PRODUCTS OF RANDOM AFFINE TRANSFORMATIONS

A. K. Grincevičius UDC 519.21

In the present note we consider a sequence of independent random matrices $\left\{\begin{pmatrix}\xi_i & \eta_i\\ 0 & 1\end{pmatrix}\right\}_1^\infty$; $\xi_i \neq 0$ almost surely, and we study the question of the continuity of the distribution function of the element in the right upper corner of the infinite product of these matrices.

We write

$$\begin{pmatrix}\rho_n & \psi_n\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\xi_1 & \eta_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\xi_2 & \eta_2\\ 0 & 1\end{pmatrix}\dots\begin{pmatrix}\xi_n & \eta_n\\ 0 & 1\end{pmatrix}.$$

Then $\psi_n = \eta_1 + \eta_2\xi_1 + \dots + \eta_n\xi_1\dots\xi_{n-1}$, and if as $n \to \infty$ the sequence of random variables ψ_n converges in probability, then we denote the limit by

$$\psi=\eta_1+\eta_2\xi_1+\dots+\eta_n\xi_1\dots\xi_{n-1}+\dots .$$

One has

THEOREM 1. The following assertions are equivalent:

1) there exists a number S_0 such that $\mathbf{P}\{\psi = S_0\} > 0$;

2) there exists a sequence of numbers $\{S_n\}_0^\infty$ such that

$$\prod_{n=1}^{\infty} \mathbf{P}\{\eta_n+\xi_n S_n=S_{n-1}\}>0 \tag{1}$$

and the random variable $\rho_n S_n$ converges in probability to zero as $n \to \infty$.

Proof. Assertion 1 follows from Assertion 2. In fact, from (1) it follows that

$$\varliminf_{n\to\infty} \mathbf{P}\{\psi_n+\rho_n S_n=S_0\}\geqslant\prod_{n=1}^{\infty}\mathbf{P}\{\eta_n+\xi_n S_n=S_{n-1}\}>0.$$

Since the random variable $\psi_n + \rho_n S_n$ converges in probability to the random variable ψ as $n \to \infty$, one also has $\mathbf{P}\{\psi = S_0\} > 0$.

We shall prove that Assertion 2 follows from Assertion 1. Let $\mathbf{P}\{\psi = S_0\} = \alpha > 0$, so for any $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$\mathbf{P}\{|\psi-S_0|<2\delta\}<\alpha+\varepsilon, \tag{2}$$

and by virtue of the convergence in probability one can find a natural number $n' \equiv n'(\varepsilon)$ such that for $n > n'$

$$\mathbf{P}\{|\psi-\psi_n|\geqslant\delta\}=\mathbf{P}\{|\rho_n\bar{\psi}_n|\geqslant\delta\}<\varepsilon, \tag{3}$$

where

$$\bar{\psi}_n=\eta_{n+1}+\eta_{n+2}\xi_{n+1}+\dots+\eta_{n+k}\xi_{n+1}\dots\xi_{n+k-1}.$$

It follows from (2) and (3) that

$$\mathbf{P}\{|\psi_n-S_0|<\delta\}\leqslant\mathbf{P}\{|\psi-\psi_n|\geqslant\delta\}+\mathbf{P}\{|\psi-S_0|<2\delta\}<\alpha+2\varepsilon \tag{4}$$

for $n > n'$.

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Now

$$\alpha = \mathbf{P}\{\psi = S_0\} \leqslant \sum \mathbf{P}\{\psi_n + \rho_n s = S_0,\ |\rho_n s| < \delta\}\,\mathbf{P}\{\psi_n = s\} + \mathbf{P}\{|\psi - \psi_n| \geqslant \delta\};$$

so one can find a number S_n such that

$$\alpha_n = \mathbf{P}\{\psi_n + \rho_n = S_0,\quad |\rho_n S_n| < \delta\} > \alpha - \varepsilon. \tag{5}$$

It is easy to see that

$$\{\psi = S_0\} \cup \{\psi_n + \rho_n S_n = S_0,\ |\rho_n S_n| < \delta\} \subset \{|\psi - \psi_n| \geqslant \delta\} \cup \{|\psi_n - S_0| < \delta\}$$

and hence

$$\mathbf{P}\{|\psi - \psi_n| \geqslant \delta\} + \mathbf{P}\{|\psi_n - S_0| < \delta\} \geqslant \mathbf{P}\{\psi = S_0\} +$$
$$+ \mathbf{P}\{\psi_n + \rho_n S_n = S_0,\ |\rho_n S_n| < \delta\} - \mathbf{P}\{\psi = S_0,\ \psi_n + \rho_n S_n = S_0,\ |\rho_n S_n| < \delta\}. \tag{6}$$

Since the events $\{\psi = S_0,\ \psi_n + \rho_n S_n = S_0\}$ and $\{\psi_n + \rho_n S_n = S_0,\ \bar{\psi}_n = S_n\}$ coincide, it follows from (3)-(6) that

$$\alpha_n \mathbf{P}\{\bar{\psi}_n = S_n\} + \alpha + 2\varepsilon + \varepsilon > \alpha + \alpha_n,$$

or

$$\mathbf{P}\{\bar{\psi}_n = S_n\} > \frac{\alpha_n - 3\varepsilon}{\alpha_n} > 1 - \frac{3\varepsilon}{\alpha - \varepsilon} \tag{7}$$

for $n > n'$, if only the number ε is sufficiently small.

Now from (7) follows the existence of a monotone increasing sequence of natural numbers $\{n_k\}_1^\infty$ such that

$$\mathbf{P}\{\bar{\psi}_{n_k} = S_{n_k},\ \bar{\psi}_{n_{k+1}} = S_{n_{k+1}}\} > 1 - \frac{\alpha}{2^{k+1}}. \tag{8}$$

To conclude the proof of the theorem, we need an auxiliary assertion. We write $\rho(j, n) = \xi_{j+1} \ldots \xi_n$, $\psi(j, n) = \eta_{j+1} + \eta_{j+2}\xi_{j+1} + \ldots + \eta_n \xi_{j+1} \ldots \xi_{n-1}$.

<u>LEMMA 1.</u> For any real numbers a_0 and a_n

$$\mathbf{P}\left\{\sup_{1 \leqslant j < n} \left(\psi_j + \rho_j m\left(\psi(j, n) + \rho(j, n)\, a_n\right)\right) > a_0\right\} \leqslant 2\mathbf{P}\{\psi_n + \rho_n a_n > a_0\},$$

where $m(X)$ is the median of the random variable X.

The proof of the lemma repeats the proof of the analogous assertion for the sum of independent variables and we omit it.

By virtue of Lemma 1 and (8), there exist real numbers S_n for $n > n_1$ (and $n \neq n_k$) such that

$$\mathbf{P}\{\psi(j, n_k) + \rho(j, n_k) S_j = S_{n_k},\ j = n_k + 1, \ldots, n_{k+1}\} > 1 - 2\{\psi(n_k, n_{k+1}) + \rho(n_k, n_{k+1}) S_{n_{k+1}} \neq S_{n_k}\} > 1 - \frac{\alpha}{2^k}.$$

Since the event on the left side of the preceding inequality coincides with the event

$$\{\eta_j + \xi_j S_j = S_{j-1},\ j = n_k + 1, \ldots, n_k\},$$

we get that

$$\mathbf{P}\{\eta_j + \xi_j S_j = S_{j-1},\ j = n_1 + 1,\ n_1 + 2, \ldots\} > 1 - \sum_{k=1}^{\infty} \alpha \frac{1}{2^k} = 1 - \alpha. \tag{9}$$

From the equation $\mathbf{P}\{\psi = S_0\} = \alpha$ and (8) it follows that

$$\mathbf{P}\{\psi_{n_1} + \rho_{n_1} S_{n_1} = S_0\} > 0,$$

so one can find constants S_k for $k < n_1$ such that

$$\mathbf{P}\{\eta_j + \xi_j S_j = S_{j-1},\ j = 1, \ldots, n_1\}. \tag{10}$$

From (9) and (10), we get (1), and the second part of Assertion 2 follows from the following inequalities

$$\mathbf{P}\{|\rho_n S_n| > \delta\} \leqslant \mathbf{P}\{\bar{\psi}_n \neq S_n\} + \mathbf{P}\{|\rho_n \bar{\psi}_n| > \delta\} \to 0$$

as $n \to \infty$ by virtue of the convergence of the series ψ in probability and the inequality

$$\mathbf{P}\{\bar{\psi}_n = S_n\} \geqslant \mathbf{P}\{\eta_j + \xi_j S_j = S_{j-1},\ j = n+1, \ldots, n_k\}\ \mathbf{P}\{\bar{\psi}_{n_k} = S_{n_k}\} \to 1 \quad \text{for} \quad n \to \infty,$$

where $n_{k-1} < n \le n_k$. The theorem is proved.

The following example shows that one cannot relinquish the condition "$\rho_n S_n$ converges to zero in probability." Let $S_n = S$, $n = 0, 1, 2, \ldots$, satisfy (1), and the series $\ln \xi_n + \ln \xi_{n+1} + \ldots$ converge and have continuous distribution function. It follows from (1) that

$$\mathbf{P}\left\{\sum_{j=1}^{m} \eta_{n+j}\, \xi_{n+1} \ldots \xi_{n+j-1} = S - S\xi_{n+1} \ldots \xi_{n+m}\right\} \to 1$$

as $n \to \infty$ uniformly in all natural m, so the distribution of the random variable ψ will be continuous, i.e., Assertion 1 will not be satisfied.

If in the preceding example it is assumed that the random walk $\ln \xi_1 + \ldots + \ln \xi_n$, $n = 0, 1, 2, \ldots$, oscillates between $+\infty$ and $-\infty$, and $\mathbf{P}\{\ln \xi_1 + \ldots + \ln \xi_n < A\} \to 1$ for all real A (cf. [3]), then the random series will converge in probability, but almost surely diverge.

Theorem 1, as well as its proof, is an extension to a more general case of the well-known theorem of Levy [2] for series of independent variables.

An analogous assertion also exists for the product of random affine transformations of a finite-dimensional space. Let R^k be k-dimensional Euclidean space (of column-vectors), as before $\{(\xi_j, \eta_j)\}_1^\infty$ is a sequence of independent random elements, here η_j assume values in R^k, and ξ_j in the group of all real invertible $k \times k$ matrices, and the series $\psi = \eta_1 + \xi_1\eta_2 + \ldots + \xi_1 \ldots \xi_{n-1}\eta_n$ converges in probability. In order not to complicate the formulation of the following theorem, points and lines of the space R^k will be called planes, as well as planes of larger dimensions.

THEOREM 2. The following assertions are equivalent:

1) there exists a plane E_0 in R^k such that

$$\mathbf{P}\{\psi \in E_0\} > 0;$$

2) there exists a sequence of planes $\{E_n\}_0^\infty$ in R^k such that

$$\prod_{n=1}^{\infty} \mathbf{P}\{\eta_n + \xi_n E_n \subset E_{n-1}\} > 0$$

and for any neighborhood V of zero in the space R^k

$$\mathbf{P}\{\xi_1 \ldots \xi_n E_n \subset V + E_0 - E_0\} \to 1 \quad \text{as} \quad n \to \infty.$$

The proof of Theorem 2 differs slightly from the proof of Theorem 1, so we shall omit it. We shall merely indicate certain differences. At the beginning of the proof of the fact that Assertion 2 follows from Assertion 1, it is necessary to find a plane $E' \subset R^k$ and a natural number n_0 such that $\mathbf{P}\{\bar{\psi}_{n_0} \in E'\} > 0$, but for all $n > n_0$ and all planes E, whose dimension is strictly less than the dimension of E', $\mathbf{P}\{\bar{\psi}_n \in E\} = 0$. Without loss of generality, one can assume that $n_0 = 0$ and $E' = E_0$. Now we give the basic inequality analogous to (6)

$$\mathbf{P}\{\psi \in E_0,\ \psi_n + \rho_n E_n = E_0,\ \rho_n E_n \subset E_0^\delta - E_0\} + \mathbf{P}\{\psi_n \in E_0^\delta\} +$$
$$+ \mathbf{P}\{\psi - \psi_n \notin E_0^\delta - E_0\} \geqslant \mathbf{P}\{\psi \in E_0\} + \mathbf{P}\{\psi_n + \rho_n E_n = E_0,\ \rho_n E_n \subset E_0^\delta - E_0\},$$

where

$$E_0^\delta = \{x \mid x \in R^k,\ |x| \leqslant \delta\} + E_0.$$

Also, we give without proof a lemma which in the multidimensional case is somewhat different from Lemma 1.

LEMMA 2. Suppose for some planes E_n and E_0, whose dimensions coincide, one has

$$\mathbf{P}\{\psi_n + \rho_n E_n = E_0\} \geqslant 1 - \varepsilon > 0.$$

Then one can find planes $E_1, E_2, \ldots, E_{n-1}$ such that

$$\mathbf{P}\{\eta_j+\xi_j E_j=E_{j-1},\ j=1,\ \ldots,\ n\}\geqslant 1-\frac{\varepsilon}{1-\varepsilon}.$$

In conclusion, we note that the problem of the continuity of the distribution of the series ψ arose in the study of the asymptotic behavior of the product of random linear transformations of the line (cf. [1, 4, and 5]).

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STRONG γ-VARIATION AND R_S-VALUED RANDOM MEASURES

N. Kalinauskaitė

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In the present paper we generalize the theorem of Blumenthal and Getoor [1] on strong γ-variation of stochastic processes with independent increments to the case of random measures with values in R_S, $s \geq 1$, defined on the Borel σ-algebra of subsets of a compact metric space.

Let $\mathcal{X}$ be a compactum with metric ρ, $\mathcal{B}$ be the σ-algebra of its subsets generated by open balls.

The Euclidean norm in the space R_S, $s \geq 1$, will be denoted by $|\cdot|$, and the scalar product by $(\cdot,\cdot)$. Everywhere below d(A) is the diameter of the set $A\in\mathcal{X}$ in the metric ρ. Let $\xi:\Omega\times\mathcal{B}\to R_s$, $s\geqslant 1$, be a homogeneous random measure for which

$$Ee^{i(z,\,\xi(A))}=\exp\{-m(A)\,\psi(z)\},\ z\in R_s,\ A\in\mathcal{B},\tag{1}$$

where $m:\ \mathcal{B}\to[0,\infty)$ is a nonatomic bounded regular measure, $\psi(z)$ is the logarithm of the characteristic function of an infinitely divisible law, viz.:

$$\psi(z)=\int\limits_{|u|>0}\left(e^{i(z,u)}-1-\frac{i(z,u)}{1+|u|^2}\right)\nu(du);\tag{2}$$

ν is a measure on R_S, satisfying the condition

$$\int\limits_{|u|\leqslant 1}|u|^2\,\nu(du)<\infty.$$

The existence of such random measures, their distributions, etc. are considered by Prékopa [3-5].

Definition. By the strong γ-variation of the measure ξ is meant the quantity

$$V_\gamma^{(s)}(\xi)=\sup_\pi\sum_i|\xi(A_i)|^\gamma,\ \gamma>0,\tag{3}$$

where the supremum is taken over all finite partitions π of the compactum $\mathcal{X}$ into disjoint sets from $\mathcal{B}$.

We write

$$\dim\xi=\inf\{\gamma>0:V_\gamma^{(s)}(\xi)<\infty\}.$$

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