where

$$\|D\| < \varepsilon + (1+\varepsilon)^2\varepsilon + \varepsilon(\|\bar{R}(u_3 - u_1, v)\| + \varepsilon)(2+\varepsilon).$$

Since ε can be arbitrarily selected, we have proved (7). This completes the proof of the lemma. Equation (8) can be proved in the same way.

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CRITERIA FOR THE REMOVABILITY OF SETS IN SPACES OF L_p^1 QUASICONFORMAL AND QUASI-ISOMETRIC MAPPINGS

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The problem of removable singularities considered in this paper arises in the work of Ahlfors and Beurling [1].

In [1] they considered removable sets in the class AD(G) of functions that are analytic in a plane region, with finite Dirichlet integral

$$\int_G |\nabla u|^2\, dx < \infty.$$

Within the region G, a relatively closed set $E \subset G$ is removable in the class AD(G) if any of the functions $u \in AD(G \setminus E)$ is continuable to a function $\tilde{u} \in AD(G)$.

The class of all such sets (NED-sets) was described in [1] in terms of extremal lengths for families of curves joining two continua F_0, F_1 in the region G; the extremal length of a family coincides with the conformal capacity of the pair F_0, F_1 with respect to G [2].

We will use the concept of (1, p)-capacity $C_{1,p}(F_0, F_1; G)$ for a pair of continua F_0, F_1 lying in a region G of Euclidean space R^n,

$$C_{1,p}(F_0, F_1; G) = \inf \int_G |\nabla u|^p\, dx.$$

The infimum is taken over all continuous functions u equal to zero on F_0, to one on F_1, and having a finite Dirichlet integral

$$\int_G |\nabla u|^p dx < \infty.$$

In plane regions, (1, 2)-capacity coincides with conformal capacity.

For a region $G \subset R^2$ a relatively closed set E is an NED-set if and only if for any pair of continua F_0, $F_1 \subset G$, $C_{1,2}(F_0, F_1; G \setminus E) = C_{1,2}(F_0, F_1; G)$.

For any pair F_0, F_1, the (1, 2)-capacity is attained for a function (an extremal). The extremal function is harmonic in $G \setminus (F_0 \cup F_1 \cup E)$. Therefore, the set $E \in$ NED if and only if any of the extremal functions can be continued from $G \setminus E$ to G without decreasing the Dirichlet integral.

The class NED is removable for plane quasiformal mappings [1, 3].

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In both the removability problems described above, the fundamental difficulty is to prove the existence of generalized first derivatives in the full region. Therefore, it seems natural to us to pose the problem of removability for spaces of functions that have generalized first derivatives.

In a region $G \subset R^n$ we consider the space $L_p^1(G)$ of locally integrable functions possessing generalized first derivatives with integrable p-th power $p > 1$. In $L_p^1(G)$ we introduce the seminorm

$$\|u\|_{L_p^1(G)} = \left(\int_G |\nabla u|^p \, dx\right)^{1/p}.$$

Regions G_1 and G_2 $(G_1 \supset G_2)$ are called $(1, p)$-equivalent if the restriction operator $\Theta: L_p^1(G_1) \to L_p^1(G_2)$ $(\Theta u = u|_{G_2})$ is an isomorphism between the vector spaces $L_p^1(G_1)$ and $L_p^1(G_2)$.

Regions G_1 and G_2 are $(1,p)$-equivalent if any function $u \in L_p^1(G_2)$ can be continued in a unique way to a function $\tilde{u} \in L_p^1(G_1)$.

A criterion for the $(1, p)$-equivalence of regions G_1 and G_2 is the membership of a set $E \in G_1 \setminus G_2$ in the class NC_p in the region G_1 (Theorem 3.1).

Definition of the Class NC_p. In a region G a relatively closed set E is called an NC_p-set if for any pair of continua F_0, $F_1 \in G \setminus E$, $C_{1,p}(F_0, F_1; G \setminus E) = C_{1,p}(F_0, F_1; G)$.

The fundamental properties of NC_p-sets are simple consequences of this criterion. From them we note the localization principle: A set $E \subset G$ is an NC_p-set in the region G if and only if it is an NC_p-set in any ball $B \subset G$.

On the plane, the class NC_2 coincides exactly with NED, i.e., the removable sets for the classes AD of quasiconformal mappings and the space L_2^1 are one and the same. In the case of dimension $n > 2$ and the classes analogous to AD, the removable sets are smaller than those for the space L_n^1 and also smaller than those for the class of quasiconformal mappings [4-6].

The criterion given above for $(1, p)$-equivalence of regions is a consequence of a theorem on approximating an arbitrary function $v \in L_p^1$ $(p > 1)$ to any desired degree of accuracy by linear combinations $c_0 + \sum_{i=1}^{l} c_i v_i$ of extremal functions for $(1, p)$-capacity, whose gradients have pairwise disjoint support (Theorem 1).

The approximation theorem presented in Sec. 1 is the fundamental result of this paper.

The results of Sec. 1 carry over to the space $W_p^1(G)$ in the case of a bounded region G.

We assume that for a homeomorphism $\varphi: G \to G'$ of regions $G, G' \subset R^n$ and any pair of continua $F_0, F_1 \subset G$ the following inequalities hold

$$K^{-1}C_{1,p}(F_0, F_1; G) \leqslant C_{1,p}(\varphi(F_0), \varphi(F_1); G') \leqslant KC_{1,p}(F_0, F_1; G), \qquad (*)$$

where the constant K does not depend on the choice of continua. The homeomorphism φ is then called quasiconformal for $p = n$, and quasi-isometric for $p \neq n$, $p > 1$.† The least of the possible coefficients K in the inequalities (*) is called the distortion coefficient $q(\varphi)$ for the mapping φ.

It is well known that each quasiconformal (quasi-isometric) homeomorphism $\varphi: G \to G'$ generates a bounded operator $\varphi^*: L_n^1(G_1) \to L_n^1(G)$ $(L_p^1(G') \to L_p^1(G)$, $p > 1)$ by the formula $(\varphi^* f)(x) = f(\varphi(x))$.

In Sec. 2 we prove a result that allows us to compute the norm of the operator φ^* exactly.

In Sec. 3 we derive some properties of NC_p-sets. In Sec. 4 we establish theorems on removability, of NC_n-sets for quasiconformal, and NC_p-sets for quasi-isometric mappings.

The removability of compact NC_n-sets for quasiconformal mappings is proved by the modulus method from [8].

The standard theorems on removability for quasiconformal mappings [5, 6, 8] are particular cases of the theorems given in Sec. 4.

The results in Secs. 3 and 4 were announced in [9].

†We use the standard definition of a quasi-isometry [7].

1. AN APPROXIMATION THEOREM

A BASIS OF EXTREMAL FUNCTIONS IN $L_p^1(G)$

1.1. Preparatory Reduction. The Classes $E_p(G)$ and $E_p(\overline{G})$. In a region $G \subset R^n$ we consider two n-dimensional sets F_0 and F_1, relatively closed with respect to the region G. In addition, we assume that there exists a continuous function from the class $L_p^1(G)$, equal to 0 on F_0 and 1 on F_1.

The space $\overset{\circ}{L}{}_p^1(F_0, F_1; G)$ is the set of those functions $u \in L_p^1(G \setminus (F_0 \cup F_1))$ such that the function $\tilde{u}: G \to R$ equal to u on $G \setminus (F_0 \cup F_1)$ and zero outside the $G \setminus (F_0 \cup F_1)$ belongs to the space $L_p^1(G)$.

The Dirichlet Problem. To minimize the integral

$$\int_{G\setminus(F_0\cup F_1)} |\nabla(f+u)|^p\,dx,$$

where $u \in \overset{\circ}{L}{}_p^1(F_0, F_1; G)$.

There exists (cf., e.g., [10]) a unique function $v \in f + \overset{\circ}{L}{}_p^1(F_0, F_1; G)$ such that

$$\int_{G\setminus(F_0\cup F_1)} |\nabla v|^p\,dx = \inf_{u\in\overset{\circ}{L}{}_p^1(F_0,F_1;G)} \int_{G\setminus(F_0\cup F_1)} |\nabla(f+u)|^p\,dx.$$

The function v is continuous and monotone in the region $G \setminus (F_0 \cup F_1)$. Its behavior near the boundary of the sets F_0 and F_1 depends on the structure of the boundary.

We extend the function v to be zero on F_0 and one on F_1. The extended function will be called extremal for the (1, p)-capacity of the pair of continua F_0, F_1 in the region G, or simply extremal where this will not lead to confusion. The extremal function belongs to the class $L_p^1(G)$.

If $\partial F_0 \cap G$ and $\partial F_1 \cap G$ are smooth manifolds, then the extremal function is continuous in the region G [11].

Throughout this paper we will understand the Dirichlet problem to mean only the problem described earlier.

We will be interested in two classes of continuous extremal functions $E_p(G)$ and $E_p(\overline{G})$.

1) $E_p(G)$ is the set of extremal functions for (1, p)-capacity for all possible pairs of n-dimensional connected compacts F_0, $F_1 \subset G$ possessing smooth boundary.

2) Among the extremal functions for the (1, p)-capacity of pairs of closed (relative to the set G) sets F_0, F_1 with smooth boundary, in the class $E_p(\overline{G})$, two conditions stand out:

a) for each function $u \in E_p(\overline{G})$ and any number $0 < a < 1$ the set $u^{-1}(-\infty, a)$ is connected;

b) the set $u^{-1}(a, +\infty)$ is connected.

LEMMA 1.1. The set $E_p(G)$ is dense in $E_p(\overline{G})$ in the sense of convergence in $L_p^1(G)$.

LEMMA 1.1'. There exists a countable set of functions $v_i \in E_p(G)$, i = 1, 2, . . . dense in $E_p(\overline{G})$.

Proof of Lemmas 1.1 and 1.1'.

I. If there exists in $E_p(G)$ a countable everywhere dense set, then Lemma 1.1' follows from Lemma 1.1.

We consider the collection of all polynomials $P: R^n \to R$ with rational coefficients. For each polynomial we choose in R the countable everywhere dense set A of regular values. The collection $\mathfrak{N}$ of (n − 1)-dimensional manifolds forming the connected components of the pre-image $P^{-1}(t)$, $t \in A$, is countable.

In $E_p(G)$ we form the set of all functions v, extremal for all possible pairs (F_0, F_1) whose boundaries belong to $\mathfrak{N}$. The set $\mathfrak{A}$ of such functions is countable.

If $u \in E_p(G)$ is an extremal function for the pair (F_0, F_1), then ∂F_0, ∂F_1 are smooth manifolds. They are level surfaces of smooth finite-valued functions. By Weierstrass' theorem there exists a pair of manifolds M_0, $M_1 \in \mathfrak{N}$ that lie sufficiently close to ∂F_0, ∂F_1. A theorem on the continuity of capacity [12] allows us to conclude that $\mathfrak{A}$ is everywhere dense in $E_p(G)$.

II. We choose a function v from the class $E_p(\overline{G})$. We fix $\varepsilon \in (0, 1/2)$. The function $v_\varepsilon = \{\min[1-\varepsilon, \max(\varepsilon, v)] - \varepsilon\}/(1-2\varepsilon)$ is extremal for the pair of connected compacta $F_{0,\varepsilon} = v^{-1}[0, \varepsilon]$, $F_{1,\varepsilon} = v^{-1}[1-\varepsilon, 1]$

Therefore, we can construct two sequences $\{F^m_{0,\varepsilon}\}$, $\{F^m_{1,\varepsilon}\}$ of n-dimensional connected compacta with smooth boundaries, "exhausting" $F_{0,\varepsilon}$ and $F_{1,\varepsilon}$ in the following sense: α) $F^m_{i,\varepsilon} \subset F_i$ for all $m \geq 1$, $i = 0, 1$; β) $F^{m_1}_{i,\varepsilon} \subset F^{m_2}_{i,\varepsilon}$ for $m_1 \leq m_2$; γ) $\bigcup_m F^m_{i,\varepsilon} \supset \operatorname{Int} F_{i,\varepsilon}$, $i = 0, 1$.

The functions v_ε are continuous, take the value zero on $F^m_{0,\varepsilon}$ and the value one on $F^m_{1,\varepsilon}$ for all m.

Therefore, the Dirichlet problem for the pair $(F^m_{0,\varepsilon}, F^m_{1,\varepsilon})$ is solvable and the extremal function v_m belongs to the class $E_p(G)$.

The following inequality is obvious

$$\int_G |\nabla v_m|^p\, dx \leqslant \int_G |\nabla v_\varepsilon|^p\, dx, \quad m = 1, 2, \ldots \tag{1}$$

The sequence $\{v_m\}$ is bounded in $L^1_p(G)$. We can assume that it converges weakly to some function $u \in L^1_p(G)$. The limit function is equal to one on $\operatorname{Int} F_{1,\varepsilon}$ and equal to zero on $\operatorname{Int} F_{0,\varepsilon}$. By a lemma on semi-continuity (cf., below) we have

$$\int_G |\nabla u|^p\, dx \leqslant \underline{\lim}_{m\to\infty} \int_G |\nabla v_m|^p\, dx. \tag{2}$$

It follows from (1), (2) that

$$\int_G |\nabla u|^p\, dx \leqslant \int_G |\nabla v_\varepsilon|^p\, dx.$$

Since the solution of the Dirichlet problem is unique, $v_\varepsilon = u$. The sequence $\{v_m\}$ converges to the function v_ε weakly and $\|v_m\|_{L^1_p(G)} \to \|v_\varepsilon\|_{L^1_p(G)}$, therefore v_m converges to v_ε in $L^1_p(G)$ (cf., Lemma 1.2). It is obvious that $\|v_\varepsilon - v\|_{L^1_p(G)} \to 0$ for $\varepsilon \to 0$.

Thus, for each function $v \in E_p(\bar{G})$ we can construct a sequence of solutions from $E_p(G)$, converging in $L^1_p(G)$ to the function v.

In the proof we made use of the following lemma:

Lemma on Semicontinuity. Suppose that a sequence of functions $\{v_m \in L^1_p(G)\}$ converges weakly to a function $v \in L^1_p(G)$. Then

$$\int_G |\nabla v|^p\, dx \leqslant \underline{\lim}_{m\to\infty} \int_G |\nabla v_m|^p\, dx.$$

Proof. If we take the factor space of $L^1_p(G)$ with respect to subspace of identically constant functions, the results is a Banach space.

The inequality then becomes a well-known property of the norm for a Banach space.

COROLLARY. Suppose that the sequence $\{v_m \in L^1_p(G)\}$ is bounded in $L^1_p(G)$ and converges almost everywhere to the function $v \in L^1_p(G)$. Then there exists a subsequence $\{v_{m_k}\}$ of the sequence $\{v_m\}$ that converges weakly in $L^1_p(G)$ to the function v. In addition the following inequality holds

$$\int_G |\nabla v|^p\, dx \leqslant \underline{\lim}_{m\to\infty} \int_G |\nabla v_{m_k}|^p\, dx.$$

Proof. Since the sequence $\{v_m\}$ is bounded, we can select from it a weakly convergent subsequent $\{v_{m_k}\}$. By an embedding theorem, the sequence $\{v_{m_k}\}$ converges to its weak limit u in L_p in any ball B ($\bar{B} \subset G$). Therefore, we may assume that $v_{m_k} \to u$ almost everywhere, i.e., $u = v$. The inequality follows from the lemma on semicontinuity.

LEMMA 1.2. If the sequence $\{v_m\}$, $v_m \in L^1_p(G)$ converges weakly to the function v and $\|v_m\|_{L^1_p} \to \|v\|_{L^1_p}$, then v_m converges to v in $L^1_p(G)$.

For the proof it is sufficient to note that the factor space $L^1_p(G)$, $p > 1$, with respect to the identically constant functions is uniformly convex.

Let F_0, $F_1 \subset G$ be two closed sets for which the Dirichlet problem is solvable in $L^1_p(G)$.

We assume that F_0 may be represented in the form of a countable union of disjoint n-dimensional sets $\{F_{0,i} \subset G\}$, $i = 1, 2, \ldots$, closed with respect to G.

It is clear that for any pair $\left(F_{0,k} = \bigcup_{i=1}^{k} F_{0,i},\ F_1\right)$ the Dirichlet problem is solvable in $L_p^1(G)$.

LEMMA 1.3. The extremal functions v_k for the pair $\{F_{0,k}, F_1\}$ converge in $L_p^1(G)$ to an extremal function v for the pair $\{F_0, F_1\}$.

The proof is analogous to the proof of Lemma 1.1.

1.2. THEOREM I. For each function $u \in L_p^1(G)$ and for every $\varepsilon > 0$ there exists a linear combination $\sum_{k=1}^{l} c_k v_k$ of functions $v_k \in E_p(\bar{G})$ satisfying the conditions

$$\begin{aligned} &\text{a)}\quad \left\| u - \sum_{k=1}^{l} c_k v_k \right\|_{L_p^1(G)} < \varepsilon, \\ &\text{b)}\quad \left\| \sum_{k=1}^{l} c_k v_k \right\|_{L_p^1(G)} = \sum_{k=1}^{l} |c_k| \, \| v_k \|_{L_p^1(G)}. \end{aligned} \tag{3}$$

We will carry out the proof in several stages.

A. It is sufficient to prove the theorem for the class $C^\infty \cap L_p^1(G)$, since this class is everywhere dense in $L_p^1(G)$ [13].

B. The Approximation of Smooth Bounded Functions from $L_p^1(G)$ by Piecewise Extremals. The concept of a piecewise extremal function to be introduced below will only be used in the proof of Theorem 1.

In the region G we consider open sets $V_0 \subset V_1 \subset \ldots \subset V_l$. We assume that for each pair $(\bar{V}_{k-1}, G \setminus V_k)$ the Dirichlet problem is solvable in $L_p^1(G)$, $k = 1, 2, \ldots, l$.

The function v is called a piecewise extremal function, associated with the set $V_0, V_1, \ldots, V_l$ and the real numbers $a_0, a_1, \ldots, a_l$, if

$$v = a_0 + \sum_{k=1}^{l} (a_k - a_{k-1}) v_k,$$

where v_k is an extremal function for the pair $(\bar{V}_{k-1}, G \setminus V_k)$.

To determine a piecewise extremal function it is sufficient to give the choice of sets $\{V_k\}$ and the choice of numbers $\{a_k\}$. A piecewise extremal function satisfies

$$v \in L_p^1(G) \text{ and } \| v \|_{L_p^1(G)} = \sum_{k=1}^{l} | a_k - a_{k-1} | \, \| v_k \|_{L_p^1(G)}.$$

We fix a smooth bounded function $f \in L_p^1(G)$ and a partition τ of the interval $[\min_{x \in G} f(x), \max_{x \in G} f(x)]$ by the real numbers $\min f(x) < a_0 < a_1 < \ldots < a_l < \max f(x)$ where we require $f^{-1}(a_i)$ to be smooth manifolds. By Sard's theorem, this condition is satisfied for almost all values of the function f.

The open sets $V_k = f^{-1}(-\infty, a_k)$ form a monotone sequence. For each pair $(\bar{V}_k, G \setminus V_k)$ the Dirichlet problem is solvable. Therefore, for the choice of sets $\{V_k\}$ and numbers $\{a_k\}$* we have the piecewise extremal function $v_\tau = a_0 + \sum_{k=1}^{l} (a_k - a_{k-1}) v_k$.

In each of the open sets $V_k \setminus \bar{V}_{k-1}$ the following inequality holds

$$\int_{V_k \setminus \bar{V}_{k-1}} | \nabla v_k |^p \, dx \leqslant (a_k - a_{k-1})^{-1} \int_{V_k \setminus \bar{V}_{k-1}} | \nabla f |^p \, dx.$$

Therefore, the following inequality holds throughout the region G

$$\int_G | \nabla v_\tau | \, dx \leqslant \int_G | \nabla f |^p \, dx. \tag{4}$$

A sequence τ_m of partitions of the interval $[\min_{x \in G} f(x), \max_{x \in G} f(x)]$ corresponds to a sequence $v_m = v_{\tau_m}$ of piecewise extremal functions.

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*Here and throughout the proof of the theorem, closure will be understood relative to the region G.

It is clear from the construction that the piecewise extremal functions satisfy

$$|f(x)-v_m|\leqslant \operatorname{diam}\tau_m,\ \operatorname{diam}\tau_m=\max_{i=\overline{1,l}}|a_i-a_{i-1}|.$$

In addition, we assume that diam $\tau_m \to 0$ for $m \to \infty$. In this case the sequence $\{v_m\}$ converges uniformly to the function f. The lemma on semicontinuity, inequality (4), and Lemma 1.2 allow us to assume that the sequence $\{v_m\}$ converges to f in the space $L_p^1(G)$. Therefore, it is sufficient to prove the theorem for piecewise extremal functions.

C. The piecewise extremal function $v=a_0+\sum_{k=1}^{l}(a_k-a_{k-1})v_k$ is given on a sequence of open sets $\{V_k\}$ each of which may be written as a countable union of connected components.

We wish to show for every $\varepsilon > 0$ there exists a piecewise extremal function v_ε satisfying the conditions

a) $\|v-v_\varepsilon\|_{L_p^1(G)}<\varepsilon$;

b) the sequence of open sets $\{V_{k-1}^s\}$ that determines v_ε consists only of sets with a finite number of connected components, therefore the open sets $G\setminus\overline{V}_k^\varepsilon$ consist of a finite number of connected components $k = 1, 2, \ldots, l$.

The proof will proceed by induction on the sequentially constructed sets V_{k-1}, $G\setminus V_k$, $k = 1, 2, \ldots, l$.

We fix $\varepsilon > 0$.

<u>The Induction Assumption.</u> We assume that there exists a piecewise extremal function $u=a_0+\sum_{k=1}^{l}(a_k-a_{k-1})u_k$ satisfying the following conditions

a) for $l \geq q \geq r > 0$ the sets V_{q-1}^u* and $G\setminus\overline{V}_q^u$ consist of a finite number of connected components;

b) $\|v-u\|_{L_p^1(G)}<\varepsilon(l-r+1)/l$.

<u>The Induction Base $r = l$.</u> We assume that the set $G\setminus\overline{V}_l$ consists of a finite number of connected components U_i, $i = 1, 2, \ldots$. For each pair $\left(\overline{V}_{l-1}, \bigcup_{i=1}^{j}\overline{U}_i\right)$ the Dirichlet problem is solvable. By Lemma 1.3 the extremal functions $v_{l,j}$ of these problems converge to a function v_l in the space $L_p^1(G)$.

We set V_l^W equal to $G\setminus\bigcup_{i=1}^{j_0}\overline{U}_i$, where j_0 is so large that $\|v_{l,j_0}-v_l\|_{L_p^1(G)}\leqslant\varepsilon/2l(a_l-a_{l-1})$.

We construct a new piecewise extremal function with respect to the set $V_0, V_1, \ldots, V_{l-1}, V_l^W$ and the numbers $a_0, a_1, \ldots, a_l$.

It is clear that $\|v-w\|_{L_p^1(G)}\leqslant\varepsilon/2l$.

Now suppose that V_{l-1} consists of a countable number of connected components U_i, $i = 1, 2, \ldots$. For each pair $\left(\bigcup_{i=1}^{j}\overline{U}_i, G\setminus V_l^w\right)$ the Dirichlet problem is solvable. By Lemma 1.3 the extremal functions $u_{l,j}$ of corresponding pairs converge to a function w_l in the space $L_p^1(G)$.

We set $V_{l-1,j}=\bigcup_{i=1}^{j}U_i$ and $V_{k,j}=V_k^v\cap V_{l-1,j}$ for all $0 \leq k \leq l-1$. We consider the sequence of piecewise extremal functions

$$u_j=a_0+\sum_{k=1}^{l}(a_k-a_{k-1})u_{k,j},\quad j=1,2,\ldots$$

The functions $u_{k,j}$ are extremal functions for the pair $(\overline{V}_{k-1,j}, G\setminus V_{k,j})$ for all $k < l$ and for all $j \geq 1$. We will show that $\|w-u_j\|_{L_p^1(G)}\to 0$ as $j\to\infty$. In fact,

$$\|w-u_j\|_{L_p^1(G)}=\sum_{k=1}^{l-1}(a_k-a_{k-1})\int_G|\nabla u_k-\nabla u_{k,j}|^p\,dx+\|w_l-u_{l,j}\|_{L_p^1(G)}^p.$$

*The index u on the set V_k^u means that V_k^u is a set from the sequence corresponding to the function u.

By Lemma 1.3, the last term converges to zero. For $k < l$, it follows from the formula

$$\int_G |\nabla u_k - \nabla u_{k,j}|^p\,dx = \int_{V_k^v \setminus V_{k,j}} |\nabla u_k - \nabla_{k,j}|^p\,dx = \int_{V_k^v \setminus V_{k,j}} |\nabla u_k|^p\,dx = \sum_{i=j+1}^{\infty} \int_{U_i} |\nabla u_k|^p\,dx$$

that $\|u_k - u_{k,j}\|_{L_p^1(G)}$ converges to zero as $j \to \infty$.

We choose j_0 so that

$$\|w - u_{j_0}\|_{L_p^1(G)} \leqslant \varepsilon/2l.$$

We have constructed a piecewise extremal function $u = u_{j_0}$ for which the induction hypothesis holds for $r = l$: the set $V_{l-1}^u, G \setminus \overline{V}_l^u$ consists of a finite number of connected components and the estimate $\|v - u\|_{L_p^1(G)} \leqslant \|v - w\|_{L_p^1(G)} + \|w - u_{j_0}\|_{L_p^1(G)} \leqslant \varepsilon/l$ holds.

<u>The Induction Step.</u> We assume that the induction hypothesis holds for $r = s < l$, $s > 1$. We will construct a function w for which the induction hypothesis holds for $r = s - 1$.

Let V_{s-2}^u consist of a countable number of connected components U_i. For each pair of $\left(\bigcup_{i=1}^{j} \overline{U}_i, G\setminus V_{s-1}^u\right)$ the Dirichlet problem is solvable. By Lemma 1.3 the extremal functions $u_{s-1,j}$ for the corresponding pairs converge to a function u_{s-1} in the space $L_p^1(G)$.

We set $V_{s-2,j} = \bigcup_{i=1}^{j} U_i$ and $V_{k,j} = V_k^u \cap V_{s-2,j}$ for all $0 \le k < s-2$. We consider a sequence of piecewise extremal functions

$$u_j = a_0 + \sum_{k=1}^{s-1} (a_k - a_{k-1})\, u_{k,j} + \sum_{k=s}^{l} (a_k - a_{k-1})\, u_k, \quad j = 1, 2, \ldots$$

The functions $u_{k,j}$ are extremal functions for the pair $(V_{k-1,j}, G \setminus V_{k,j})$ for all $k < s-1$ and for all $j \geq 1$.

We will prove that $\|u - u_j\|_{L_p^1(G)} \to 0$.

In fact,

$$\|u - u_j\|_{L_p^1(G)} = \sum_{k=1}^{s-2} (a_k - a_{k-1}) \int_G |\nabla u_k - \nabla u_{k,j}|^p\,dx + \|u_{s-1} - u_{s-1,j}\|^p_{L_p^1(G)}.$$

By Lemma 1.3 the last term converges to zero. For $k < s - 1$ it follows from the formula

$$\int_G |\nabla u_k - \nabla u_{k,j}|^p\,dx = \int_{V_k^u \setminus V_{k,j}} |\nabla u_k - \nabla u_{k,j}|^p\,dx = \int_{V_k^u \setminus V_{k,j}} |\nabla u_k|^p\,dx = \sum_{i=j+1}^{\infty} \int_{U_i} |\nabla u_k|^p\,dx$$

that $\|u_k - u_{k,j}\|_{L_p^1(G)}$ converges to zero for $j \to \infty$.

We choose j_0 so that

$$\|u - u_{j_0}\|_{L_p^1(G)} < \varepsilon/2l. \tag{5}$$

We will construct a piecewise extremal function $\tilde{w} = u_{j_0}$, for which the induction function will hold for $r = s$, the set $V_{s-2}^{\tilde{w}}$ will consist of a finite number of connected components, and inequality (5) will hold.

The complement of the set $\overline{V}_{s-1}^{\tilde{w}}$ may consist of a countable number of components W_j, $j = 1, 2, \ldots$. In this case the set $V_{s-2}^{\tilde{w}}$ must undergo a further manipulation.

By the induction assumption, each of the sets $G \setminus \overline{V}_k^u = G \setminus \overline{V}_k^{\tilde{w}}$ for $k \geq s$ consists of a finite number of connected components. The inclusion $G \setminus \overline{V}_k^u \subset G \setminus \overline{V}_{s-1}^{\tilde{w}}$, $k \geq s$, implies that for j larger than some j_0, $W_j \cap (G \setminus \overline{V}_k^u) = \emptyset$ for all $k \geq s$. From this it follows $W_j \subset V_s^{\tilde{w}}$ for $j > j_0$.

We consider the sets $W_{s-1,j}$ representing the interior of the closure relative to G of the set $V_{s-1}^{\tilde{w}} \cup \left(\bigcup_{k=j}^{\infty} W_k\right)$ for $j > j_0$.

The number of connected components of the set $W_{s-1,j}$ does not exceed the number of connected components of the sets $V^{\tilde{w}}_{s-1}$ for all $j > j_0$.

It follows from the inclusion $W_j \subset V^u_s = V^{\tilde{w}}_s$ for $j > j_0$ that the extremal function for the pair $(\overline{V}^{\tilde{w}}_{s-1}, G \setminus V^{\tilde{w}}_s)$ vanishes on W_j. Therefore, it is an extremal function for any of the pairs $(\overline{W}_{s-1,j}, G \setminus V^{\tilde{w}}_s)$ for $j > j_0$.

For such a pair $(\overline{V}^{\tilde{w}}_{s-2}, G \setminus W_{s-1,j})$ the Dirichlet problem is solvable. The extremal functions $w_{s-1,j}$ for these problems converge to an extremal function $\tilde{w}_{s-1}$ for the pair $(V^{\tilde{w}}_{s-2}, G \setminus V^{\tilde{w}}_{s-1})$ in the space $L^1_p(G)$. This allows us to choose a $j_1 > j_0$ such that the piecewise extremal function w for the decomposition $V^{\tilde{w}}_0 \subset \ldots \subset V^{\tilde{w}}_{s-2} \subset W_{s-1,j_1} \subset V^{\tilde{w}}_s \subset \ldots \subset V^{\tilde{w}}_l$ and the numbers $a_0, a_1, \ldots, a_l$ satisfies the inequality

$$\|\tilde{w} - w\|_{L^1_p(G)} \leqslant \varepsilon/2l.$$

From this and inequality (5),

$$\|u - w\|_{L^1_p(G)} \leqslant \varepsilon/l.$$

This proves the induction hypothesis.

D. In the preceeding three sections we proved the possibility of approximating an arbitrary function $u \in L^1_p(G)$ by piecewise extremal functions $w = a_0 + \sum_{k=1}^{l} (a_k - a_{k-1}) w_k$. Each of the functions w_k is a solution of the Dirichlet problem in $L^1_p(G)$ for the pair of closed, relative to G, sets (F^k_0, F^k_1), whose interiors consist of a finite number of connected components and $F^k_0 = G \cap \overline{\operatorname{Int} F^k_0}$, $F^k_1 = G \cap \overline{\operatorname{Int} F^k_1}$. The functions w_k are continuous and $\|w\|_{L^1_p(G)} = \sum_{k=1}^{l} |a_k - a_{k-1}| \|w\|_{L^1_p(G)}$.

To complete the proof of the theorem it is sufficient to demonstrate the possibility of representing any of the functions w_k in the form $w_k = c^k_0 + \sum_{i=1}^{l_k} c^k_i w^k_i$, where the functions $w^k_i \in E_p(\bar{G})$ and $\|w_k\|_{L^1_p(G)} = \sum_{i=1}^{l_k} |c^k_i| \|w^k_i\|_{L^1_p(G)}$.

We consider a continuous extremal function v for the pair of closed (relative to G) sets F_0, F_1.

For a real number a, we choose a corresponding pair of sets $V_a = v^{-1}(-\infty, a)$, $W_a = v^{-1}(a, \infty)$. We will study the behavior of the two functions: $\tau_0 = \tau_{0,v}(a)$, equal to the number of connected components of the set V_a, and $\tau_1 = \tau_{1,v}(a)$, equal to the number of connected components of the set W_a.

1. If $\operatorname{Int} F_0$ and $\operatorname{Int} F_1$ consist of a finite number of connected components, and if $F_0 = G \cap \overline{\operatorname{Int} F_0}$, $F_1 = G \cap \overline{\operatorname{Int} F_1}$, then $0 \le \tau_0 \le k_0$, $0 \le \tau_1 \le k_1$, where k_0 is the number of connected components of the set F_0, k_1 is the number of connected components of the set F_1.

We assume the opposite: that there exists a number a, for which the set V_a consists of $l > k$ connected components. Clearly, $0 < a < 1$, $V_a \supset F_0$. Thus, there exists a connected component $\tilde{V}$ of the set V_a disjoint from F_0. For the function $u \equiv v$ outside $\tilde{V}$ and $u \equiv a$ on $\tilde{V}$,

$$\int_G |\nabla u|^p \, dx \leqslant \int_G |\nabla v|^p \, dx.$$

It follows from the uniqueness of extremals that $u \equiv v$ — a contradiction. The proof for τ_1 is analogous.

2. The function τ_0 is nonincreasing, while τ_1 is nondecreasing.

3. If $0 \le \tau_0 \le 1$ and $0 \le \tau_1 \le 1$, then $v \in E_p(\bar{G})$. Property 3 is a reformulation of the definition of the class $E_p(G)$.

4. $\tau_{1,v} = \tau_{0,1-v}$ for any extremal function v for a pair of closed (relative to G) sets F_0, F_1.

We assume that the function τ_0 is not constant on the interval (0,1). Let $0 = a_0 < a_1 < \ldots < a_s < 1$ be the points of discontinuity of τ_0, and $a_{s+1} = 1$.* The corresponding functions $v_k = (a_k - a_{k-1})^{-1} \min[\max(v, a_{k-1}), a_k]$ for the discrete functions $\tau_{0,k} = \tau_{0,v_k}$ are constant on (0,1), $k = 1, 2, \ldots, s+1$. Clearly, $v = \sum_{k=1}^{s+1} (a_k - a_{k-1}) v_k$, where v_k are the extremal functions for the pair of sets $F_{0,k} = v_k^{-1}(0)$, $F_{1,k} = v_k^{-1}(1)$ for all k and $\|v\|_{L^1_p(G)} = \sum_{k=1}^{s+1} |a_k - a_{k-1}| \|v_k\|_{L^1_p(G)}$.

*The point 1 might not be a point of discontinuity for τ_0.

For the functions $w_k = 1 - v_k\tau_{1,w_k}$ constant, $\tau_{0,w_k} = \tau_{1,1-w_k} = \tau_{1,v_k}$. Repeating the arguments given above, we obtain

$$w_k = \sum_{j=1}^{s_k} (a_{k,j} - a_{k,j-1})\, w_{k,j},$$

where $w_{k,j} = (a_{k,j} - a_{k,j-1})^{-1} \min[a_{k,j}, \max(a_{k,j-1}, w_k)]$, $0 = a_{0,j} < a_{1,j} < \ldots < a_{s_k-1,j} < 1$ are the points of discontinuity of τ_{0,w_k} and $a_{s_k,j} = 1$.

The functions $\tau_{1,w_{k,j}}$ and $\tau_{0,w_{k,j}}$ are constant for all k, j on (0,1) and

$$\| w_k \|_{L_p^1(G)} = \sum_{j=1}^{s_k} | a_{k,j} - a_{k,j-1} | \| w_{k,j} \|.$$

Consequently,

$$v = \sum_{k=1}^{s+1} (a_k - a_{k-1}) \left(1 - \sum_{j=1}^{s_k} (a_{k,j} - a_{k,j-1})\, w_{k,j} \right) = 1 - \sum_{k=1}^{s+1} \sum_{j=1}^{s_k} (a_k - a_{k-1})(a_{k,j} - a_{k,j-1})\, w_{k,j} = 1 - \sum_{k=1}^{s+1} \sum_{j=1}^{s_k} c_{k,j} w_{kj},$$

where $c_{k,j} = (a_k - a_{k-1})(a_{k,j} - a_{k,j-1})$. It is clear that

$$\| v \|_{L_p^1(G)} = \sum_{k=1}^{s+1} \sum_{j=1}^{s_k} | c_{k,j} | \, \| w_{k,j} \|_{L_p^1(G)}. \tag{6}$$

Proposition 1.4. Each continuous extremal function w with corresponding functions $\tau_{0,w}$, $\tau_{1,w}$ constant on (0,1) can be represented in the form $w = \sum_{k=1}^{l} w_k$, where w_k are continuous extremal functions corresponding to the following conditions: a) for all $a \in (0, 1)$, $\tau_{1,w_k}(a) = 1$ for all $k = 1, 2, \ldots, l$; b) $\tau_{0,w_k}(a) \le \tau_{0,w}(a)$ for all a; c) τ_{0,w_k} is constant on (0,1); d) $\| w \|_{L_p^1(G)} = \sum_{k=1}^{l} \| w_k \|_{L_p^1(G)}$.

Proof. The set $G \setminus v^{-1}(0)$ consists of a finite number of connected components $G_1, G_2, \ldots, G_l$, since each of the connected components of this set must intersect the set $W_a = v^{-1}(a, \infty)$ for all $0 < a < 1$, while W_a by assumption has a finite number of components.

For the pair $(F_{0,k} = G \setminus G_k, F_{1,k} = G_k \cap F_1)$ the function w_k, equal to zero on $F_{0,k}$ and equal to w at the remaining points, is extremal. Clearly,

$$\| w \|_{L_p^1(G)} = \sum_{k=1}^{l} \| w_k \|_{L_p^1(G)}.$$

The functions $\tau_{0,k} = \tau_{0,w_k}$ are constant. In fact, pick a fixed but arbitrary function w_k. We assume that $\tau_{0,k}$ is nonconstant. We consider the point of discontinuity a, closest to the origin. The set $V_{k,a_1} = w_k^{-1}(-\infty, a_1)$ $(a_1 < a)$ has a larger number of connected components than $V_{k,a_2} = w_k^{-1}(-\infty, a_2)$ $(a_2 > a)$. Therefore, as we cross a, two connected components of V_{k,a_1} must unite. Then two connected components of the set $V_{a_1} = w^{-1}(-\infty, a_1)$ must unite as we cross a, i.e., τ_0 is discontinuous. From the construction of the function w_k, $w_k^{-1}(0, 1] \cap w_j^{-1}(0, 1] = \emptyset$, if $k \ne j$. Therefore,

$$\tau_{1,w} = \sum_{k=1}^{l} \tau_{1,w_k}.$$

The functions τ_{1,w_k} are nondecreasing, while $\tau_{1,w}$ are constant, therefore, it follows from this that τ_{1,w_k} are constant, $k = 1, 2, \ldots, l$.

We assume that for some k_0, $\tau_{1,w_{k_0}} > 1$. Then for any $a \in (0,1)$ the set $W_a = w_{k_0}^{-1}(a, \infty)$ is not connected. Clearly $\bigcup_{0<a<1} W_a = w_{k_0}^{-1}(0, 1] = G_{k_0}$. The set G_{k_0} is connected, therefore $\bigcup_{0<a<1} W_a$ is also connected, which is not possible if any of the sets W_a is not connected.

In conclusion, the functions $\tau_{1,w_k}(a) = 1$ for any k and all $a \in [0, 1)$. This proves the proposition.

For each of the functions $w_{k,j}$ in representation (6), we can apply Proposition 1.4, to obtain a representation for the function v in the form of a sum $1 + \sum_{k=1}^{r} \tilde{c}_k \tilde{v}_k$, where $\tilde{v}_k$ are continuous extremal functions with the following properties:

a) $\tau_{1,\tilde{v}_k}(a) = 1$ for all $a \in [0, 1)$;

b) $\tau_{0,\tilde{v}_k}(a)$ is constant on $(0,1)$;

c) $\|v\|_{L^1_p(G)} = \sum_{k=1}^{r} |\tilde{c}_k| \|\tilde{v}_k\|_{L^1_p(G)}$.

Applying Proposition 1.4 to each of the functions $1 - \tilde{v}_k$, we obtain a representation for the function v in the form $c_0 + \sum_{k=1}^{t} c_k v_k$, where $v_k \in E_p(\tilde{G})$ and $\|v\|_{L^1_p(G)} = \sum_{k=1}^{t} |c_k| \|v_k\|_{L^1_p(G)}$. This proves the theorem.

Remark. If the function $u \in L^1_p(G)$ is bounded, then the function constructed in Theorem I $w = c_0 + \sum_{k=1}^{l} c_k v_k$ is also bounded and $|w(x)| < 2|u(x)|$ for all $x \in G$.

COROLLARY. The linear hull of $E_p(G)$ is everywhere dense in $L^1_p(G)$.

This follows from Theorem I and Lemma 1.1'.

1.2. THEOREM II. Each function $u \in L^1_p(G)$ is representable in the form $u = c_0 + \sum_{i=1}^{\infty} c_i v_i$, where c_i are real numbers, and the functions v_i belong to the class $E_p(G)$ for all $i \geq 1$. In addition, $\left\| u - \sum_{i=1}^{l} c_i v_i \right\|_{L^1_p(G)} \to 0$ as $l \to \infty$.

Proof. We fix $\varepsilon > 0$. According to Theorem I there exist functions $\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{l_0} \in E_p(\bar{G})$ for which $\left\| u - \sum_{i=1}^{l_0} c_i \tilde{v}_i \right\|_{L^1_p(G)} < \varepsilon/2$. Lemma 1.1 allows us to find for each of the functions $\tilde{v}_i \in E_p(\tilde{G})$ a function $v_i \in E_p(G)$ such that $\|\tilde{v}_i - v_i\|_{L^1_p(G)} \leqslant \varepsilon/2|c_i| l_0$. We set $u_1 = u - \sum_{i=1}^{l_0} c_i v_i$. Obviously, $\|u_1\|_{L^1_p(G)} < \varepsilon$.

Suppose that u_k has been constructed and $\|u_k\| < \varepsilon/2^{k-1}$. Applying Theorem I to u_k we construct functions $\tilde{v}_{l_{k-1}+1}, \ldots, \tilde{v}_{l_k} \in E_p(\bar{G})$ for which $\left\| u_k - \sum_{i=l_{k-1}+1}^{l_k} c_i \tilde{v}_i \right\|_{L^1_p(G)} \leqslant \varepsilon/2^{k+1}$. Lemma 1.1 allows us to find for each of the functions $\tilde{v}_i \in E_p(\tilde{G})$ a function $v_i \in E_p(G)$ such that $\|\tilde{v}_i - v_i\|_{L^1_p(G)} \leqslant \varepsilon/2^{k+1} c_i (l_k - l_{k-1})$, $i = l_{k-1}+1, \ldots, l_k$. We define $u_{k+1} = u_k - \sum_{i=l_{k-1}+1}^{l_k} c_i v_i$. Clearly, $\|u_{k+1}\|_{L^1_p(G)} \leqslant \varepsilon/2^k$.

The series $\sum_{i=1}^{\infty} c_i v_i$ converges absolutely in $L^1_p(G)$.

In fact, by construction $\left\| \sum_{i=l_{k-1}+1}^{l_k} c_i \tilde{v}_i \right\|_{L^1_p(G)} < \varepsilon/2^{k-1} + \varepsilon/2^{k+1}$ and by Theorem I $\left\| \sum_{i=l_{k-1}+1}^{l_k} c_i \tilde{v}_i \right\|_{L^1_p(G)} = \sum_{i=l_{k-1}+1}^{l_k} |c_i| \|\tilde{v}_i\|_{L^1_p(G)}$. Then

$$\sum_{i=l_0+1}^{N} \|c_i v_i\|_{L^1_p(G)} \leqslant (l_k = \min_{l_i > N} l_i) \leqslant \sum_{i=l_0+1}^{l_k} \|c_i v_i\|_{L^1_p(G)}$$

$$= \sum_{j=1}^{k} \sum_{i=l_{j-1}+1}^{l_j} \|c_i v_i\|_{L^1_p(G)} \leqslant \sum_{j=1}^{k} \sum_{i=l_{j-1}+1}^{l_j} \left(\|c_i \tilde{v}_i\|_{L^1_p(G)} + \frac{\varepsilon}{2^{j+1}(l_j - l_{j-1})} \right)$$

$$= \sum_{j=1}^{k} \left(\left\| \sum_{i=l_{j-1}+1}^{l_j} c_i \tilde{v}_i \right\|_{L^1_p(G)} + \frac{\varepsilon}{2^{j+1}} \right) \leqslant \sum_{j=1}^{k} \left(\frac{\varepsilon}{2^{j-1}} + \frac{\varepsilon}{2^{j+1}} + \frac{\varepsilon}{2^{j+1}} \right) \leqslant 3\varepsilon/2.$$

Since N is arbitrary, the series converges absolutely and we can combine similar terms. By [14] the sum of the series differs from u by some constant c_0. This concludes the proof.

THEOREM III. There exists a countable collection of functions $v_i \in E_p(G)$, $i = 1, 2, \ldots$,* such that for any function $u \in L^1_p(G)$ and any $\varepsilon > 0$ there exists a representation of u in the form $u = c_0 + \sum_{i=1}^{\infty} c_i v_i$ for which

$$\|u\|_{L^1_p(G)} \leqslant \sum_{i=1}^{\infty} \|c_i v_i\|_{L^1_p(G)} \leqslant \|u\|_{L^1_p(G)} + \varepsilon.$$

*We can choose the countable set $\{v_i\}$ so that it is linearly independent. Then $\{v_i\}$ will be a Schauder basis.

Proof. If in the proof of Theorem II we were to use Lemma 1.1' in place of Lemma 1.1, we would prove the existence of a basis satisfying the inequality $\sum_{i=1}^{\infty}\|c_iv_i\|_{L_p^1(G)}\leqslant\|u\|_{L_p^1(G)}+\varepsilon$. On the other hand, $\|u\|_{L_p^1(G)}\leqslant\sum_{i=1}^{\infty}\cdot\|c_iv_i\|_{L_p^1(G)}$. This concludes the proof.

1.3. For a bounded region G, the corollary to Theorem I carries over to the space $W_p^1(G)$ of functions whose p-th power is integrable over G, with generalized derivatives in G whose p-th powers are integrable. In the space W_p^1 we consider the norm

$$\|u\|_{W_p^1(G)}=\|u\|_{L_p(G)}+\|u\|_{L_p^1(G)}.$$

THEOREM II'. The linear hull of the set $E_p(G)$ is everywhere dense in the space $W_p^1(G)$.

Proof. We consider a bounded function $u\in W_p^1(G)$. By Theorem II and the remark below Theorem I there exists a uniformly bounded sequence of functions $\{u_k\}$, each of which is a linear combination of elements from the set $E_p(G)$, converging to u in $L_p^1(G)$.

Then by [14] we can choose a bounded sequence of real numbers $\{c_k\}$ such that the sequence $\{u_k+c_k\}$ converges to u almost everywhere. By Lebesgue's theorem $\{u_k+c_k\}\to u$ in $L_p(G)$. This proves the theorem.

2. THE INVARIANCE OF THE CLASSES $L_p^1(L_n^1)$ FOR QUASI-ISOMETRIC (QUASICONFORMAL) MAPPINGS

The standard proof of invariance is well known; it uses the metric definition of the mappings and their differential properties.

We present a new proof of this result (using Theorems I and II), allowing an exact computation of the norm of the operator induced by the mapping.

THEOREM 2.1. Let $\varphi:G\to G'$ be a K-quasi-isometric (K-quasiconformal) mapping of a region $G\subset R^n$ onto $G'\subset R^n$. Then the operator φ^*, defined by the formula $(\varphi^*u)(x)=u(\varphi(x))$ almost everywhere for any function $u\in L_p^1(G')(L_n^1(G'))$, is a bounded operator from $L_p^1(G')(L_n^1(G'))$ onto $L_p^1(G)(L_n^1(G))$ and $\|\varphi^*\|=K^{1/p}$.

Proof. For any pair of n-dimensional continua with smooth boundaries F_0', $F_1'\subset G'$, $F_0'\cap F_1'=\emptyset$ and their images $F_0=\varphi^{-1}(F_0')$, $F_1=\varphi^{-1}(F_1')$, the following inequality holds

$$\int_G|\nabla v|^p\,dx\leqslant K\int_{G'}|\nabla u|^p\,dx \tag{7}$$

for extremal functions v, u corresponding to the pairs (F_0,F_1), (F_0',F_1'), respectively.

1. We extend inequality (7) to a broader class of functions.

Let F_0', $F_1'\subset G'$ be closed (relative to G') connected sets, satisfying the condition $\overline{\operatorname{Int}F_i'}=F_i'$, $i=0,1$. We construct two sequences $\{F_{0,m}'\subset F_0'\}$, $\{F_{1,m}'\subset F_1'\}$ of n-dimensional connected compacta "exhausting" F_0', and F_1', i.e., $\bigcup_{m=1}^{\infty}\operatorname{Int}F_{i,m}'\supset\operatorname{Int}F_i'$. $\operatorname{Int}F_{i,k}\supset F_{i,l}$ for $k>l$, $i=1,2$.

If $\tilde{u}$ is an extremal function for the pair (F_0',F_1'), and $\tilde{u}_m$ is an extremal function for the pair $(F_{0,m}',F_{1,m}')$, then the following inequality holds

$$\int_{G'}|\nabla\tilde{u}_m|^p\,dy\leqslant\int_{G'}|\nabla\tilde{u}|^p\,dy. \tag{8}$$

The pre-images $F_{i,m}=\varphi^{-1}(F_{i,m}')$ of the sets $F_{i,m}'$ "exhaust" the sets $F_i=\varphi^{-1}(F_i')$, $i=0,1$.

For the extremal functions $\tilde{v}_m$ related to the pair $(F_{0,m},F_{1,m})$, it follows from (7) and (8) that $\|\tilde{v}_m\|_{L_p^1(G)}\leqslant K^{1/p}\|\tilde{u}_m\|_{L_p^1(G')}\leqslant K^{1/p}\|\tilde{u}\|_{L_p^1(G')}$. The boundedness in $L_p^1(G)$ of the sequence $\{v_m\}$ allows us to assume that it converges weakly to some function $\tilde{v}\in L_p(G)$ equal to zero on $\operatorname{Int}F_0$, to one on $\operatorname{Int}F_1$, $0\leq\tilde{v}(x)\leq1$ for all $x\in G$. From the above, and the lemma on semicontinuity we obtain

$$\|\tilde{v}\|_{L_p^1(G)}\leqslant\varliminf_{m\to\infty}\|\tilde{v}_m\|_{L_p^1(G)}<K^{1/p}\|\tilde{u}\|_{L_p^1(G')}.$$

2. We return to the beginning of the proof. We fix $\varepsilon>0$ and a natural number $l\geq2$. The function $u_\varepsilon=\{\min[\max(v,\varepsilon),1-\varepsilon]-\varepsilon\}/(1-2\varepsilon)$ is extremal for the pair $F_{0,\varepsilon}=u_\varepsilon^{-1}(0)$, $F_{1,\varepsilon}=u_\varepsilon^{-1}(1)$. We associate with the numbers ε, l two systems of sets

$$F'_{0,1} = F_{0,\varepsilon},\ F'_{0,2} = u_\varepsilon^{-1}([0, 1/l]),\ \ldots,\ F'_{0,l} = u_\varepsilon^{-1}([0, (l-1)/l]);$$
$$F'_{1,1} = u_\varepsilon^{-1}([1/l, 1]),\ \ldots,\ F'_{1,l-1} = u_\varepsilon^{-1}([(l-1)/l, 1]),\ F'_{1,l} = u_\varepsilon^{-1}$$

and

$$F_{0,i} = \varphi^{-1}(F'_{0,i}),\ F_{1,i} = \varphi^{-1}(F'_{1,i}),\quad i = 1, 2, \ldots, l.$$

Using part one of the proof, each of the extremal functions $u_i = \min(i, \max(i-1, lu_\varepsilon) - (i-1)$ for the pairs $(F'_{0,i}, F'_{1,i})$ corresponds to a function $v_i \in L^1_p(G)$ equal to zero on $\operatorname{Int} F_{0,i}$, to one on $\operatorname{Int} F_{1,i}$ and satisfying the conditions

$$0 \leqslant v_i(x) \leqslant 1,\quad \|v_i\|_{L^1_p(G)} \leqslant K^{1/p}\|u_i\|_{L^1_p(G')}. \tag{9}$$

This can be done, since by the monotonicity of the function u_ε on the region $G' \setminus (F_{0,\varepsilon} \cup F_{1,\varepsilon})$, the pairs $(F'_{0,i}, F'_{1,i})$ satisfy the conditions of part one.

It is easy to see that $u_\varepsilon = l^{-1}\sum_{i=1}^{l} u_i\ \|u_\varepsilon\|_{L^1_p(G)} = l^{-1}\sum_{i=1}^{l}\|u_i\|_{L^1_p(G)}$. As $l \to \infty$, the function $w_l = l^{-1}\sum_{i=1}^{l} v_i$ converges uniformly to $\varphi^* u_\varepsilon = u_\varepsilon \circ \varphi$ [this follows from (9)]. Using (9), we obtain

$$\|w_l\|_{L^1_p(G)} \leqslant l^{-1}\sum_{i=1}^{l}\|v_i\|_{L^1_p(G)} \leqslant K^{1/p} l^{-1}\sum_{i=1}^{l}\|u_i\|_{L^1_p(G')} = K^{1/p}\|u_\varepsilon\|_{L^1_p(G')}. \tag{10}$$

From (10) and the corollary to the lemma on semicontinuity we obtain

$$\|\varphi^* u_\varepsilon\|_{L^1_p(G)} \leqslant K^{1/p}\|u_\varepsilon\| \leqslant K^{1/p}\|u\|/(1-2\varepsilon). \tag{11}$$

As $\varepsilon \to 0$, $u_\varepsilon \to u$ uniformly and $\varphi^* u_\varepsilon \to \varphi^* u$ uniformly. From inequality (11) it follows that the following sequences are bounded $\{\|u_{\varepsilon_n}\|_{L^1_p(G')}\}$, $\{\|\varphi^* u_{\varepsilon_n}\|_{L^1_p(G)}\}$, $\varepsilon_n \to 0$. Taking the sequence $\{\varphi^* u_{\varepsilon_n}\}$ as converging weakly in $L^1_p(G)$, it follows from (11) and the lemma on semicontinuity that we finally obtain

$$\|\varphi^* u\|_{L^1_p(G)} \leqslant K^{1/p}\|u\|_{L^1_p(G)} \tag{12}$$

for any function $u \in E_p(G')$.

3. We pick an arbitrary function $u \in L^1_p(G')$ and $\varepsilon > 0$. By Theorem III the function u is representable in the form $u = c_0 + \sum_{i=1}^{\infty} c_i v_i$, $v_i \in E_p(G')$. Thus, the following inequality holds

$$\|u\|_{L^1_p(G')} \leqslant \sum_{i=1}^{\infty}\|c_i v_i\|_{L^1_p(G')} \leqslant \|u\|_{L^1_p(G')} + \varepsilon. \tag{13}$$

The series $c_0 + \sum_{i=1}^{\infty} c_i v_i(y)$ converges to u almost everywhere. As is well known, quasi-isometric and quasi-conformal mappings take sets of measure zero into sets of measure zero. Therefore, the series $c_0 + \sum_{i=1}^{\infty} c_i v_i \cdot (\varphi(x))$ converges almost everywhere to the function $u(\varphi(x))$, $x \in G$.

From (12) and (13) we obtain

$$\sum_{i=1}^{\infty}\|c_i v_i(\varphi(x))\|_{L^1_p(G)} = \sum_{i=1}^{\infty}\|c_i \varphi^* v_i\|_{L^1_p(G)} \leqslant K^{1/p}\sum_{i=1}^{\infty}\|c_i v_i\|_{L^1_p(G')} \leqslant K^{1/p}\|u\|_{L^1_p(G')} + \varepsilon \cdot K^{1/p}. \tag{14}$$

Consequently, the series $c_0 + \sum_{i=1}^{\infty} c_i \varphi^* v_i$ converges in $L^1_p(G)$ and its sum coincides with the function $u(\varphi(x)) = (\varphi^* u)(x)$, $x \in G$.

From (14) we obtain the estimate

$$\|\varphi^* u\|_{L^1_p(G)} \leqslant K^{1/p}\|u\|_{L^1_p(G)},$$

since in (14) ε is arbitrary.

3. (1, p)-EQUIVALENCE OF REGIONS AND NC_p-SETS

3.1. THEOREM 3.1. Regions G_1 and G_2 ($G_1 \supset G_2$) are (1, p)-equivalent if and only if the set $G_1 \setminus G_2$ is an NC_p-set in G_1.

Proof. Necessity. Let the spaces $L_p^1(G_1)$ and $L_p^1(G_2)$ be isomorphic as linear spaces under the restriction map $\theta u = u|_{G_2}$, $u \in L_p^1(G_1)$. Passing to the factor spaces $L_p^1(G_1)/R$ and $L_p^1(G_2)/R$, and using Banach's theorem, we see that the operator Θ^{-1} is bounded.

We will show that $m(G_1 \setminus G_2) = 0$. Suppose the opposite. Then the set $G_1 \setminus G_2$ has at least one point of density x_0. We consider the sequence of open cubes $Q_m = Q(x_0, 1/m)$ with centers at the point x_0, with edge of length $1/m$, and boundaries parallel to the coordinate planes. We consider the functions u_m, equal to zero outside the cube Q_m, equal to $1/2m$ at the point x_0, and linear on each segment joining the point x_0 with an arbitrary point of the boundary of the cube Q_m. It is clear that $|\nabla u_m(x)| = 1$ almost everywhere in Q_m.

It follows from the boundedness of the operator Θ^{-1} that

$$m(Q_m) = \int_{G_1} |\nabla u_m|^p \, dx \leqslant \|\theta^{-1}\| \int_{G_2} |\nabla u_m|^p \, dx = \|\theta^{-1}\| \cdot m(Q_m \setminus (G_1 \setminus G_2)).$$

If the point x_0 is a point of density, then as $m \to \infty$ the inequality fails. This contradiction shows that $m(G_1 \setminus G_2) = 0$.

Consequently, Θ is an isometric operator and $G_1 \setminus G_2$ is an NC_p-set.

Sufficiency. Let $E = G_1 \setminus G_2$ be an NC_p-set in G_1. For pairs of connected compacta with smooth boundary F_0, $F_1 \subset G_2$, $F_0 \cap F_1 = \emptyset$, we consider the extremal functions u_1 in G_1 and u_2 in G_2. From the definition of NC_p-sets,

$$\int_{G_1} |\nabla u_1|^p \, dx = \int_{G_2} |\nabla u_2|^p \, dx.$$

The function u_1 is equal to zero on F_0 and equal to one on F_1, with

$$\int_{G_2} |\nabla u_1|^p \, dx \leqslant \int_{G_1} |\nabla u_1|^p \, dx = \int_{G_2} |\nabla u_2|^p \, dx.$$

By the uniqueness of the extremal function, $u_1 \equiv u_2$ on G_2. Therefore, each function $u \in E_p(G_2)$ can be continued to the region G_1 with preservation of norm. Lemma 1.1 and the lemma on semicontinuity allow us to extend this conclusion to the entire class $E_p(\bar{G}_2)$.

We choose an arbitrary function $v \in L_p^1(G_2)$.

By Theorem I, for each $\varepsilon > 0$ there exists a function $v_\varepsilon = \sum_{k=1}^{l} c_{k,\varepsilon} v_{k,\varepsilon}$, satisfying the conditions: a) $\|v - v_\varepsilon\|_{L_p^1(G)} < \varepsilon$; b) $\|v_\varepsilon\|_{L_p^1(G)} = \sum_{k=1}^{l} |c_{k,\varepsilon}| \|v_{k,\varepsilon}\|$; c) $v_{k,\varepsilon} \in E_p(\bar{G}_2)$ for all k. It follows from the above that each of the functions $v_{k,\varepsilon}$ is continuable to G_2 with preservation of norm. For the continuation $\tilde{v}_\varepsilon$ of the function v_ε, the following inequality holds

$$\|\tilde{v}_\varepsilon\|_{L_p^1(G_1)} = \left\| \sum_{k=1}^{l} c_{k,\varepsilon} \tilde{v}_{k,\varepsilon} \right\|_{L_p^1(G_1)} \leqslant \sum_{k=1}^{l} |c_{k,\varepsilon}| \|\tilde{v}_{k,\varepsilon}\|_{L_p^1(G_1)} = \sum_{k=1}^{l} |c_{k,\varepsilon}| \|v_{k,\varepsilon}\|_{L_p^1(G_2)} = \|v_\varepsilon\|_{L_p^1(G_2)} = \|v\|_{L_p^1(G_2)} + \varepsilon.$$

Choosing a sequence $\varepsilon_n \to 0$ as $n \to \infty$, we construct a sequence of functions $v_{\varepsilon_n} \to v$ [in $L_p^1(G_2)$] such that the sequence $\left\{\sum c_{k,\varepsilon_n} \tilde{v}_{k,\varepsilon_n}\right\}$ weakly converges in $L_p^1(G_1)$ to some function $\tilde{\tilde{v}}$. Therefore [14], $\tilde{\tilde{v}} - v = T = \text{const}$ on G_2. Setting $\tilde{v} = \tilde{\tilde{v}} - T$, we obtain an extension $\tilde{v}$ of the function v onto G_1. By the lemma on semicontinuity,

$$\|\tilde{v}\|_{L_p^1(G_1)} = \|v\|_{L_p^1(G_2)}.$$

We have shown that each function $v \in L_p^1(G_2)$ can be continued to G_1 with preservation of norm.

To complete the proof of the (1,p)-equivalence of the regions G_1 and G_2, we have to show that the extension operator is bijective. For this it is sufficient to show that the measure of the set $G_1 \setminus G_2$ is equal to zero.

Let $x \in \partial(G_1 \setminus G_2) \cap G_1$. We choose a spherical ring $D = \{y \in R^n : 0 < a < |y - x| < b\}$, $a<b$, lying in G_1. The set $F_1 = \{y \in R^n : |y - x| \le a\}$ has a nonempty intersection with the region G_2. For sufficiently small b the set $F_0 = \{y \in R^n : |y - x| \ge b\}$.

The gradient of the extremal function u for the pair (F_0, F_1) is different from zero on D. By the above, the function $u|_{G_2}$ may be continued to a function $\tilde{u} \in L_p^1(G_1)$ with $\|\tilde{u}\|_{L_p^1(G_1)} = \|u\|_{L_p^1(G_2)}$.

Therefore,

$$\int_{G_1 \setminus G_2} |\nabla \tilde{u}|^p \, dx = 0. \tag{15}$$

If $m(G_1 \setminus G_2) \neq 0$, then $|\nabla \tilde{u}| = 0$ almost everywhere on $G_1 \setminus G_2$, i.e., $|\nabla \tilde{u}| = 0$ a.e. on $F_0 \cup F_1$, $\tilde{u} = 0$ a.e. on F_0, $\tilde{u} = 1$ a.e. on F_1 (since $F_0 \cap G_2 \neq \emptyset$, $F_1 \cap G_2 \neq \emptyset$).

The fact that the function $\tilde{u}$ is zero on F_0, one on F_1, and that $\|\tilde{u}\|_{L_p^1(G_1)} = \|u\|_{L_p^1(G_2)}$, leads to the inequality

$$\int_D |\nabla \tilde{u}|^p \, dx \leqslant \int_D |\nabla u|^p \, dx.$$

It follows from the uniqueness of the extremal function that $\tilde{u} \equiv u$. Inequality (15), and the fact that $|\nabla u| > 0$ on D, imply that $m(D \cap (G_1 \setminus G_2)) = 0$. The countable additivity of the measure and the arbitrary nature of the ring D allow us to conclude that the set $G_1 \setminus G_2$ has no interior points and therefore has measure zero.

This completes the proof.

COROLLARY. The restriction operator in the definition of (1,p)-equivalence of regions is an isometry of the spaces L_p^1.

It follows from this that the measure of the difference of (1,p)-equivalent regions is equal to zero.

3.2. Properties of NC_p-Sets. The Localization Principle. A set $E \subset G$ is an NC_p-set in G if and only if for any open ball $B(x, r) \subset G$ the set $E \cap B(x, r)$ is an NC_p-set in the ball $B(x, r)$.

Proof. Sufficiency is obvious.

Necessity. We choose an arbitrary ball $B_1 = B(x, r_1)$ $(0 < r_1 < r)$ and a function $v \in L_p^1(B \setminus E)$. Multiplying v by a smooth finite-valued function ψ, equal to one on the ball B_1 and equal to zero outside the ball $B(x, r)$, and defining $\tilde{v} = v \cdot \psi$ in the ball $B(x, r)$ and $\tilde{v} = 0$ outside the ball $B(x, r)$, we obtain a function that belongs to the class $L_p^1(G \setminus E)$. By Theorem 3.1, $\tilde{v}$ can be continued with preservation of class and without increasing the norm, to a function $\tilde{w}$ defined on the region G. Thus, we obtain a unique extension of a function $v \in L_p^1(B(x, r_1) \setminus E)$ to a function $w \in L_p^1(B(x, r_1))$. Since r_1 was arbitrary, it follows that the regions $B(x, r) \setminus E$ and $B(x,r)$ are (1,p)-equivalent, i.e., E is an NC_p-set in $B(x, r)$.

We fix a region G and a set $E \subset G$. With the exception of Property 3.2, the properties of NC_p-sets are consequences of the localization principle.

Property 3.2. If there exists a sequence $\{B_n\}_{n \geq 1}$ covering a set $E \subset G$, and if the set $E_n = E \cap B_n$ is an NC_p-set in B_n, then E is an NC_p-set in G.

Property 3.3. Any closed subset of an NC_p-set is an NC_p-set.

Proof. Let E_1 be a closed subset of the set E. We choose an arbitrary function $v \in L_p^1(G \setminus E_1)$. Then $v \in L_p^1(G \setminus E)$, and by Theorem 3.1 it is continuable in a unique way to a function $\tilde{v} \in L_p^1(G)$. Consequently, the regions G and $G \setminus E_1$ are (1,p)-equivalent, and by the same theorem E_1 is an NC_p-set.

Property 3.4. The intersection of any number of NC_p-sets is an NC_p-set.

COROLLARY 3.5. Let G be a region in R^n, $\{E_m\}$, $m = 1, 2, \ldots, M$ be NC_p-sets in G. Then their union $E = \bigcup_{m=1}^{M} E_m$ is an NC_p-set.

It is sufficient to carry out the proof for two NC_p-sets E_1 and E_2. The intersection $E_1 \cap E_2$ is an NC_p-set by Property 3.3. We consider the region $G_1 = G \setminus (E_1 \cap E_2)$. In this region the set $(E_1 \cup E_2) \setminus (E_1 \cup E_2)$ satisfies the conditions of Property 3.2. We choose an arbitrary function $v \in L_p^1(G \setminus (E_1 \cup E_2))$. In the region G_1, we can apply Property 3.2 and Theorem 3.1 to continue v to a continuous function $\tilde{v} \in L_p^1(G_1)$. Since $E_1 \cap E_2$ is an NC_p-set in G_1, it follows that $\tilde{v}$ can be continued to a function $w \in L_p^1(G)$. This completes the proof.

Property 3.6. Let G_1 be a subregion of G and E an NC_p-set in G. Then $E_1 = G_1 \cap E$ is an NC_p-set in G_1.

This follows from the localization principle.

Property 3.7. Each compact subset and NC_p-set E in the region G is an NC_p-set in any region.

Proof. It follows from Property 3.6 that it is sufficient to prove that any compact subset of a set E is an NC_p-set in R^n. This easily follows from Theorem 3.1.

Property 3.8. Every NC_q-set E in a region G is an NC_p-set in G for all $p > q$.

Proof. By the localization principle, it is sufficient to verify the assertion of the theorem for a ball $B \subset G$. If the function $v \in L_p^1(B \setminus E)$, then $v \in L_q^1(B \setminus E)$, $q < p$. By assumption E is an NC_q-set in B. Therefore v has a generalized derivative in the ball B. Since $m(E) = 0$, it follows that $v \in L_p^1(B)$.

Property 3.9. Let E be a closed set in the region G. Then

a) if E is an NC_p-set, it follows that $m(E) = 0$;

b) if E is an NC_p-set, then $\dim E \leq n-2$;

c) if the $(n-1)$-dimensional Hausdorff measure $\Lambda_{n-1}(E) = 0$, then E is an NC_p-set.

Proof. Property a) was proved in Theorem 3.1.

It follows from the localization principle that the intersection of the set with any ball $B \subset G$ is an NC_p-set in B. We assume that there exists a ball B dividing the set into two nonempty open sets B_0 and B_1 ($B \setminus E = B_0 \cup B_1$). We choose in each of these sets B_0 and B_1 a closed ball $F_0 \subset B_0$, $F_1 \subset B_1$. Then there is a function v that is equal to zero on F_0 and equal to one on F_1 and attains the capacity $C_p(F_0, F_1; B \setminus E)$. Consequently, $C_p(F_0, F_1; B \setminus E) = 0$. At the same time, it is well known that $C_p(F_0, F_1; B) > 0$. This contradicts the fact that $E \cap B$ is an NC_p-set in B.

This contradiction shows that for any ball $B \subset G$ the set $B \setminus E$ is connected. This proves assertion b).

If the set E satisfies property c), then each function that has generalized derivatives in the region $G \setminus E$ can be continued to a function $\tilde{v}$ which has generalized derivatives in the region G. Since $m(E) = 0$, it follows that the regions G and $G \setminus E$ are $(1,p)$-equivalent. Therefore, E is an NC_p-set.

Remark. Property 3.9, for the case $p = n$ and $G = R^n$, was proved in [15].

Remark. There exists an example of an NC_n-set which has nonzero $(n-1)$-dimensional Hausdorff measure [16].

4. REMOVABILITY OF SETS FOR QUASICONFORMAL AND QUASI-ISOMETRIC MAPPINGS

In this section we will show that NC_n-sets are removable for quasiconformal mappings. Theorems on removability that appear in [5, 6, 8] are particular cases of this result. We also show that NC_p-sets are removable for quasiconformal mappings.

THEOREM 4.1. Let G be a region in R^n, and let E be an NC_p-set in the region G. Then any quasiconformal homeomorphism φ of the region $G \setminus E$ onto the bounded region $G' \subset R^n$ can be continued to a quasiconformal homeomorphism $\tilde{\varphi}: G \to R^n$ without increasing the distortion coefficient.

Remark. This result is also true for unbounded regions. To show this it is sufficient to carry out the arguments below on a sphere.

THEOREM 4.2. Let G be a region in R^n, let E be an NC_n-set in G, and let $\varphi: G \setminus E \to R^n$ be a mapping with bounded distortion [17]. We assume that for any point $x \in E$ there exists a ball $B(x, r)$ such that φ belongs to the class $L_n^1(B \setminus E)$. Then there exists a unique continuation of the mapping φ to a mapping with bounded distortion $\tilde{\varphi}: G \to R^n$ with no increase in the distortion coefficient.

Proof of Theorems 4.1 and 4.2. Theorem 4.1 follows from Theorem 4.2. In fact, in the case of a sphere the condition is satisfied for any quasiconformal mapping. By Property 3.9 the set $G \setminus E$ is connected, and therefore a quasiconformal mapping φ is on $G \setminus E$ a mapping with bounded distortion. For each compact region V ($\bar{V} \subset G$) the mapping $\varphi \in L_n^1(V \setminus E)$. By Property 3.6 and Theorem 3.1, φ can be continued to a mapping $\tilde{\varphi}_V: V \to R^n$, belonging to the class L_n^1. Therefore, φ can be continued to a mapping $\tilde{\varphi}: G \to R^n$, belonging to the class $L_{n,loc}^1(G)$.* Since E has measure zero, it follows that $\tilde{\varphi}$ is a mapping with bounded distortion [17]. If the mapping φ is a homeomorphism, then $\tilde{\varphi}$ is also a homeomorphism. [This easily follows from the fact that $m(E) = 0$ and the fact that the mapping $\tilde{\varphi}$ is open.]

Remark. Theorem 4.1 contains a corresponding result from [8] for the particular case when E is compact.

*With regard to the definition of a mapping with bounded distortion and the properties used here, cf., [17].

THEOREM 4.3. Let G be a region in R^n, and let E be an NC_p-set ($p > 1$) in the region G. Then any quasi-isometric mapping $\varphi: G \setminus E \to R^n$ can be continued in a unique way to a quasi-isometric mapping $\tilde{\varphi}: G \to R^n$.

Proof. Applying Property 3.8, we can reduce the proof to the case $p > n$. The mapping $\circledS^{-1}\varphi^*: L_p^1(G') \to L_p^1(G)$, where $G' = \varphi(G)$; $\varphi^*: L_p^1(G') \to L_p^1(G \setminus E)$, $\varphi^* f = f \circ \varphi$, $f \in L_p^1(G')$; $\circledS: L_p^1(G) \to L_p(G \setminus E)$, $\circledS g = g|_{G \setminus E}$, $g \in L_p^1(G)$, is a structural isomorphism between the space $L_p^1(G')$ and $L_p^1(G)$ [18]. By a result from [18], $\circledS^{-1}\varphi^*$ can be continued to a quasi-isometric mapping $\tilde{\varphi}$ which is obviously the continuous extension of the mapping φ.

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