TESTS FOR HOLOMORPHY IN SYMMETRIC REGIONS

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In this article we obtain conditions for holomorphy of functions in symmetric regions, connected with the action of the group of analytic automorphisms of this region. The results proved in Sec. 3 were published in a shortened form in [1].

1. Formulation of the Results

Let D be an open region in $\mathbf{C}^n$. Denote by G(D) the group of analytic automorphisms, i.e., biholomorphic mappings of the region D onto itself. We recall that D is called symmetric if each point $z \in D$ is the unique fixed point of some holomorphic involution $\sigma_z \in G(D)$ ($\sigma_z \circ \sigma_z$ is the identity mapping). Each irreducible symmetric region, excluding two special types of region of complex dimension 16 and 27, is analytically equivalent to one of the classical regions (defined below).

If F is a compactum in $\mathbf{C}^n$, then as usual C(F) denotes the space of all continuous complex functions on F with the sup-norm. The union of the uniform limits on F of sequences of polynomials in complex variables is denoted by P(F).

THEOREM 1. Let D be a classical region, and F a compactum lying in D such that $P(F) \neq C(F)$.

If the function f is continuous in D and for any $\omega \in G(D)$ the restriction $f \circ \omega|_F$ belongs to P(F), then f is holomorphic in the region D.

THEOREM 2. Let D be a classical region, and R a compact Riemannian surface with nonempty boundary ∂R, analytically imbedded in D. If the function f is continuous in D and $f \circ \omega|_{\partial R}$ admits holomorphic continuation in R for any $\omega \in G(D)$, then f is holomorphic in the region D.

For the special case when $D = D^n$ is the unit sphere in $\mathbf{C}^n$ and f is continuous in the closed sphere $\overline{D^n}$, results analogous to these were obtained in [2, 3]. They are a direct corollary of the following theorem ([2] for n = 1 and [3] for n > 1).

Let A be a closed subalgebra of the algebra $C(\overline{D^n})$, invariant under the action of the group $G(D^n)$ and containing all complex polynomials (or at least one nonconstant polynomial in complex variables). Then A coincides with one of the following algebras: 1) $P(\overline{D^n})$, 2) $C(\overline{D^n})$, 3) $P(\overline{D^n}) \oplus C_0(D^n)$, where $C_0(D^n)$ is the subalgebra of $C(\overline{D^n})$ consisting of functions equal to zero on the boundary of the sphere.

For the case $D = D^n$, $f \in C(\overline{D^n})$, Theorems 1 and 2 are obtained easily from the above theorem, if as A we take the algebra of all functions satisfying the conditions of the corresponding theorem.

Nagel and Rudin [4] proved that the above list contains all invariant, closed subalgebras of $C(\overline{D^n})$ containing constants and nonconstant functions, and they also mentioned its application to obtaining criteria for holomorphy.

In the complex-one-dimensional case, we may give Theorems 1 and 2 the following formulation: if a function is continuous in the open unit disk $D^1 \subset \mathbf{C}$ and

$$\int_{\omega(\gamma)} z^k f(z)\, dz = 0, \quad k = 0, 1, \ldots, \tag{1.1}$$

for some closed contour $\gamma \subset D^1$ and all conformal automorphisms $\omega \in G(D^1)$, then f is holomorphic in D^1.

In [5] Zalcman posed the problem of the sufficiency of condition (1.1) with k = 0 for f to be holomorphic. In other words, we are concerned with the validity of Morer's theorem for an arbitrary conformally invariant family of contours lying in the region.

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The following theorem gives a positive answer to this question.

THEOREM 3. Let $f \in L^2(D^1, \sigma)$ (σ is the Lebesgue measure), and for some pointwise-smooth closed contour $\gamma \subset D^1$ and almost all (in the Haar measure) $\omega \in G(D^1)$, we have

$$\int_{\omega(\gamma)} f(z)\, dx = 0. \tag{1.2}$$

Then almost everywhere, f coincides with a holomorphic function.

For extensions of the functional space to which f may belong, condition (1.2) can be strengthened:

THEOREM 4. Let $f \in L^1_{\text{loc}}(D^1, \sigma)$ and let there exist circles $C_{r_i} = \{z \in \mathbf{C} : |z| = r_i\}$, $0 < r_i < 1$, $i = 1, 2$, such that

$$\int_{\omega(C_{r_1})} f(z)\, dz = \int_{\omega(C_{r_2})} f(z) dz = 0 \tag{1.3}$$

for almost all $\omega \in G(D^1)$.

Set $J_r(s) = F(2 - is, 3/2, 3, -4r(1-r)^{-2})$, where F is a hypergeometric function.

Then almost everywhere on D^1, f coincides with a holomorphic function in each of the two following cases: 1) J_{r_1} and J_{r_2} have no common zeros in the complex plane, and 2) J_{r_1} and J_{r_2} have no common zeros in the strip $|\mathrm{Re}\, s| < 1$, and near the boundary of the disk f satisfies (almost everywhere) the estimate of growth

$$|f(z)| \leqslant c(1 - |z|^2)^{-1}$$

For the complex-multidimensional case, we have an extension of Theorem 3 in the following form:

THEOREM 5. Let D be a bounded symmetric region in $\mathbf{C}^n$ and $f \in C_1(D)$, and moreover let

$$\frac{\partial f}{\partial \bar{z}_k} \in L^2(D, \rho\sigma), \quad k = 1, \ldots, n, \tag{1.4}$$

where $\rho(z) = K(z, \bar{z})$, and K is the core function of the region D. Let Δ be a region lying together with its closure in the region D, with smooth boundary $\partial\Delta$.

If for all $\omega \in G(D)$ we have

$$\int_{\omega(\partial\Delta)} f(z)\, dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_{k-1} \wedge d\bar{z}_{k+1} \wedge \ldots \wedge d\bar{z}_n = 0, \quad k = 1, \ldots, n, \tag{1.5}$$

then f is holomorphic in D.

Clearly Theorem 5 contains Theorem 1 for the case when f satisfies (1.4) and F is a compact subregion of D.

2. Proof of Theorems 1 and 2

We recall the definition of classical regions.

1) $D^{I}_{p,q}$ is the region formed by complex $p \times q$-matrices z, satisfying the condition $e_p - z\bar{z}' > 0$, where e_p is the unit matrix of dimension p, z' is the transposed matrix, $\bar{z}$ is the matrix with complex conjugated elements, and the sympol >0 denotes a positive definite matrix.
2) D^{II}_p is the region formed by complex symmetric matrices z of dimension p, satisfying the condition $e_p - z\bar{z} > 0$.
3) D^{III}_p is the region formed by complex cosymmetric matrices z of dimension p, satisfying the condition $e_p + z\bar{z} > 0$.
4) $D^{IV}_p = \{z \in \mathbf{C}^p : |z_1^2 + \ldots + z_p^2|^2 - 2(|z_1|^2 + \ldots + |z_p|^2) + 1 > 0,\ |z_1^2 + \ldots + z_p^2| < 1\}$.

LEMMA 2.1. Let D be a classical region, and K a stationary subgroup of the group G(D) at the point $0 \in D$. Then for each point $z \in D$, there exists an element $k \in K$ such that $kz = \bar{z}$.

Proof. We verify this lemma for regions of each type in turn.

1) For the region $D = D^{I}_{p,q}$ the group K consists of transformations of the form

$$k_{u,v} : z \to uzv',$$

where u and v are unitary matrices of dimensions p and q, respectively, and moreover $\det u \cdot \det v = 1$.

We may represent the arbitrary matrix $z \in D$ in the form $z = u_1 d v_1$, where u_1 and v_1 are unitary and d is diagonal and real. Then $k = k_{u,v}$ is the required element, where $u = \bar{u}_1 u_1^{-1}$, $v = v_1^{-1}\bar{v}$.

2) Let $D = D^{II}_p$ or $D = D^{III}_p$. Then K consists of transformations

$$k_u : z \to uzu',$$

where u is unitary. Each matrix $z \in D$ can be written in the form $z = usu'$, where u is unitary and s is real [6]. In this case we take $k = k_{uu'}$.

3) For the region $D = D^{IV}_p$, the group K consists of transformations

$$k_{\lambda, u} : z \to \lambda uz,$$

where $\lambda \in \mathbf{C}$, $|\lambda| = 1$, and u is an orthogonal $p \times p$-matrix.

Let $z = x + iy$, $x, y \in \mathbf{R}^p$. Set $\lambda = e^{i\varphi}$, where $\varphi = \arctan[2(x, y)/\|y\|^2 - \|x\|^2]$. Then for the vector $w = \lambda^{1/2} z$, we have $(\mathrm{Re}\, w, \mathrm{Im}\, w) = 0$. There exists an orthogonal transformation u of the space $\mathbf{R}^p$ such that $u(\mathrm{Re}\, w) = \mathrm{Re}\, w$, $u(\mathrm{Im}\, w) = -\mathrm{Im}\, w$, i.e., $uw = \bar{w}$. Then the required element is $k_{\lambda,u}$.

LEMMA 2.2. Let D be a bounded irreducible symmetric region, and X a family of continuous functions on D which are invariant with respect to analytic automorphisms, i.e., $f \circ \omega \in X$ for any $f \in X$, $\omega \in C(D)$. Then either all the functions in X are constant, or X separates points on D.

Proof. Considered with the topology of uniform convergence on compacta, the group G(D) is a semisimple Lie group. Fix the point $z_0 \in D$. The stationary subgroup K at the point z_0 is a maximal compact subgroup of G(D). Set $D_0 = \{z \in D : f(z) = f(z_0)$ for all $f \in X\}$, and let H be the subgroup consisting of all elements of G(D) which map D_0 into itself. Since for any $f \in X$ and $k \in K$ we have $f \circ k \in X$, then $K \subset H$. Since D_0 is closed, the group H is closed in G(D) and therefore H is a Lie subgroup.

As D is irreducible, the Lie algebra $\mathfrak{K}$ of the group K is a maximal subalgebra of the Lie algebra $\mathfrak{G}$ of the group G(D) [7, p. 337], and since the Lie algebra $\mathfrak{H}$ of the group H contains $\mathfrak{K}$, then either $\mathfrak{K} = \mathfrak{H}$ or $\mathfrak{H} = \mathfrak{G}$. In the latter case H = G, and hence $D_0 = D$ and $X = \mathbf{C}$. If $\mathfrak{K} = \mathfrak{H}$, then the groups H and K are locally holomorphic and the factor-space $H/K = D_0$ is discrete. Since K is connected and takes D_0 into itself, then it follows from the fact that D_0 is discrete that each point of D_0 is fixed with respect to K. However, the involution σ_{z_0} at the point z_0 has a unique fixed point z_0. Therefore, $D_0 = \{z_0\}$, and this means that X separates points.

LEMMA 2.3. Let D be a classical region, let f satisfy the conditions of Theorem 1 and for any $z \in D$ and $k \in K$, let

$$f(kz) = f(z). \tag{2.1}$$

Then f is constant.

Proof. Denote by X_K the algebra of all the functions satisfying the conditions of the lemma. It is easily seen that for any $f \in X_K$, the function $z \to \overline{f(\bar{z})}$ belongs to X_K. But by (2.1) and Lemma 2.2, $f(z) = f(\bar{z})$ and therefore $\bar{f} \in X_K$. Then the algebra X formed by polynomials in the functions $f \circ \omega$, $f \in X_K$, $\omega \in G(D)$, is closed with respect to complex conjugation, and therefore does not separate points on F, since otherwise by the Stone–Weierstrass theorem $X|_F$ is dense in C(F), and it would follow from the inclusion $X|_F \subset P(F)$ that P(F) = C(F), which contradicts the condition. Applying Lemma 2.2, we obtain $X = \mathbf{C}$, and hence $X_K = \mathbf{C}$.

Proof of Theorem 1. Denote by X(F) the space of functions satisfying the conditions of Theorem 1. Let $f \in X(F)$, and suppose that $f \in C^1(D)$. Set

$$\tilde{f}(z) = \int_K f(kz)\, dk$$

(dk is the Haar measure). Then $\tilde{f} \in X(F)$ and satisfies (2.1). By Lemma 2.3, $\tilde{f} = \mathrm{const}$. Since $\tilde{f}(z_0) = f(z_0)$, then

$$f(z_0) = \int_K f(kz)\, dk. \tag{2.2}$$

Consider the G(D)-invariant Laplace operator

$$\Delta = \sum_{i,j=1}^{n} g_{ij}(z) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.$$

Applying the differential operator Δ to both parts of equation (2.2) and bearing in mind that Δ commutes with the action of K, we see that for $z = z_0$, $(\Delta f)(z_0) = 0$. Since $z_k f \in X(F)$, $k = 1, \ldots, n$, then we also have $\Delta(z_k f)(z_0) = 0$. Therefore

$$\sum_{j=1}^{n} g_{kj}(z_0) \frac{\partial f}{\partial \bar{z}_j}(z_0) = \Delta(z_k f)(z_0) - z_{0k}\Delta f(z_0) = 0.$$

As the matrix $(g_{kj}(z_0))_n^{k,j=1}$ is irreducible, $(\partial f/\partial \bar{z}_j)(z_0) = 0$, $j = 1, \ldots, n$. Applying the same arguments to the functions $f \circ \omega$, $\omega \in G(D)$, as D is homogeneous we see that $\partial f/\partial z_j = 0$ identically, i.e., f is holomorphic.

If we do not suppose earlier that f is differentiable, we have to consider the modified functions

$$(\varphi * f)(z) = \int_{G(D)} \varphi(\omega) f(\omega^{-1} z)\, d\omega,$$

where φ are finite functions on G(D) in the class C^∞. These convolutions approximate f in the topology of uniform convergence on compacta and belong to X(F). Their differentiability follows from the existence of a differentiable right inverse of the mapping $p: G(D) \ni \omega \to \omega(z_0) \in D$. By the above $\varphi * f$ are holomorphic, and therefore f is also holomorphic.

The proof of Theorem 2 is completely analogous.

3. Proof of Theorems 3, 4, and 5

1. The group $G(D^1)$ of conformal automorphisms of the open unit disk D^1 in the complex plane is isomorphic to the group of complex matrices of the form

$$\omega = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with the condition $\alpha\bar{\alpha} - \beta\bar{\beta} = 1$. The action on the disk is defined by the formula

$$\omega z = (\alpha z + \beta)(\bar{\beta} z + \bar{\alpha})^{-1}.$$

Kelly's transformation $w \to (w - i)(w + i)^{-1}$ of the upper half plane Π_+ induces on D^1 an isomorphism of groups of conformal automorphisms. The group G(D) under this isomorphism goes to the group $SL_2(\mathbf{R})$ of unimodular real matrices of order 2, which acts on Π_+ by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d}.$$

It will be convenient for us to parametrize conformal mappings of the disk by the standard method:

$$\omega_{\theta,s}(z) = e^{i\theta}(z + s)(1 + \bar{s}z)^{-1}, \quad \theta \in [0, 2\pi], \quad |s| < 1. \tag{3.1}$$

If $\omega = \omega_{\theta,s}$, then write $\theta = \theta(\omega)$, $s = s(\omega)$.

Set also $\omega_z = \omega_{0,z}$. The element ω_z corresponds to the matrix

$$\omega_z = \begin{pmatrix} (1 - |z|^2)^{-1/2} & z(1 - |z|^2)^{-1/2} \\ \bar{z}(1 - |z|^2)^{-1/2} & (1 - |z|^2)^{-1/2} \end{pmatrix}.$$

2. Let γ be a pointwise-smooth curve lying in D_1. Denote by $H(\gamma)$ the subspace of $L^2(D^1, \sigma)$ consisting of functions f for which

$$\int_{\omega(\gamma)} f(z)\, dz = 0 \tag{3.2}$$

for almost all (in the Haar measure) $\omega \in G(D^1)$.

Let L^2_{hol} be the subspace of holomorphic functions in $L^2(D^1, \sigma)$. By Cauchy's theorem $L^2_{hol} \subset H(\gamma)$. Denote by $H^\perp(\gamma)$ the orthogonal complement of L^2_{hol} in $H(\gamma)$. Let $f \in H^\perp(\gamma)$. Then the function

$$(R_\omega f)(z) = r(\omega, z) f(\omega^{-1} z), \tag{3.3}$$

where $r(\omega, z) = (-\bar{\beta} z + \alpha)^{-2}$, $\omega^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ -\bar{\beta} & \alpha \end{pmatrix}$, also belongs to $H^{\perp}(\gamma)$.

Formula (3.3) defines a unitary representation of the group $G(D^1)$ in the space $H^{\perp}(\gamma)$.

Set $\rho(z) = (1 - |z|^2)^{-2}$. We note that the measure $\rho\sigma$ is $G(D^1)$-invariant.

LEMMA 3.1. The operator $\bar{\partial} = \rho^{-1}(\partial/\partial\bar{z})$ maps the dense subspace $H_{\bar{\partial}} \subset H^{\perp}(\gamma)$ into the space $L^2(D^1, \rho\sigma)$.

Proof. For each finite function φ on $G(D^1)$ in the class C^{∞}, set

$$R_{\varphi} f = \int_{G(D^1)} \varphi(\omega) R_{\omega} f d\omega, \quad f \in H^{\perp}(\gamma). \tag{3.4}$$

The functions $R_{\varphi} f$ belong to $H^{\perp}(\gamma)$, and their linear hull $H_{\bar{\partial}}$ is dense in $H^{\perp}(\gamma)$.

Using the change of variables $\eta = \omega^{-1}\omega_z$, we rewrite (3.4) in the form

$$(R_{\varphi} f)(z) = \int_{G(D^1)} \varphi(\omega_z \eta^{-1}) f(\eta(0)) r(\omega_z \eta^{-1}, z) d\eta. \tag{3.5}$$

It follows from this notation that $R_{\varphi} f \in C^{\infty}(D^1)$. It remains to prove that $\partial R_{\varphi} f \in L^2(D^1, \rho\sigma)$.

From (3.5) we have

$$\frac{\partial}{\partial \bar{z}} (R_{\varphi} f)(z) = \int \left[\frac{\partial}{\partial \bar{z}} \varphi(\omega_z \eta^{-1}) r(\omega_z \eta^{-1}) + \varphi(\omega_z \eta^{-1}) \frac{\partial}{\partial \bar{z}} r(\omega_z \eta^{-1}, z) \right] f(\eta(0)) d\eta. \tag{3.6}$$

We estimate the terms under the integral separately. Set $\theta_0 = \theta(\eta^{-1})$, $s_0 = s(\eta^{-1})$. Then

$$e^{i\theta(\omega_z \eta^{-1})} = (e^{i\theta_0} + \bar{s}_0 z)(1 + e^{i\theta_0} s_0 \bar{z})^{-1},$$
$$s(\omega_z \eta^{-1}) = (e^{i\theta_0} s_0 + z)(e^{i\theta_0} + \bar{s}_0 z)^{-1}. \tag{3.7}$$

From (3.7),

$$\frac{\partial}{\partial \bar{z}} \theta(\omega_z \eta^{-1}) = i e^{i\theta_0} s_0 (1 + e^{i\theta_0} s_0 \bar{z})^{-1},$$
$$\frac{\partial}{\partial \bar{z}} \bar{s}(\omega_z \eta^{-1}) = e^{-i\theta_0}(1 - |s_0|^2)(1 + e^{i\theta_0} s_0 \bar{z})^{-2}.$$

Hence, bearing in mind that $|s_0| \leq 1$, $|z| \leq 1$, we have

$$\left| \frac{\partial}{\partial \bar{z}} \theta(\omega_z \eta^{-1}) \right| \leqslant 2(1 - |s_0|^2)^{-1},$$
$$\left| \frac{\partial}{\partial \bar{z}} \bar{s}(\omega_z \eta^{-1}) \right| \leqslant 4(1 - |s_0|^2)^{-1}. \tag{3.8}$$

Since $s_0 = e^{-i\theta_0} \omega_z^{-1} \omega(0)$, then

$$1 - |s_0|^2 = 1 - |\omega(0) - z|^2 |1 - \omega(0)\bar{z}|^{-2} =$$
$$= (1 - |z|^2)(1 - |\omega(0)|^2)|1 - \omega(0)\bar{z}|^{-2} \geqslant (1 - |z|^2)(1 - |\omega(0)|)(1 + |\omega(0)|)^{-1}.$$

The integration in (3.4) takes place over the compact subset $G(D^1)$, and therefore in the domain of integration we have $|\omega(0)| \leq c_1 < 1$. From the above inequality we have $1 - |s_0|^2 \geq (1 - c_1)(1 + c_1)^{-1}(1 - |z|^2)$. Substituting this inequality in (3.8), we obtain

$$\left| \frac{\partial}{\partial \bar{z}} \theta(\omega_z \eta^{-1}) \right| \leqslant 2(1 + c_1)(1 - c_1)^{-1}(1 - |z|^2)^{-1},$$

$$\left| \frac{\partial}{\partial \bar{z}} \bar{s}(\omega_z \eta^{-1}) \right| \leqslant 4(1 + c_1)(1 - c_1)^{-1}(1 - |z|^2)^{-1}.$$

Introducing the function $\tilde{\varphi}(\theta, s) = \varphi(\omega_{\theta, s})$, on the basis of the last two inequalities we obtain

$$\left| \frac{\partial}{\partial \bar{z}} \varphi(\omega_z \eta^{-1}) \right| = \left| \frac{\partial \tilde{\varphi}}{\partial \theta} (\theta(\omega_z \eta^{-1}), s(\omega_z \eta^{-1})) \cdot \frac{\partial}{\partial \bar{z}} \theta(\omega_z \eta^{-1}) + \right.$$

$$+\frac{\partial\tilde{\varphi}}{\partial s}(\theta(\omega_z\eta^{-1}),\ s(\omega_z\eta^{-1}))\cdot\frac{\partial}{\partial\bar{z}}\bar{s}(\omega_z\eta^{-1})\Big|\leqslant 2(1+c_1)(1-c_1)^{-1}(1-|z|^2)^{-1}\times$$

$$\times\left\{\left|\frac{\partial\tilde{\varphi}}{\partial\theta}(\theta(\omega_z\eta^{-1}),\ s(\omega_z\eta^{-1})\right|+2\left|\frac{\partial\tilde{\varphi}}{\partial s}(\theta(\omega_z\eta^{-1}),\ s(\omega_z\eta^{-1})\right|\right\}. \tag{3.9}$$

We estimate the function $(\partial/\partial\bar{z})r(\omega_Z\eta^{-1}, z)$. We have

$$r(\omega_z\eta^{-1},\ z)=(1-|z|^2)[(a\bar{z}-\bar{b})z+(a+\bar{b}z)]^{-2},$$

where $\eta^{-1}=\begin{pmatrix} a & b\\ \bar{b} & \bar{a}\end{pmatrix}$. Hence

$$\left|\frac{\partial}{\partial\bar{z}}r(\omega_z\eta^{-1},z)\right|\leqslant|r(\omega_z\eta^{-1},z)|(1-|z|^2)^{-1}+2|a|(1-|z|^2)^{-1/2}|r(\omega_z\eta^{-1},z)|^{3/2}. \tag{3.10}$$

From the relation $\eta^{-1}=\omega_Z^{-1}\omega$ we define the element a of the matrix $\eta^{-1}: a=(1-|z|^2)^{-1/2}(\omega_1-z\bar{\omega}_2)$, where $\omega=\begin{pmatrix}\omega_1 & \omega_2\\ \bar{\omega}_2 & \bar{\omega}_1\end{pmatrix}$. As a result of the compactness of the domain of integration in (3.4), the elements ω_1, ω_2 and the function $r(\omega_Z\eta^{-1}, z)$ are bounded: $|\omega_1|$, $|\omega_2|$, $|r|\leq c_2$. Then from (3.10) we have

$$\left|\frac{\partial}{\partial\bar{z}}r(\omega_z\eta^{-1},z)\right|\leqslant c_3(1-|z|^2)^{-1},\quad c_3=c_2+4c_2^{5/2}. \tag{3.11}$$

Using formula (3.6) and inequalities (3.9) and (3.11), we obtain

$$\left|\frac{\partial}{\partial\bar{z}}(R_\varphi f)(z)\right|\leqslant(1-|z|^2)^{-1}F(z),$$

where $F(z)=\int_{G(D^1)}\psi(\omega)|f(\omega^{-1}z)|d\omega$, and ψ is some continuous function on the group $G(D^1)$ with compact carrier. Since $f\in L^2(D^1,\sigma)$, then $F\in L^2(D^1,\sigma)$ and by the last inequality, $(1-|z|^2)^2\frac{\partial}{\partial\bar{z}}R_\varphi f\in L^2(D^1,(1-|z|^2)^{-2}\sigma)$.

Proof of Theorem 3. Denote by $H_1^\perp(\gamma)$ the subspace of the space $H(\gamma)$ consisting of all $f\in H^\perp(\gamma)$ such that

$$f(e^{i\theta}z)=e^{-i\theta}f(z),\quad z\in D^1,\quad \theta\in[0,2\pi).$$

The operator π_1 defined by the formula

$$(\pi_1 f)(z)=\frac{1}{2\pi}\int_0^{2\pi}f(e^{i\theta}z)e^{i\theta}d\theta$$

is a projector of the space $H^\perp(\gamma)$ onto the subspace $H_1^\perp(\gamma)$. The operator π_1 is continuous, and therefore since $H_{\bar{\partial}}$ is dense in $H^\perp(\gamma)$, the subset $H_{\bar{\partial}}\cap H_1^\perp(\gamma)$ is dense in $H_1^\perp(\gamma)$.

Let $f\in H_{\bar{\partial}}\cap H_1^\perp(\gamma)$. Apply Green's formula to (1.2). We obtain

$$\int_{\omega(\Delta)}\frac{\partial f}{\partial\bar{z}}dz\wedge d\bar{z}=\int_{\omega(\gamma)}f(z)\,dz=0,$$

where Δ is the region bounded by the curve γ. We rewrite the last equation as:

$$\int_{D^1}\bar{\partial}f\cdot\chi_\Delta\circ\omega^{-1}\cdot\rho\,d\sigma=0, \tag{3.12}$$

where χ_Δ is the characteristic function of the region Δ.

Denote by $p: G(D^1)\to D^1$ the natural projection, $p(\omega)=\omega(0)$, $\omega\in G(D^1)$. Using the mapping p, we may carry Eq. (3.12) over to the group $G(D^1)$:

$$\int_{G(D^1)}(\bar{\partial}f\circ p)(\eta)(\chi_\Delta\circ p)(\omega^{-1}\eta)\,d\eta=0,$$

($d\eta$ is the Haar measure). The latter expression can be written as an equation of convolutions:

$$(\bar{\partial}f\circ p)*(\chi_\Delta\circ p)^\sim=0, \tag{3.13}$$

where $(\chi_\Delta\circ p)^\sim(\omega)=(\chi_\Delta\circ p)(\omega^{-1})$.

Finally, using the isomorphism of the groups $G(D^1)$ and $SL_2(\mathbf{R})$ induced by Kelly's transformation of the upper half plane into a disk, we carry the function $\bar{\partial} f \circ p$, $(\chi_\Delta \circ p)^\sim$ over to $SL_2(\mathbf{R})$. By (3.13) the functions f' and χ' that we obtain are connected by the relation

$$f' * \chi' = 0. \tag{3.14}$$

The function χ' is finite, since χ_Δ has compact carrier lying in D^1.

Let $K = SO(2)$ be an orthogonal subgroup of the group $SL_2(\mathbf{R})$. It follows from the easily verified relation

$$\partial R_\omega f = \bar{\partial} f \circ \omega^{-1},\ \omega \in G(D^1), \tag{3.15}$$

that if $f \in H_1^\perp(\gamma)$, then $\bar{\partial} f$ is invariant with respect to rotations of the disk. Then f' is biinvariant with respect to K, i.e., $f'(k_1\omega k_2) = f'(\omega)$ for all $k_1, k_2 \in K$, $\omega \in SL_2(\mathbf{R})$. Moreover, by Lemma 3.1 $f' \in L^2(SL_2(\mathbf{R}))$.

We apply the K-spherical Fourier transform to (3.14) (see, e.g., [8]). Bearing in mind that one of the functions in the convolution is K-biinvariant, we obtain

$$\hat{f}' \cdot \hat{\chi}' = 0. \tag{3.16}$$

Recall the Harish–Chandra formula for calculating K-spherical transformations:

$$\hat{g}(s) = \widehat{Hg^K}(s),\ g \in L^2(SL_2(\mathbf{R})), \tag{3.17}$$

where on the right-hand side we have the ordinary Fourier functions

$$(Hg^K)(t) = e^{-t} \int_{-\infty}^{\infty} g^K(x, e^t)\, dx$$

and g^K is obtained by carrying the following function over to the upper half plane $H_+ = SL_2(\mathbf{R}) / K$:

$$g^K(\omega) = \iint_K g(k_1\omega k_2)\, dk_1 dk_2,$$

and this is constant on two-sided residue classes by K. Since the function $H\chi'^K$ has compact carrier, $\hat{\chi}'$ can be continued to the complex plane as an entire function. Therefore, the set of zeros of the function $\hat{\chi}'$ on $\mathbf{R}$ has measure zero. By (3.16), $\hat{f}' = 0$ almost everywhere. The Plancherel measure for a Fourier transform on $SL_2(\mathbf{R})$ is absolutely continuous relative to Lebesgue measure on a real straight line. Since $f' \in L^2(SL_2(\mathbf{R}))$, it follows from $\hat{f}' = 0$ that $f' = 0$. Returning to the original function, we obtain $\bar{\partial} f = 0$. The operator $\bar{\partial}$ is one-to-one on $H^\perp(\gamma)$, and therefore $f = 0$. The function $f \in H_\partial \cap H_1^\perp(\gamma)$ was taken arbitrarily, and therefore we can conclude that $H_{\bar{\partial}} \cap H_1^\perp(\gamma) = 0$.

Now let f be an arbitrary function on $H_{\bar{\partial}} \cap H^\perp(\gamma)$. From (3.15) we have the relation

$$\pi_0 \bar{\partial} f = \bar{\partial} \pi_1 f$$

where $(\pi_0 \bar{\partial} f)(z) = \frac{1}{2\pi} \int_0^{2\pi} (\bar{\partial} f)(e^{i\theta} z)\, d\theta$. Since $\bar{\partial}\pi_1 f = 0$, then $\bar{\partial} f(0) = (\pi_0 \bar{\partial} f)(0) = 0$. The same is true for the functions $R_\omega f$, $\omega \in G(D^1)$, and therefore, using (3.15), we obtain $\bar{\partial} f(\omega^{-1}(0)) = 0$ and as ω is arbitrary, $\bar{\partial} f = 0$. Then $f = 0$ and we see that $H_{\bar{\partial}} \cap H^\perp(\gamma) = 0$. Hence, since $H_{\bar{\partial}} \cap H^\perp(\gamma)$ is dense in $H^\perp(\gamma)$, we have $H^\perp(\gamma) = 0$, i.e., $H(\gamma) = L^2_{hol}$.

4. Proof of Theorem 4. First let $f \in C^\infty(D^1)$. Apply Green's formula to (1.3):

$$\int_{D^1} \bar{\partial} f \cdot \chi_{r_i} \circ \omega^{-1} \cdot \rho\, d\sigma = 0,\quad i = 1, 2. \tag{3.18}$$

Here χ_r is the characteristic function of the disk with radius r and center zero. Bearing in mind the invariance of the functions χ_{r_i} under rotations of the disk, as in the previous section, we obtain

$$(\bar{\partial} f \circ p) * (\chi_{r_i} \circ p) = 0,\quad i = 1, 2. \tag{3.19}$$

We find the Fourier transform of the function χ_r in explicit form. To do this we go over to the function $\tilde{\chi}_r(w) = \chi_r[(w - i)/(w + i)]$, defined in the upper half plane. Using (3.17) and elementary transformations, we find

$$\hat{\chi}_r(s) = \int_{-\infty}^{\infty} e^{(is-1)t} \tilde{\chi}_r(x + ie^t)\, dx dt = 16r^2(1-r)^{is-4}(1+r)^{-is} \int_0^1 [1 + 4rt(1- r)^{-2}]^{is-2} t^{1/2}(1-t)^{1/2} dt = 2\pi r^2 (1-r)^{is-4}(1+r)^{-is} F(2 - is, 3/2, 3, -4r(1-r)^{-2}), \tag{3.20}$$

where F is a hypergeometric function.

Suppose now that condition 1) of Theorem 4 is satisfied. By (3.20), the functions $\hat{\chi}_{r_1}$, $\hat{\chi}_{r_2}$ have no common zeros in the complex plane. Consider the space $\mathscr{E}_K$ of finite K-invariant distributions on D^1, i.e., the dual space of $C_K^\infty(D^1)$, the space of infinitely differentiable K-invariant functions in the disk D^1 with the topology of uniform convergence on compacta. We recall that here K is the group of rotations of the disk.

Let $\hat{\mathscr{E}}_K$ be the space of Fourier transforms

$$\hat{T}(s) = \langle T, \varphi_s \rangle$$

of distributions $T \in \mathscr{E}_K(D^1)$. Here φ_s is the K-spherical function corresponding to the value of the parameter s. The space $\hat{\mathscr{E}}_K(D^1)$ consists [9] of entire even functions with the following conditions on the growth:

$$\sigma_{n,R}(f) = \sup_{z \in \mathbf{C}} (1 + |z|)^{-n} e^{-R|\operatorname{Im} z|} |f(z)| < \infty.$$

The seminorms $\sigma_{n,R}$ define a topology in $\hat{\mathscr{E}}_K(D^1)$ in which Fourier transformation is a topological isomorphism. This topology coincides with the topology in the space $\hat{\mathscr{E}}(\mathbf{R})$ of ordinary Fourier transformations of finite distributions on a straight line.

Since the functions $\hat{\chi}_{r_1}$, $\hat{\chi}_{r_2}$ have no common zeros, then by Schwartz' theorem [10], the ideal generated by the functions χ_{r_1}, χ_{r_2} is dense in $\hat{\mathscr{E}}(\mathbf{R})$. In this case the ideal generated by these functions in the subspace $\hat{\mathscr{E}}_K(D^1) \subset \hat{\mathscr{E}}(\mathbf{R})$ of even functions is dense in $\hat{\mathscr{E}}_K(D^1)$. Therefore, by (3.19) we see that the mean $(\pi_0 \bar{\partial} f)(z) = \frac{1}{2\pi} \int_0 (\bar{\partial} f)(e^{i\theta} z)\, d\theta = 0$. Then $\bar{\partial} f(0) = 0$, and taking the function $R_\omega f$ instead of f, by (3.15) we obtain $\bar{\partial} f(\omega^{-1}(0)) = 0$. Since $\omega \in G(D^1)$ is arbitrary, we have $\bar{\partial} f = 0$, i.e., f is holomorphic.

The case $f \in L^1_{\mathrm{loc}}(D^1, \sigma)$ can be reduced by a standard method to the smooth case. We may choose a sequence φ_n of smooth finite functions on $G(D^1)$ such that the functions $R_{\varphi_n} f$ converge as $n \to \infty$ to f in the L^1-norm on each compact subset of D^1. The functions $R_{\varphi_n} f$ belong to $C^\infty(D^1)$ and satisfy the conditions of the theorem, and by the above proof, are holomorphic. On almost every circle $|z| = r$, $0 < r < 1$ the sequence $R_{\varphi_n} f$ converges to f in the L^1-norm. Hence, on the basis of Cauchy's integral formula, it follows that f coincides with a holomorphic function almost everywhere.

To prove Theorem 4 for condition 2), we consider the space L_K^1 of K-invariant functions belonging to $L^1(D^1, \sigma)$. This space is a commutative Banach algebra with respect to the operation of convolution

$$(f * g)(z) = \int_{G(D^1)} f(\omega(0))\, g(\omega^{-1} z)\, d\omega.$$

The space of maximal ideals of this algebra coincides with the set of bounded K-spherical functions φ_s, and moreover the multiplicative linear functionals m_s on the algebra L_K^1 are of the form $m_s : f \to \hat{f}(s)$. The boundedness of the function φ_s is equivalent to the condition $|\operatorname{Re} s| < 1$, and therefore condition 2) of Theorem 4 together with formula (3.20) means that Gel'fand transformations of the elements $\chi_{r_1}, \chi_{r_2} \in L_K^1$ do not vanish simultaneously on the space of maximal ideals. Hence it follows that the ideal $I(\chi_{r_1}, \chi_{r_2})$ generated by the elements χ_{r_1}, χ_{r_2} is dense in the algebra L_K^1.

Let f, r_1 and r_2 satisfy condition 2) of Theorem 4. Consider the function $R_\varphi f$ [see (3.4)], where φ is a smooth function on $G(D^1)$ with compact carrier. It follows from condition 2) for the function f and the final estimate for $(\partial / \partial \bar{z}) R_\varphi f$, obtained in the proof of Lemma 3.1, that the function $\bar{\partial} R_\varphi f$ is bounded. Moreover, an equation of the form (3.19) holds for the function $\bar{\partial} R_\varphi f$, since the function $R_\varphi f$ satisfies condition (1.3). Hence, and from the fact that (χ_{r_1}, χ_{r_2}) is dense in L_K^1, we see that the mean by the group of rotations $\pi_0(\bar{\partial} R_\varphi f) = 0$. Then $\bar{\partial} R_\varphi f(0) = 0$, and since (1.3) is also satisfied for the function $R_\omega R_\varphi f$, then also $\bar{\partial} R_\omega R_\varphi f(0) = 0$. As ω was arbitrary, from formula (3.15) we obtain $\bar{\partial} R_\varphi f = 0$. Approximating f by functions of the form $R_\varphi f$, as we did above, we obtain the required result.

5. Proof of Theorem 5. This is completely analogous to the proof of Theorem 3. By Stocks' theorem, condition (1.5) is equivalent to the following:

$$\int_D [\rho(z)]^{-1} \frac{\partial f}{\partial \bar{z}_k}(z)\, \chi_\Delta(\omega^{-1} z)\, \rho(z)\, d\sigma(z) = 0, \quad \omega \in G(D), \quad k = 1, \ldots, n, \tag{3.21}$$

where χ_Δ is the characteristic function of the region Δ. The conditions placed on the derivatives of the function f mean that the functions $1/\rho \cdot (\partial f/\partial \bar{z}_k)$ belong to $L^2(D, \rho\sigma)$. Therefore, the holomorphy of f is proved, if we prove that the equation

$$\int_D g(z)\chi_\Delta(\omega^{-1}z)\rho(z)\,d\sigma(z) = 0, \quad \omega \in G(D), \tag{3.22}$$

has no nontrivial solutions is the space $L^2(D, \rho\sigma)$. We note that the measure $\rho\sigma$ is invariant with respect to the action of the group G(D). Denote by K the stationary subgroup of G(D) at some fixed point z_0, and set $(\pi_K g)(z) = \int_K g(uz)\,du$, where du is the Haar measure. Let $g \in L^2(D, \rho\sigma)$ satisfy (3.22). Then

$$\int_D (\pi_K g)(z)(\pi_K\chi_\Delta)(\omega^{-1}z)\rho(z)\,d\sigma(z) = 0. \tag{3.23}$$

The left-hand side of (3.23) can be rewritten as the convolution of two functions on the group G(D). To this equation [of type (3.13)], we may apply the K-spherical Fourier transformation (see [7]) on the symmetric space G(D)/K. This transformation is defined on K-invariant functions $h \in L^2(D, \rho\sigma)$, and is connected with the classical Fourier transformation using the Harish–Chandra formula: $\hat{h}(s) = \widehat{\mathcal{H}h}(s)$. Here s belongs to the Cartan subalgebra $\mathcal{A} \cong \mathbf{R}^p$ of the Lie algebra of the group G(D), $\mathcal{H}$ is the Harish–Chandra transformation:

$$(\mathcal{H}h)(\lambda) = \varkappa(\lambda)\int_Z h(z\exp\lambda\cdot z_0)\,dz, \quad \lambda \in A,$$

Z is the subgroup in the Iwasawa decomposition $G(D) = Z\exp\mathcal{A}\cdot K$, and $\varkappa(\lambda)$ is some factor. It is easily established that the function $\mathcal{H}\pi_K\chi_\Delta$ has compact carrier in $\mathbf{R}^p$, and therefore $\widehat{\pi_K\chi_\Delta} = \widehat{\mathcal{H}\pi_K\chi_\Delta}$ may be continued to an entire function in $\mathbf{C}^p$. Hence it follows that the set of zeros of the function $\widehat{\pi_K\chi_\Delta}$ in $\mathbf{R}^p$ has zero Lebesgue measure, and it follows from the equation $\widehat{\pi_K g}\cdot\widehat{\pi_K\chi_\Delta} = 0$, obtained from (3.23) by applying the Fourier transformation, that $\widehat{\pi_K g} = 0$ almost everywhere. Since $\pi_K g \in L^2(D, \rho\sigma)$, then by Plancherel's formula we obtain $\pi_K g = 0$, and in particular, $g(z_0) = 0$. For any $\omega \in G(D)$, the function $g\circ\omega$ also satisfies (3.22), and therefore $g(\omega(z_0)) = 0$ and thus $g = 0$, by the homogeneity of the region D with respect to the group G(D). The theorem is proved.

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