(m)] \wedge \neg([x]R[x_i(m-1)]$ for e). Let $V(\psi(i, m)) \rightleftharpoons \{[x] / [x] \Vdash_V \psi(i_j, m)\}$ and $\mathfrak{A}$ be the subalgebra of the algebra $\langle V(p_1), \ldots, V(p_n) \rangle$ generated by the elements $V(\psi(i, m))$. Take a mapping $f \langle V(p_1), \ldots, V(p_n) \rangle$ into $(\langle T_j, R \rangle \oplus 1)^+$, where

$$f(\varphi(V(p_1), \ldots, V(p_n)) = V(\varphi(p_1, \ldots, p_n)) \cap (\langle [x] \rangle \cup \langle [y] \rangle,$$

where $\langle y \rangle = \langle \{1\}, R \rangle$ and $\langle y \rangle \neq C_k$. Clearly $\langle [x] \rangle \cup \langle [y] \rangle \cong \langle T_j, R \rangle \oplus 1$. Clearly, f is a homomorphism, and in view of the choice of $\psi(i, m)$, maps $\mathfrak{A}$ "onto" $(\langle [x] \rangle \cup \langle [y] \rangle)^+$. In order to prove that f is one-to-one, it is sufficient to show that $z \neq 0 \Rightarrow f(z) \neq 0$, and this follows from the choice of the formulas $\psi(i, m)$. The Lemma is proved.

LEMMA 3. If the quasiidentity q is false in $\mathcal{F}_n(\lambda)$, then for some j, $(\langle \mathcal{T}_j, R \rangle \oplus 1)^+ \Vdash \neg q$.

Proof. Suppose that $\mathcal{F}_n(\lambda) \Vdash \neg(f = 1 \Rightarrow g = 1)$, i.e., there exist $\varphi_j(V(p_i))$ such that $f(\varphi_j(V(p_i))) = 1$ and $g(\varphi_j(V(p_i))) \neq 1$. Then there exists $[x] \in \mathcal{T}$, $[x] \Vdash_V \neg g(\varphi_j)$ and for all $[y] \in \mathcal{T}$ $[y] \Vdash_V f(\varphi_j)$. By (2), $\langle [x] \rangle \cong \mathcal{T}_j$ and there exists $\mathcal{T}_n \cong \langle \{t\}, R, V \rangle$, and moreover, $\neg([x]R[t])$. Therefore, $\ulcorner(\mathcal{T}_j \Vdash_V \neg g(\varphi_j))$ and $\mathcal{T}_j \oplus 1 \Vdash f(\varphi_i)$, and therefore $(\langle T_j, R \rangle \oplus 1)^+ \vDash \neg(f = 1 \Rightarrow g = 1)$; the Lemma is proved.

LEMMA 4. If $\mathfrak{B}$ is finitely generated and $\mathfrak{B} \Vdash \Diamond x \wedge \Diamond \neg x \Rightarrow y = 1$, then in the Stone representing set $Q_{\mathfrak{B}}$ there exists a single-element maximal clot.

Proof. We recall that $Q_{\mathfrak{B}}$ is the set of ultrafilters on $\mathfrak{B}$,

$$\forall \nabla_1, \nabla_2 \in Q_{\mathfrak{B}} (\nabla_1 R \nabla_2 \Leftrightarrow (\Box x \in \nabla_1 \Rightarrow \Box x \in \nabla_2)),$$

and $\mathrm{Im} : \mathfrak{B} \to Q_{\mathfrak{B}}^+$, where $\mathrm{Im}(a) = \{\nabla / a \in \nabla\}$ is an isomorphic embedding of $\mathfrak{B}$. Moreover, if a_i are generators of $\mathfrak{B}$ and $V(p_i) \rightleftharpoons \{\nabla / a_i \in \nabla\}$, then

$$\nabla \Vdash_V \varphi(p_i) \Leftrightarrow \varphi(a_i) \in \nabla. \qquad (3)$$

By Zorn's lemma, any clot $Q_{\mathfrak{B}}$ is contained in some maximal clot. It follows from (3) that maximal clots have no more than 2^n elements and that the set of maximal clots is finite. Suppose that there are no single-element clots among the maximal clots $C_1, \ldots, C_k$. Fix an element ∇_i, $i \le k$ from each clot. Let $\psi(\nabla_i)$ be an element in ∇_i but not in the clot C_i, and let $\Box\varphi(i, j)$ be an element in ∇_i but not in the clot C_j, $i \ne j$.

Take an element $a \rightleftharpoons \bigvee_i \left(\psi(\nabla_i)) \wedge \left(\bigwedge_{j\neq i} \Box\,\varphi(i, j)\right)\right)$. Suppose that $\Box a \ne 0$. Then $\Box a$ appears in some ultrafilter ∇, and ∇ is contained in some maximal clot C_m. In this case $\Box a \in \Delta$ for any $\Delta \in C_m$. Let $\Delta \in C_m$ and $\Delta \ne V_m$, then $\Box a \in \Delta$ and for some i, $(\nabla_i) \wedge \left(\bigwedge_{j\neq i} \Box\,\varphi(i, j)\right) \in \Delta$. From the definition of $\Box\varphi(i, j)$ we see that $\Delta \in C_i$, i.e., $m = i$. Then $\psi(\nabla_i) \notin \Delta$; contradiction. Thus $\Box a = 0$.

Suppose that $\Box\neg a \ne 0$. Then as before, there exists a maximal clot C_m, where $\forall \Delta \in C_m (\Box\neg a \in \Delta)$. Let $\Delta = \nabla_m$. Then $\psi(\nabla_m) \in \nabla_m$ and $\Box\varphi(m, j) \in \nabla_m$ for $m \ne j$, in this case $a \in \nabla_m$, which contradicts $\Box\neg a \in \nabla_m$. Therefore $\Box\neg a = 0$, $\Box a = 0$ and $\Diamond a \wedge \Diamond\neg a = 1$, which contradicts $\mathfrak{B} \Vdash \Diamond x \wedge \Diamond\neg x = 1 \Rightarrow y = 1$. Thus the assumption that there do not exist any single-element clots leads to a contradiction. The Lemma is proved.

<u>THEOREM 5.</u> For any $\lambda \supseteq S4.3$ $\mathcal{F}_\omega(\lambda)$ has a finite basis of quasiidentities, obtained by joining the quasi-identity $\Diamond x \wedge \Diamond\neg x = 1 \Rightarrow y = 1$ to the basis of identities of $\mathcal{F}_\omega(\lambda)$.

<u>Proof.</u> We know that the basis of identities of $\mathcal{F}_\omega(\lambda)$ is finite [4]. The truth of the quasiidentity $\Diamond x \wedge \Diamond\neg x = 1 \Rightarrow y = 1$ on $\mathcal{F}_\omega(\lambda)$ follows from Lemma 3. Suppose that there exists a finitely generated algebra $\mathfrak{B} \in \mathrm{eq}(\lambda)$, on which the quasiidentity in the formulation of the Theorem is true and

$$\mathfrak{B} \models \neg(f(x_1, \ldots, x_n) = 1 \Rightarrow g(x_1, \ldots, x_n) = 1).$$

Moreover, we may assume that $\mathfrak{B}$ is generated by the elements a_i, $i \le n$, where $f(a_1, \ldots, a_n) = 1$ and $g(a_1, \ldots, a_n) \ne 1$. In this case $\Box f \to \Box g \notin \lambda$ and since λ is finitely approximable [4], there exists $\mathcal{T}_i$ such that $\neg(\mathcal{T}_i \models \Box f \to \Box g)$, and then there exists $\mathcal{T}_j$ such that $\forall x \in \mathcal{T}_j x \models_V \Box f$ and $\exists y \in \mathcal{T}_j (y \Vdash_V \neg\Box g)$. Thus $\langle T_j, R\rangle^+ \models \neg(f = 1 \Rightarrow g = 1)$.

Consider the embedding of $\mathfrak{B}$ in the algebra $Q^+_{\mathfrak{B}}$. By (3), $\nabla \Vdash_V f(p_1, \ldots, p_n)$ for all $\nabla \in Q_{\mathfrak{B}}$ and by Lemma 4, $Q_{\mathfrak{B}}$ contains a single-element maximal clot, so therefore the formula f is satisfied on the single-element scale. Together with the above, this gives us $(\langle T_j, R\rangle)^+ \models \neg(f = 1 \Rightarrow g = 1)$. Then by Lemma 2 we have $\mathcal{F}_\omega(\lambda) \models \neg(f = 1 \Rightarrow g = 1)$. The Theorem is proved. From this theorem and Lemma 3, we have:

<u>COROLLARY 6.</u> If $\lambda \supseteq S4.3$, then the quasiequational theory $\mathcal{F}_\omega(\lambda)$ and the admissibility problem of the rules for λ are soluble.

<u>Proposition 7.</u> Let $\lambda \supseteq S4.3$ and let f/g be an admissible nonderived rule of λ; then for any φ_i, $i \le n$, $\neg f(\varphi_i) \in \lambda(\bar{1})$, where $\bar{1}$ is a single-element scale, in particular $f(\varphi_i) \notin \lambda$.

<u>Proof.</u> As in the Theorem, we show that for some j, $\langle T_j, R\rangle^+ \models \neg(f = 1 \Rightarrow g = 1)$. Suppose that $f(\varphi_i)$ is satisfied on a single-element scale. Then $(\langle T_j, R\rangle \oplus 1)^+ \models \neg(f = 1 \Rightarrow g = 1)$, which by Lemma 2 implies that $\mathcal{F}_\omega(\lambda) \models \neg(f = 1 \Rightarrow g = 1)$, and therefore f/g is inadmissible, which is a contradiction. The Proposition is proved.

In [1], Porte posed two problems:

a) Does an admissible nonderived rule of the Lewis system S5 always have an unsatisfiable premise?

b) is the system S5 together with the rule of deduction $\neg(\Diamond x \to \Box x)/y$ structurally complete?

An affirmative answer to problem a) follows from Proposition 7, and an affirmative answer to problem b) is given by the following:

<u>Proposition 8.</u> For all $\lambda \supseteq S.4.3$, the logic λ together with the rule of deduction $\Diamond x \wedge \Diamond\neg x/y$ is structurally complete.

<u>Proof.</u> Let f/g be an admissible rule in the logic $\lambda + (\Diamond x \wedge \Diamond\neg x/y)$, then by Theorem 5 f/g is admissible in λ. Then by Proposition 7, f/g is derived in λ, or $\neg f(\varphi_i) \in \lambda(\bar{1})$. In the latter case $\Box f \to \Box \bigvee_{i\le n} (\Diamond p_i \wedge \Diamond\neg p_i) \in \lambda$ (where p_i are the variables of f), since otherwise there would exist $\mathcal{T}_i$, on which the above formula is not true, and then $\neg f \notin \lambda(1)$; contradiction. Consider $\mathcal{F}_n(\lambda)$. By Lemma 1, any element $\langle T, R, V\rangle$ is formular. Choose one element from each maximal clot, and let φ_i, $i \le k$, be the formulas distinguishing them. Then

$$\langle T, R, V\rangle \Vdash \Box\left(\bigvee_{i\le n} (\Diamond p_i \wedge \Diamond\neg p_i)\right) \to \Diamond(\bigvee \varphi_i) \wedge \Diamond\neg(\bigvee \varphi_i),$$

and therefore the latter formula is a formula in λ. Then $\Box f \to \Diamond\psi \wedge \Diamond\neg\psi \in \lambda$, where $\psi = \vee \varphi_i$, and therefore from the premise of f, using the rules and axioms of λ, we obtain $\Diamond\psi \wedge \Diamond\neg\psi$, and using the rule $\Diamond x \wedge \Diamond\neg x/y$, from the latter we obtain g. Thus, f/g is derived in $\lambda + (\Diamond x \wedge \Diamond\neg x/y)$. The Proposition is proved.

If as well as finite rules of deduction we also consider rules with infinite premises, we get negative answers to the above problems; namely, it is easily seen that the rule

$$\left(\bigwedge_{i<n} \Diamond \left(p_i \wedge \left(\bigwedge_{j\neq i} \neg p_j\right)\right)\right) \vee \Box x, \quad n<\omega/\Box x$$

is admissible and nonderived in $S5 + (\Diamond x \wedge \Diamond\neg x/y)$ and clearly its premise for $x = p \to p$ consists of formulas in S5.

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HIERARCHY OF LIMITING COMPUTATIONS

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UDC 517.11:518.5

In [1] and [2] Ershov introduced a hierarchy of subsets of the natural sequence N and, in particular, established a connection between this hierarchy and limiting computations. In [3] a certain hierarchy of functions computable in the limit was proposed and defined in analogy with Ershov's hierarchy. In the present paper we shall prove two propositions about the hierarchy of limiting computations. The first of them will imply the possibility of a natural extension of the results in Sec. 5 of [1]. The second proposition establishes an interesting property of closedness of the hierarchy of limiting computations.

"Function" means "an everywhere defined function from N into N." If φ is a partial function, then dom φ is its domain of definition, while $\varphi(x)\downarrow$ and $\varphi(x)\uparrow$ mean that φ is respectively defined and not defined at the point x. Let $\lambda x, y.\langle x, y\rangle$ be the Cantor function coding pairs, l and r be their inverse functions, $\varkappa$ the Kleene numeration of all partly recursive functions (prf), and $(0; <_0)$ the Kleene system of ordinal notations (see [2, 4, and 5]). The symbol $\Box$ means the end of a proof.

To any partial function φ and any $m \in N$ we shall assign a function f called the m-extension of the function φ, and defined as follows: $f(x) \simeq m$ when $\varphi(x)\uparrow$ and $f(x) \simeq \varphi(x)$ otherwise. To every partial function φ and every $a \in 0$ we shall assign a new partial function ψ, called the a-minimization of the function φ and defined for any $x \in N$ as follows:

$$\psi(x) \simeq \begin{cases} \uparrow, \text{ if } & \varphi\langle v, x\rangle\uparrow \quad \text{for any} \quad v <_0 a, \\ \varphi\langle u, x\rangle & \text{otherwise,} \end{cases}$$

where u is the $<_0$-smallest element of the set $\{b \mid b <_0 a\}$, for which $\varphi\langle u, x\rangle\downarrow$.

<u>Definition 1.</u> For any $a \in 0$ and $m \in N$ we define the sets P_a, C_a^m, and D_a as follows: P_a is the set of all a-minimizations of all prf; C_a^m is the set of all m-extensions of functions in P_a; D_a is the class of all everywhere defined functions in P_a.

The basic objects of study in the present paper are the classes of everywhere defined functions C_a^m and D_a. We note some of their properties: 1) $D_a \subseteq C_a^m$ for any $a \in O$ and $m \in N$. 2) $D_a = C_a^m \cap C_a^n$ for any $a \in O$, m, $n \in N$, $m \neq n$. 3) $C_a^m \not\subseteq C_a^n$ for any $a \in O$, m, $n \in N$, $m \neq n$. 4) For any $a, b \in O$ and $m \in N$, if $a <_0 b$, then $C_a^m \subseteq D_b$. 5) The class $\bigcup \{C_a^m \mid a \in O,\ m \in N\} = \bigcup\{D_a \mid a \in O\}$ coincides with the class of all functions T-convergent to the creative set, and also with the class of all functions computable in the limit.

Kazan. Translated from Sibirskii Matematicheskii Zhurnal, Vol. 25, No. 5, pp. 146-156, September-October, 1984. Original article submitted June 16, 1982.

0037-4466/84/2505-0798$08.50 © 1985 Plenum Publishing Corporation