CONDITIONAL IDENTITIES IN FINITE GROUPS

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A theorem of Oates and Powell [1] states that all the identical relations in a finite group are consequences of a finite number of identities in this group. Here we shall consider conditional identities (quasi-identities), i.e., relations of the form

$$v_1(x_1, x_2, \ldots, x_k)=1 \& v_2(x_1, x_2, \ldots, x_k)=1 \& \ldots$$
$$\ldots \& v_l(x_1, x_2, \ldots, x_k)=1 \Rightarrow w(x_1, x_2, \ldots, x_k)=1. \quad (1)$$

Let us recall that the class of groups given by a system of formulas of the type (1) is called a quasivariety. We shall denote by qvar G the quasiprimitive closure of a group G, i.e., the smallest quasivariety containing the group G (it is defined by all the conditional identities which hold good in G). The quasiprimitive closure of a finite group G is constructed rather simply (see [2], p. 295): It consists of the subgroups of the cartesian powers of the group G. (In this sense the smallest variety var G containing the group G is complex since according to a theorem of Birkhoff we must also add all the homomorphic images of the subgroups of the cartesian powers of the group G). Therefore it is interesting in the first place to clarify as to when these classes are finitely defined, i.e., in which cases the set of all the conditional identities of a finite group is equivalent to a finite subset (has a finite base). In the present note we shall prove the following theorem.

THEOREM. The conditional identities of a finite group G have a finite base if and only if all the Sylow subgroups of the group G are abelian.

Proof. 1. "Only if". Let us assume that the group G contains a nilpotent nonabelian subgroup. Let us choose a minimal such subgroup H. Let p be a prime divisor of the order of the group H and F_n be the free group of rank n in the variety var H, which, as we know, lies in qvar H and hence also in qvar G.

LEMMA 1. For $n > 4m + 3$ there exists an element a in the commutant $[F_n, F_n]$ of the group F_n which is not equal to

$$[x_1, x_2][x_3, x_4] \ldots [x_{2m-1}, x_{2m}]y^p \quad (2)$$

for any $x_1, x_2, \ldots, x_{2m}, y \in F_n$.

Proof. By virtue of the minimality, the subgroup H is nilpotent of class 2 and $|[H, H]| = p$. Hence it follows that the values of the commutator $[x, y]$ and the power z^p in the variety var H depend only on the cosets relative to the product of the commutant with the p^{th} power of the group which contain the elements x, y, and z. Since $|F_n/F_n^p[F_n, F_n]| \le p^n$, the product (2) does not assume more than $p^{(2m+1)n}$ different values in the group F_n. On the other hand, $[F_n, F_n]$ is generated by the commutators $[f_i, f_j]$, $i > j$, of the free generators of the group F_n, where these commutators are independent over $Zp = Z/pZ$ (in the contrary case we would easily get the identity $[x, y] \equiv 1$), so that $|[F_n, F_n]| = p^{n(n-1)/2}$; whence the lemma is proved.

Moreover, let us observe that if $b = a^k \neq 1$, then the element b also cannot be represented in the form (2) since $gp\{a\} = gp\{b\}$, and in a nilpotent group of class 2 a power of the produce (2) is again a product of the same form.

LEMMA 2. Let L be a group with generators $g_1, g_2, \ldots, g_t$ lying in var H. Every element of the Frattini subgroup of the group L can be represented in the form (2) if $m \ge t(t-1)/2$.

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Proof. Let $l \in \Phi(L)$, where $\Phi(L)$ is the Frattini subgroup of L. As we know (see, e.g., [3], p. 198), $\Phi(L) = L^p[L, L]$ for the nilpotent p-group L, so that $l = y^p k$, where $y \in L$ and $k \in [L, L]$. It remains to observe that $[g_i, g_j]$, $1 \leq i < j \leq t$, generate [L, L] and that every power of a commutator is a commutator in a nilpotent group of class 2.

Let us now fix a natural number t and put $m = t(t-1)/2$, $n = 4m + 4$, $s = 2^{|G|} + 1$. We take s copies of the group F_n: $D_i \cong F_n$, $i = 1, 2, \ldots, s$; moreover, a_i are the images of the element a of Lemma 1 under these isomorphisms. Let us form the direct product C of the groups D_i with the amalgamated central subgroups gp $\{a_i\}$.

$$1 \to N \xrightarrow{\alpha} D \xrightarrow{\beta} C \to 1$$

is the corresponding exact sequence, where $D = D_1 \times D_2 \times \ldots \times D_s$ and $a_i\beta = a_j\beta = c$ ($\neq 1$ in C), $1 \leq i, j \leq s$. Let K be a subgroup of C such that the number of its generators is $\leq t$, L be the minimal preimage of the subgroup K under the epimorphism β. It is clear that L has not more than t generators and $N\alpha \cap L \subseteq \Phi(L)$ (otherwise we would be able to find a maximal subgroup M of L such that $L = M(N\alpha \cap L)$, i.e., $M\beta = K$ contradicting the minimality of L). According to Lemma 2 every element of $\Phi(L)$ can be represented in the form (2). On the other hand, $N\alpha \subset \text{gp}\{a_1\} \times \text{gp}\{a_2\} \times \ldots \times \text{gp}\{a_s\}$. Therefore, by virtue of Lemma 1 and the choice of n, no element of N, other than the identity, can be represented in the form (2). Consequently, $L \cap N\alpha = 1$ and $K \cong L$. This means that $K \in \text{qvar } H \subseteq \text{qvar } G$.

Let us now show that $C \notin \text{qvar}\, G$. Indeed, in the contrary case the group C would be contained in a direct power of the group G, and consequently, there would exist a homomorphism φ such that $C \to G$, $c\varphi \neq 1$. Since $s = 2^{|G|} + 1$, we can find an index $i \neq j$ such that $D_i\beta\varphi = D_j\beta\varphi$. In this case

$$[D_i\beta, D_i\beta]\varphi = [D_i\beta\varphi, D_j\beta\varphi] = [D_i, D_j]\beta\varphi = 1,$$

where $c\varphi = 1$ since $c = a_i\beta \in [D_i, D_i]\beta$. The contradiction ($c\varphi \neq 1$, $c\varphi = 1$) so obtained proves that $C \notin \text{qvar}$ G.

Thus, if G has a nonabelian Sylow subgroup, then for an arbitrary natural number t there exists a group which does not belong to the quasivariety qvar G whereas every t-generated subgroup of it belongs to qvar G. It is clear that this cannot happen if the quasivariety qvar G is finitely defined.

II. Sufficiency. Let us formulate at first some lemmas. Let p be a prime number, $q = p^e$, Z_q be the quotient ring, G be a finite group such that $p \nmid |G|$ and, finally, let the group ring $R = Z_qG$.

LEMMA 3. The number of nonisomorphic nondecomposable modules over R is finite.

Proof. Let M be a nondecomposable R-module. Let us recall that the ring R is quasi-Frobenius (see, e.g., [4], Exercise 58.2 (2)), and observe moreover that the right socle of the ring R is equal to $p^{e-1}R$ since $p^{e-1}R$ is the lowest stratum of the additive group of the ring R and it can be regarded as a module over the semisimple ring Z_pG. Therefore, if $p^{e-1}M \neq 0$, then it follows from Theorem 58.12 and Lemma 59.1 of [4] that the module M is isomorphic to a principal right ideal of the ring R. This means that $|M| \leq |R|$. If $p^{e-1}M = 0$, then M can be regarded as a $Z_{p^{e-1}}G$-module and the proof of this lemma 3 is completed by induction over e.

LEMMA 4. If N is a submodule of a finite R-module M, then N is contained in a direct summand D of the module M, where the order of the submodule D is bounded in terms of |N| and R.*

Proof. Let $M = \dot{\sum}_{1 \leqslant i \leqslant n} M_i$, where M_i are nondecomposable submodules. Let φ_{ij}, $i \leq j$, be an isomorphism of M_i onto M_j if such an isomorphism exists. Moreover, we choose these isomorphisms such that $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$. If the isomorphism φ_{ij} closes the graph of the projections of the submodule N onto M_i and M_j upto commutativity, then we put i and j in the same equivalence class T_α. Corresponding to this partition we can represent M is the form of the double sum: $M = \dot{\sum}_{\alpha} \dot{\sum}_{i \in T_\alpha} M_i$. The number of classes is bounded obviously in terms of $|N_j|$ and $|M_i|$; which means (according to Lemma 3) in terms of |N| and R.

* I.e, |D| is bounded by a quantity which depends only on |N| and R.

The diagonal D_α of the "block" $M_\alpha = \dot{\sum\limits_{i \in T_\alpha}} M_i$, determined by the isomorphisms φ_{ij}, is a direct summand of M_α, and the sum $D = \dot{\sum\limits_\alpha} D_\alpha$ is a direct summand of the module M containing (according to the definition of the classes T_α) the submodule N. The lemma 4 is proved.

LEMMA 5. If G is a finite group with an abelian Sylow p-subgroup G_p and M is a normal factor in G, where $M \subseteq G_p$ and $M^q = 1$, then there exists a normal factor L contained in M and having a complement in the group G whose index is bounded in terms of $|G/M|$ and q.

Proof. Let H be the smallest subgroup such that $G = HM$. It is easily seen that in this case the intersection $M \cap H$ is contained in $\Phi(H)$. Therefore $|H/\Phi(H)| \leq |G/M|$ and the number $|H|$ is bounded in terms of $|G/M|$ and q. Since the subgroup G_p is abelian, we have $G_p \subseteq C = C_G(M)$ and $p \nmid |G/C|$. We can transform M into an R-module in a standard manner, where $R = Z_q(G/C)$ and use Lemma 4 for the submodule $M \cap H$: $M \cap H \subseteq D$, $M = D \times L$, where $D \triangleleft G$, $L \triangleleft G$ and the order of the group D is bounded for fixed $|G/M|$ and q since $C \supseteq M$. The product $G = (HD)L$ is semidirect by virtue of the inclusion $M \cap H \subseteq D$, and the index of HD is bounded, which was to be proved.

In the sequel we shall use the following lemma due to W. Gaschütz (see [5], p. 426).

LEMMA 6. If N is a normal factor of a finite group G with the following properties

1) G/N is a p-group,

2) N has an abelian Sylow p-subgroup,

3) N has no nontrivial factor p-group,

then N has a complement in G.

LEMMA 7. If M is a normal factor of a finite group G having an abelian Sylow subgroup G_p and $G^m = 1$, then there exists a normal factor L of the group G such that

1) $L \subseteq M$ and M/L is a p-group,

2) $G_p \cap L$ has a complement in G_p,

3) the index of the subgroup L is bounded in terms of $|G/M|$ and m.

Proof. Let N be the smallest normal factor of the subgroup M such that M/N is a p-group. Obviously N is normal in G and the pair $N \triangleleft G_pN$ satisfies the conditions of Lemma 6. Therefore N has a complement S in G_pN, where according to Sylow's theorem we choose $S \subseteq G_p$. In such a case $G_p = S(G_p \cap N)$, where the product is semidirect. Now, applying Lemma 5 to the group G/N we have $G/N = (H/N)(L/N)$, where $L \triangleleft G$, the product is semidirect, $L \subseteq M$, and the index $|G: L|$ is bounded in terms of $|G/M|$ and m. Since SN/N is a Sylow subgroup of the group G/N, the subgroup H/N can be chosen such that $(SN \cap H)N$ is a Sylow subgroup in H/N; whence $SN/N = (SN \cap H/N)(SN \cap L/N)$. The natural isomorphism of the last equation onto S (since $S \cap N = 1$) gives a semidirect factorization $S = (S \cap H)(S \cap L)$. Hence from the equation $G_p = S(G_p \cap N)$ we obtain the semidirect product $G_p = (S \cap H)[(S \cap L)(G_p \cap N)] = (S \cap H)(G_p \cap L)$, which completes the proof of the Lemma 7.

The following lemma is the statement of L. A. Shemetkov's theorem [6]. Here G_p is a Sylow p-subgroup of a finite group G.

LEMMA 8. A normal subgroup K of a group G has a complement in G if for every prime factor p of the index $|G: K|$ the subgroup $G_p \cap K$ is abelian and complemented in G_p.

LEMMA 9. If M is a normal factor of a finite group G, the orders of whose elements divide m and all of whose Sylow subgroups are abelian, then there exists a normal factor K of the group G such that

1) $K \subseteq M$,

2) K has a complement in G,

3) the index $|G: K|$ is bounded in terms of $|G/M|$ and m.

Proof. Let $p_1, p_2, \ldots, p_s$ be the different prime factors of the number $|G/M|$. Let $p = p_1$ and $L = L_1$ be a normal factor taken according to Lemma 7. Let us once more apply Lemma 7 now for $p = p_2$ and the normal factor L_1, i.e., let us find $L_2 \subseteq L_1$ such that $G_{p_2} \cap L_2$ has a complement in G_{p_2} and the index $|G: L_2|$ is bounded in terms of $|G: L_1|$ and m, which means, also in terms of $|G: M|$ and m. Since L_1/L_2 is a p_2-group, we have $G_{p_1} \cap L_1 = G_{p_1} \cap L_2$, i.e., this intersection is complemented in G_{p_1}. Let us now construct $L_3, \ldots, L_s$ in a similar manner and put $K = L_s$. In consequence of the boundedness of the number s and Lemma 8 the subgroup K satisfies all the conditions of Lemma 9.

Let us now directly proceed to the proof of the sufficiency of the condition of the theorem. Let G be a finite group with all its Sylow subgroups abelian. In this case we can find a finite group H (also with all of its Sylow subgroups abelian) such that var G = qvar H (see [7], Theorems 1 and 2). So that if $T \in \mathrm{var}\, G$ is a finite group, then T can be approximated by groups of a bounded order ($\leq |H|$), i.e. $\bigcap_{|T/M_\alpha| \leqslant |H|} M_\alpha = 1$.

Since the Sylow subgroups of T are abelian and the orders of the elements do not exceed the order of G, according to Lemma 9 we can find complemented normal subgroups K_α of bounded indices such that $\bigcap_\alpha K_\alpha = 1$. This means that the group T can be imbedded in the direct product of its subgroups of a bounded (by a number n_G) order. Consequently, T from var G belongs to the quasivariety qvar G if qvar G contains all subgroups of T of orders $\leq n_G$. The variety var G consists, as we know, of locally finite groups. Therefore a group A belongs to qvar G if and only if it belongs to var G (this variety is finitely defined since G is finite (see [8], 52.11), and all of its subgroups of orders $\leq n_G$ belong to qvar G. The fulfillment of the last condition can be ensured by a finite number of quasi-identities of the group G. With this the theorem is proved.

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