ZONES OF ATTRACTION OF SELF-SIMILAR MULTIPLE INTEGRALS

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The goal of this paper is the proof of the theorem announced in [20]. Here we consider the "multidimensional case," i.e., convergence to self-similar fields. We give a short survey of the separate sections. In Sec. 1 the concepts needed are formulated as well as the basic result of the paper (Theorem 1). The connection of this theorem with the result of Dobrushin–Major [8] is discussed as well as some similar questions. In Sec. 2 the term making the basic contribution to the distribution of the sums considered is isolated. Here we explain the idea of the following proof, which is broken up into several lemmas, and their formulations are given. The proofs of these lemmas (except for Lemma 5) are carried out to section 3. In section 4 there is proved a lemma (Lemma 1 of [20]) on the convergence with respect to distribution of "discrete multiple integrals" to "continuous" integrals of Ito–Wiener. With the help of this lemma, the remaining Lemma 5 is proved.

1. Notation for what follows: R^d is d-dimensional Euclidean space, $x \cdot y$, $|x|$ are respectively the scalar product and norm in R^d, Z^d is the integer-valued d-dimensional lattice. We shall write $t_1 < t_2$ $(t_1, t_2 \in \mathbf{R}^d)$, if $t_1^{(1)} > t_2^{(1)}, \ldots, t_1^{(d)} < t_2^{(d)}$, $t = (t^{(1)}, \ldots, t^{(d)})$. We also write $\mathbf{R}_+^d = \{t \in \mathbf{R}^d: 0 < t\}$, $K_t = \{s \in \mathbf{R}_+^d : s < t\}$, $[K_t] = K_t \cap \mathbf{Z}^d$.

A random field $X = (X(t))_{t \in \mathbf{R}_+^d}$ will be called *self-similar with index* $\chi \in \mathbf{R}$ if its finite-dimensional distributions are invariant with respect to scale transformations ("transformations of the renormalization group") $X(t) \to \lambda^\chi X(\lambda t), \lambda > 0$ [6]. For d = 1 we shall speak of a *self-similar process*. Usually self-similar fields are considered on the whole space R^d; the restriction to R_+^d is connected with the character of the summation problem studied in this paper.

Self-similar processes (familiar now under the name of fractional Brownian motion) were first considered by Kolmogorov [14]. Since then self-similar processes and fields have gained wide familiarity thanks to the role which they play in limit theorems of probability theory, and also in a series of physical theories (cf. [6, 19, 24]). The connection of the concept of self-similarity with limit theorems was discussed by Lamperti [15] and Dobrushin [6]. Following [15, 24], we shall say that a stationary field (in the narrow sense) $Y = (Y_j)_{j \in \mathbf{Z}^d}$ *belongs to the domain of attraction of some random field* $Z = (Z(t))_{t \in \mathbf{R}_+^d}$ if the finite-dimensional distributions of the field $Z^{(N)} = \left(A_N^{-1} \sum_{0 < j < Nt} Y_j\right)_{t \in \mathbf{R}_+^d}$ converge as $N \to \infty$ to the corresponding distributions of the field Z, where $0 < A_N \to \infty$ $(N \to \infty)$ is a sequence of normalizing constants. It is known [15, 6] that under quite general conditions the limit field Z is necessarily self-similar with index $\chi < 0$, and the normalizing constants have the form $A_N = N^{-\chi} L(N)$, where $L:[1, \infty) \to R_+$ is a slowly varying function. In the case of finite second moments the value $\chi < -d/2$ gives the long range dependence in the sequence Y, in contrast with the independent or weakly dependent case, corresponding to the value $\chi = -d/2$. In the latter case when the limiting self-similar field is a Brownian sheet (i.e., a Gaussian field $Z = (Z_t)_{t \in \mathbf{R}_+^d}$ with zero mean and covariance function $E[Z_t Z_s] = \prod_{j=1}^d \min(s^{(j)}, t^{(j)})$, we shall say that the sequence Y *is subordinate to the central limit theorem* (CLT). A detailed investigation of the condition for applicability of the CLT for "*one-dimensional*" (d = 1)

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stationary sequences is recounted in Ibragimov–Linnik [12] (cf. also Gnedenko–Kolmogorov [2]). Much attention is now devoted to the CLT for random fields (cf., e.g., [10]).

Starting with the familiar work of Rosenblatt [17], in the majority of papers on convergence to "strongly dependent" self-similar processes have considered the case of sequences which are nonlinear functionals of Gaussian fields with power asymptotic correlation function. The only exceptions here are perhaps the results of Davydov [5] and Gorodetskii [3] on the convergence of sums of linear sequences to processes of fractional Brownian motion and the recent paper of Kesten and Spitzer [13]. In Dobrushin and Major [8] (cf. also Taqqu [24, 25], Gorodetskii [4]) the following beautiful theorem was proved. Following [4, 24], we shall mean by the Hermitian rank of the function $F\in L_X^2\equiv L^2(\mathbf{R}, e^{-x^2/2}dx/\sqrt{2\pi})$ the index of the first nonzero coefficient of the expansion of $F(x)=\sum_{k=0}^{\infty} d_k H_k(x)$ in a series of Hermite polynomials with leading coefficient 1. We write $S_{d-1} = \{x \in R^d: |x| = 1\}$, $1_A(\cdot)$ is the indicator of the set A, $\hat{f}$ is the Fourier transform of the function $f:R^d \to C$.

THEOREM 1 [8]. Let $X=(X_j)_{j\in\mathbf{Z}^d}$ be a stationary Gaussian field with mean 0, variance 1, and correlation function

$$E[X_0X_t]=L(|t|)a(t/|t|)|t|^{-\alpha},\quad t\in\mathbf{Z}^d, \tag{1}$$

where $\alpha\in(0, d)$, $L:[1, \infty)\to\mathbf{R}_+$ is a slowly varying (s.v.) function, $a(\cdot)$ is a nonnegative continuous function defined in S_{d-1}. Let the function $F\in L_X^2$ have Hermitian rank $m \geq 1$ and $\alpha < d/m$. Then the finite-dimensional distributions of the field

$$N^{-d+\alpha m/2}L^{-m/2}(N)\sum_{0<j<Nt} F(X_j) \tag{2}$$

converge as $N \to \infty$ to the corresponding distributions of the self-similar field

$$Z_m(t)=d_m\int \hat{1}_{K_t}(x_1+\ldots+x_m)\mathfrak{z}(dx_1)\ldots\mathfrak{z}(dx_m). \tag{3}$$

Here d_m is the coefficient of the expansion of F in Hermite polynomials, $\mathfrak{z}(dx)$ is the random Gaussian spectral measure corresponding to the self-similar spectral measure $F(dx)=E[|\mathfrak{z}(dx)|^2]$, defined by

$$2^d\int_{\mathbf{R}^d} e^{ix\cdot t}\Big(\prod_{j=1}^{d}(1-\cos x^{(j)})/(x^{(j)})^2\Big)F(dx) = \int_{[-1,1]^d}\prod_{j=1}^{d}(1-|x^{(j)}|)a\big((x+t)/|x+t|\big)|x+t|^{-\alpha}dx,\quad t\in\mathbf{R}^d. \tag{4}$$

In (3) and in what follows the integral $\int$ denotes integration in a domain of $(R^d)^m$. We note that (4) in the sense of functions of slow growth is equivalent with

$$\int_{\mathbf{R}^d} e^{it\cdot x}F(dx)=a(t/|t|)|t|^{-\alpha},\quad t\in\mathbf{R}^d, \tag{5}$$

i.e., $\mathfrak{z}(dx)$ is the random spectral measure of a Gaussian generalized random field with covariance function $(t/|t|)|t|^{-\alpha}$, $t\in\mathbf{R}^d$. On the right side of (3) there is a multiple Wiener–Ito integral with respect to the Gaussian complex even random measure $\mathfrak{z}$ (cf., e.g., [16]).

We formulate the basic result of this paper.

Let $X=(X_j)_{j\in Z^d}$ be a linear field

$$X_j=\sum_{k\in\mathbf{Z}^d} h(j-k)\xi_k, \tag{6}$$

$(\xi_k)_{k\in Z^d}$ be a sequence of independent, identically distributed random variables with mean 0, variance 1, and finite moments of any order, and let the function h have the form

$$h(t)=\Lambda(|t|)\ b(t/|t|)|t|^{-\beta},\ t\in\mathbf{Z}^d, \tag{7}$$

where $\beta\in(d/2,\ d)$, $\Lambda(\cdot):[1,\ \infty)\to\mathbf{R}$ is a s.v. function, $b(\cdot)$ is a continuous function defined in S_{d-1}. We shall assume that

$$E[X_j^2]=\sum h^2(t)=1. \tag{8}$$

Let (A_N), (B_N) be two sequences of real numbers. We shall write $A_N\asymp B_N$, if $0<\underline{\lim}\ A_N/B_N\le\overline{\lim}\ A_N/B_N<\infty$, and $A_N\sim B_N$, if $\lim A_N/B_N=1$.

THEOREM 2. Let the field X of (6) satisfy the conditions listed above and let there be an entire function* $F(x)=\sum_{k=0}^{\infty}c_kx^k$ such that the series

$$\sum_{k,\ j=0}^{\infty}|c_k|\ |c_j|(k!\ j!)^2 2^{2(k+j)}\bar{\mu}_{k+j}<+\infty \tag{9}$$

converges, where $\bar{\mu}_k=E[|\xi_j|^k]$, $k\ge 0$. Let m be the smallest of the numbers 1, 2, ... such that

$$e_m=E[F^{(m)}(X_j)]\neq 0, \tag{10}$$

$F^{(m)}=d^m F/dx^m$ and $\alpha\equiv 2\beta-d<d/m$. Then

$$E\Big[\Big(\sum_{j\in[K_N]}F(X_j)-E[F(X_j)]\Big)^2\Big]\asymp\Lambda^{2m}(N)N^{2d-\alpha m} \tag{11}$$

and the finite-dimensional distributions of the random field

$$N^{-d+\alpha m/2}\Lambda^{-m}(N)\sum_{0<j<Nt}F(X_j)-E[F(X_j)] \tag{12}$$

converge as $N\to\infty$ to the corresponding distributions of the self-similar field

$$Z_m'(t)=(e_m/m!)\int\Big\{\int_{K_t}\prod_{j=1}^{m}b\big((s-x_j)/|s-x_j|\big)|s-x_j|^{-\beta}ds\Big\}\ \xi(dx_1)\ldots\xi(dx_m),\quad t\in\mathbf{R}_+^d, \tag{13}$$

where $\xi(dx)$ is real Gaussian white noise in R^d with variance dx.

It is easy to show (cf. the proof of Lemma 3 below) that the correlation function of the field X of (6) is equal to

$$E[X_0X_t]=\Lambda^2(|t|)|t|^{-\alpha}\big(a(t/|t|)+o(1)\big),\quad(|t|\to\infty), \tag{14}$$

where

$$a(t)=\int_{\mathbf{R}^d}b(s/|s|)b\big((s-t)/|s-t|\big)|s|^{-\beta}|t-s|^{-\beta}ds,\quad t\in S_{d-1}. \tag{15}$$

If the functions $a(\cdot)$, $b(\cdot)$ are related by (15) and $d_m=e_m/m!$, then from the "Parseval inequality for Ito—Wiener multiple integrals" [25] it follows that the self-similar fields $Z_m(t)$ and $Z_m'(t)$ have identical distributions. In the case when the variables (ξ_k) in Theorem 2 are Gaussian, it follows from the formula for differentiation of Hermite polynomials $H_n'(x)=nH_{n-1}(x)$ that the number m, defined by (10), coincides with the Hermitian

*It is easy to note that from (9) there follows the absolute convergence of the power series of the function F on the whole line.

rank of the function F, and $d_m = e_m/m!$. Thus, Theorem 2 generalizes Theorem 1 to the class of (nongaussian) linear fields X.

In [22] there is cited a theorem on zones of attraction of polynomial self-similar processes, subordinate to a Poisson random measure, in some sense analogous to Theorem 2. In it there also figures a condition of the type of (10), determining the "degree" of the limiting polynomial process. An essential point for the validity of this kind of theorem is the presence of a linear structure of the "underlying" field X.

There is interest in the generalization of the results of the present paper for Fourier transforms of processes $F(X_j)$, and also for more general asymptotics of the weight function h, admitting a periodic component. In the case of Gaussian processes (X_j), results of this kind are obtained by Rosenblatt [18]. These results were generalized by Giraitis. The convergence of sums of linear and quadratic functionals of linear processes X under the assumptions that the random variables (ξ_k) have infinite variance, but belong to the domain of attraction of a stable law, was investigated by Astrauskas.

The following unsolved problem also seems interesting. Suppose one has a stationary renewal process $\{\tau\}=\{\ldots, \tau_{-1}, \tau_0, \tau_1, \ldots\}$ and a function h(t), interpretable as the "response" of some system at time t to the "event" happening at time t = 0. We define the random process $I(t)=\sum_{\tau_i\le t} h(t-\tau_i)$ as the "total response" of the system at time t [26]. Let F(x), $x\in\mathbf{R}$, be some function. We pose the question of the limit distribution as $N\to\infty$ for the process $J_N(t)=A_N^{-1}\int_0^{Nt} F(I(s))\,ds,\ t\ge 0$, where $A_N\to\infty (N\to\infty)$ are normalizing constants. Let us assume that the function h(t) has the form $h(t) = \Lambda(t)t^{-\beta}$, where $\beta\in(1/2, 1)$, Λ is a s.v. function. In case the renewal process is a Poisson flow, one can prove a result analogous to Theorem 2, i.e., show that under certain additional restrictions on F and a corresponding choice of A_N the processes $J_N(t)$ converge in distribution to definite Hermitian processes $Z'_m(t)$ (13) (where d = 1 and $b(\cdot) = 1_{R_+}(\cdot)$). Now if the flow $\{\tau\}$ is not Poisson, our methods for investigating the distributions of $J_N(t)$ are not applicable, although apparently the process F(I(t)) in this case too has "degree diminishing dependence," necessary for convergence to a self-similar limit.

2. We turn to the proof of Theorem 2. First we isolate the basic term in the sum (12). We rewrite $(X_j)^k$ in the form

$$(X_j)^k=\sum_{p_1,\ldots,p_k} h(j-p_1)\xi_{p_1}\ldots h(j-p_k)\xi_{p_k}=\sum_{l=0}^{k} C_k^l \sum_{(p)_l}{}' h(j-p_1)\xi_{p_1}\ldots h(j-p_l)\xi_{p_l} \tag{16}$$

$$\sum_{(V)(k-l)}\sum_{(q)_r}{}'' (h(j-q_1)\xi_{q_1})^{v_1}\ldots(h(j-q_r)\xi_{q_r})^{v_r},$$

where $C_k^l = k!/l!(k-l)!$, the sum $\sum'_{(p)_l}$ is taken over all collections $(p)_l=(p_1,\ldots,p_l)\in(\mathbf{Z}^d)^l$ such that $p_i\ne p_j$ for $i\ne j$, i, j = 1, ..., l (the set of such collections we denote by $(\mathbf{Z}^d)_0^l$), the sum $\sum_{(V)(n)}$ for $n\ge 2$ is taken over all partitions (V) of the set {1, ..., n} into disjoint subsets $V_1, \ldots, V_r$, r = 1, 2, ..., such that $2\le v_i = |V_i|$ (= the number of elements in the set V_i), $i=1,\ldots,r$, $\sum_{(V)(1)}\ldots=0$ and $\sum_{(V)(0)}\ldots=1$, finally, the summation of the last sum in (16) goes over all collections $(q)_r=(q_1,\ldots,q_r)\in(\mathbf{Z}^d)_0^r$ such that in the collections $(p_1, \ldots, p_l)$ and $(q_1, \ldots, q_r)$ there are no common elements. We introduce new random variables:

$$\eta_p(v;\ i)=\begin{cases} E[\xi_p^v], & \text{if}\quad i=0,\\ \xi_p^v-E[\xi_p^v], & \text{if}\quad i=1,\end{cases} \tag{17}$$

$v = 2, 3, \ldots$, and in (16) we replace ξ_q^v by the sum $\eta_q(v; 0) + \eta_q(v; 1)$. We have

$$(X_j)^k = (X_j)_1^k + (X_j)_2^k, \tag{18}$$

where

$$(X_j)_1^k = \sum_{l=0}^{k} C_k^l \sum_{(p)_l}{}' h(j-p_1)\xi_{p_1}\ldots h(j-p_l)\xi_{p_l} \sum_{(V)(k-l)} \sum_{(q)_r}{}' h^{v_1}(j-q_1)\mu_{v_1}\ldots h^{v_r}(j-q_r)\mu_{v_r}, \tag{19}$$

where $\mu_v = E[\xi_p^v]$. Thus

$$F(X_j) = \sum_{k=0}^{\infty} c_k (X_j)^k = F_1(X)_j + R_j, \tag{20}$$

where

$$F_1(X)_j = \sum_{k=0}^{\infty} c_k (X_j)_1^k. \tag{21}$$

We note that

$$E[F(X_j)] = E[F_1(X)_j] = \sum_{k=0}^{\infty} c_k \sum_{(V)(k)} \sum_{(q)_r}{}' h^{v_1}(j-q_1)\mu_{v_1}\ldots h^{v_r}(j-q_r)\mu_{v_r}. \tag{22}$$

Analogously

$$e_l \equiv E[F^{(l)}(X_j)] = \sum_{k=l}^{\infty} c_k k(k-1)\ldots(k-l+1) \sum_{(V)(k-l)} \sum_{(q)_r}{}' h^{v_1}(j-q_1)\mu_{v_1}\ldots h^{v_r}(j-q_r)\mu_{v_r}. \tag{23}$$

From (19)-(23) follows the equation

$$F_1(X)_j = \sum_{l=0}^{\infty} \sum_{(p)_l}{}' h(t-p_1)\xi_{p_1}\ldots h(t-p_l)\xi_{p_l} e_l/l! \tag{24}$$

which is important for what follows. The rest of the proof consists of (a) the proof of the fact that the basic contribution to the distribution of the sum (12) is carried by the summand corresponding to $F_1(X)_j$, more precisely its term

$$(e_m/m!) \sum_{0<j<Nt} \sum_{(p)_m}{}' h(t-p_1)\xi_{p1}\ldots h(t-p_m)\xi_{p_m} \tag{25}$$

("discrete multiple integral" of least order not equal to zero) and (b) the proof of convergence with respect to distribution of the "discrete integral" (25) to the corresponding "continuous" multiple Ito—Wiener integral (13) (cf. Sec. 4 below, also Lemma 1 of [20]).

The rest of the proof is divided into several lemmas. In the formulations of these lemmas, the hypotheses of Theorem 2 are assumed to hold. We shall denote by l.i.m. the limit in $L^2(\Omega)$, and by the letters C, C(·) various constants, depending on the quantities cited in the brackets. We write

$$F_M(x) = \sum_{k=0}^{M} c_k x^k,\ F_M^{(l)} = d^l F_M/dx^l,\ l \leqslant M,\ F_M^{(0)} = F_M$$

and we set

$$F_M(X_j) = F_{M,1}(X)_j + F_{M,2}(X)_j - F_{M,3}(X)_j, \tag{26}$$

where $F_{M,1}(X)_j = \sum_{k=0}^{M} c_k (X_j)_1^k$ (cf. (21)),

$$F_{M,2}(X)_j = \sum_{k=2}^{M} c_k \sum_{l=0}^{k} C_k^l \sum_{(p)_l}{}' h(j-p_1)\xi_{p_1}\ldots h(j-p_l)\xi_{p_l} \sum_{(V)(k-l)} \sum_{(q)_r}{}'' \sum_{(i)\neq 0} h^{v_1}(j-q_1)\eta_{q_1}(v_1;\ i_1)\ldots h^{v_r}(j-q_r)\eta_{q_r}(v_r;\ i_r), \tag{27}$$

the summation in $\sum_{(i)\neq 0}$ goes over all collections $(i)=(i_1, \ldots, i_r)\in\{0, 1\}^r$, except for the value (i) = (0, ..., 0); finally

$$F_{M,3}(X)_j=\sum_{k=2}^{M} c_k \sum_{l=0}^{k} C_k^l \sum_{(p)_l}' h(j-p_1)\xi_{p_1}\ldots h(j-p_l)\xi_{p_l} \sum_{(V)(k-l)} \sum_{(q)_r}'' h^{v_1}(j-q_1)\mu_{v_1}\ldots h^{v_r}(j-q_r)\mu_{v_r}, \tag{28}$$

and the latter sum is taken over all collections $(q)_r\in(\mathbf{Z}^d)_0^r$ such that the intersection $(p)_l\cap(q)_r$ is nonempty.

LEMMA 1. The following limits exist as $M\to\infty$:

$$\text{l.i.m.}\, F_M^{(l)}(X_j)=F^{(l)}(X_j),\quad l=0,\ 1,\ \ldots, \tag{29}$$

$$\text{l.i.m.}\, F_{M,i}(X)_j\equiv F_i(X)_j,\quad i=1,\ 2,\ 3 \tag{30}$$

and

$$F(X_j)=F_1(X)_j+F_2(X)_j-F_3(X)_j. \tag{31}$$

LEMMA 2.

$$E\Big[\Big(\sum_{j\in[K_N]} F_i(X)_j\Big)^2\Big]\leqslant CN^d,\ i=2,\ 3. \tag{32}$$

LEMMA 3.

$$E\Big[\Big(\sum_{j\in[K_N]} \sum_{(p)_m}' \prod_{k=1}^{m} h(j-p_k)\xi_{p_k}\Big)^2\Big]\asymp\Lambda^{2m}(N)\,N^{2d-\alpha m}, \tag{33}$$

$$E\Big[\Big(\sum_{j\in[K_N]} \sum_{l=m+1}^{\infty} \sum_{(p)_l}' \prod_{k=1}^{l} h(j-p_k)\xi_{p_k}e_l/l!\Big)^2\Big]=o\big(\Lambda^{2m}(N)\,N^{2d-\alpha m}\big). \tag{34}$$

LEMMA 4.

$$N^{\alpha m-2d}\int\Big(\Lambda^{-m}(N)\sum_{s\in[K_{Nt}]} \prod_{j=1}^{m}{}' h(s-[x_j]) - \int_{K_{Nt}} ds \prod_{j=1}^{m} \bar h(s-x_j)\Big)^2 dx^m\to 0\quad (N\to\infty), \tag{35}$$

where

$$\begin{aligned}&\bar h(t)=b(t/|t|)\,|t|^{-\beta},\quad t\in\mathbf{R}^d,\\ &[t]=([t^{(1)}],\ \ldots,\ [t^{(d)}])\in\mathbf{Z}^d,\quad t=(t^{(1)},\ \ldots,\ t^{(d)})\in\mathbf{R}^d,\end{aligned} \tag{36}$$

$[a]$ is the greatest integer in $a\in\mathbf{R}$,

$$\prod_{j=1}^{m}{}' h(s-p_j)=\begin{cases} h(s-p_1)\ldots h(s-p_m), & \text{if}\quad p_i\neq p_j,\ i\neq j,\\ 0 & \text{otherwise.}\end{cases} \tag{37}$$

LEMMA 5. As $N\to\infty$ the finite-dimensional distributions of the random field

$$e_m/m!\,\big(\Lambda^m(N)\,N^{d-\alpha m/2}\big)^{-1}\sum_{j\in[K_{Nt}]} \sum_{(p)_m}' \prod_{k=1}^{m} h(j-p_k)\xi_{p_k},\quad t\in\mathbf{R}_+^d \tag{38}$$

converge to the corresponding distributions of the field $Z_m'(t)$ of (13).

Considering (24), Theorem 2 follows from Lemmas 1-5.

3. Proof of Lemma 1. We shall prove (29). Let M' < M". According to (16)

$$\delta=\delta(M',\ M'')\equiv E\left[\left(F^{(l)}_{M'}(X_0)-F^{(l)}_{M''}(X_0)\right)^2\right]=$$

$$=\sum_{k,\ k'} c_k c_{k'}\left(k!/(k-l)!\right)\left(k'!/(k'-l)!\right)\sum_{i=0}^{k-l}\sum_{i'=0}^{k'-l} C^i_{k-l}C^{i'}_{k'-l}\times$$

$$\times\sum\nolimits^{*} h(p_1)\dots h(p_i)h^{v_1}(q_1)\dots h^{v_r}(q_r)h(p'_1)\dots h(p'_{i'})h^{v'_1}(q'_1)\dots h^{v'_{r'}}(q'_{r'})\times$$

$$\times E[\xi(p_1)\dots\xi(p_i)\xi^{v_1}(q_1)\dots h^{v_r}(q_r)\xi(p'_1)\dots\xi(p'_{i'})\xi^{v'_1}(q'_1)\dots\xi^{v'_{r'}}(q'_{r'})], \tag{39}$$

where $\xi(p) = \xi_p$, the sum $\sum_{k,\ k'}$ is taken over all $k,\ k' = \max(M',\ l),\ \dots,\ M''$, the summation in Σ^* is over all $(p)_i\in(\mathbf{Z}^d)^i_0$, $(p')_{i'}\in(\mathbf{Z}^d)^{i'}_0$, over all partitions $(V)\in(V)(k-l-i)$, $(V')\in(V)(k'-l-i')$, respectively $(V) = (V_1,\ \dots,\ V_r)$, finally, over all collections $r \geq 1$, $(V') = (V'_1,\ \dots,\ V'_{r'})$, $r' \geq 1$ such that $(q)_r\in(\mathbf{Z}^d)^r_0$, $(q')_{r'}\in(\mathbf{Z}^d)^{r'}_0$ and $(p)_i\cap(q')_{r'}=\varnothing$ and $(p')_{i'}\cap(q')_{r'}=\varnothing$. We note that the expectation in (39) is equal to zero if any of the indices $p_1,\ \dots,\ p_i$ ($p'_1,\ \dots,\ p'_{i'}$) does not appear in the collection $(p')_{i'}\cup(q')_{r'}$ (in the collection $(p)_i\cup(q)_r$, respectively), and the number of all arrangements of the indices $p_1,\ \dots,\ p_i$ in the set $(p')_{i'}\cup(q'_{r'})$ does not exceed $(i' + r')! \geq k'!$ In accord with this remark and also thanks to the inequalities

$$\sum_{t\in\mathbf{Z}^d}|h(t)|^k\leqslant\sum_{t\in\mathbf{Z}^d}|h(t)|^2=1,\quad k\geqslant 2 \tag{40}$$

and

$$\bar{\mu}_{n_1}\dots\bar{\mu}_{n_s}\leqslant\bar{\mu}_n,\quad n_1+\dots+n_s\leqslant n, \tag{41}$$

$n_1,\ \dots,\ n_s = 1,\ 2,\ \dots$ (we recall that $\bar{\mu}_n = E|\xi(p)|^n$ and $\bar{\mu}_2 = 1$), we get from (39) that

$$\delta\leqslant\sum_{k,\ k'}|c_k|\,|c_{k'}|\left(k!/(k-l)!\right)\left(k'!/(k'-l)!\right)\sum_{i=0}^{k-l}\sum_{i'=0}^{k'-l}C^i_{k-l}C^{i'}_{k'-l}\ \bar{\mu}_{k+k'-2l}k!\ k'!\sum_{(V)(k-l-i)}\ \sum_{(V)(k'-l-i')}1. \tag{42}$$

We note that

$$\sum_{(V)(k)}1=E[(\zeta-1)^k],\quad k\geqslant 1, \tag{43}$$

where ζ is a random variable with Poisson distribution with mean 1. One has the estimate [1]

$$|E[(\zeta-1)^k]|\leqslant 2^k k!. \tag{44}$$

Considering (9), we get from (40)-(44) that $\delta(M',\ M'') \to 0$ $(M',\ M'' \to \infty)$ and thus (29) is true. (30) and (31) are proved analogously.

Proof of Lemma 2. We prove (32) for $i = 2$. We have

$$E\left[\left(\sum_{j\in[K_N]}F_2(X)_j\right)^2\right]=\sum_{j,\ j'\in[K_N]}E[F_2(X)_jF_2(X)_{j'}].$$

We consider the mean

$$E[\xi(p_1)\dots\xi(p_l)\eta(v_1;\ i_1,\ q_1)\dots\eta(v_r;\ i_r,\ q_r)\xi(p'_1)\dots\xi(p'_{l'})\eta(v'_1;\ i'_1,\ q'_1)\dots\eta(v'_{r'};\ i'_{r'},\ q'_{r'})], \tag{45}$$

$\eta(v;\ i,\ q) = \eta_q(v;\ i)$, occurring in the expectation $E[F_2(X)_jF_2(X)_{j'}]$ (cf. (27)). Here

$$(p)_l=(p_1, \ldots, p_l)\in(\mathbf{Z}^d)_0^l,\ (p')_{l'}=(p'_1, \ldots, p'_{l'})\in(\mathbf{Z}^d)_0^{l'},\ (q)_r=(q_1, \ldots, q_r)\in(\mathbf{Z}^d)_0^r$$
$$(q')_{r'}=(q'_1, \ldots, q'_{r'})\in(\mathbf{Z}^d)_0^{r'},\ (V_1, \ldots, V_r)\in(V)(k-l),\ (V'_1, \ldots, V'_{r'})\in(V)(k'-l')$$

and in addition $(p)_l \cap (q)_r=\varnothing$ and $(p')_{l'}\cap(q')_{r'}=\varnothing$. The following cases are possible:

(1) $(q)_r\cap(q')_{r'}\neq\varnothing$,
(2) $(p)_l\cap(q')_{r'}\neq\varnothing$ and $(p')_{l'}\cap(q)_r\neq\varnothing$,
(3) all other cases.

From the conditions $(i_1, \ldots, i_r)\neq 0$, $(i'_1, \ldots, i'_{r'})\neq 0$ and $E[\eta(v; 1, q)] = 0$ it follows that in case (3) the expectation (45) is equal to zero. It is also clear that this expectation is nonzero only when $(p)_l\subseteq(p')_{l'}\cup(q')_{r'}$ and $(p')_{l'}\subseteq(p)_l\cup(q)_r$. Using (41) it is easy to show that in any case (45) does not exceed the numbers $\bar{\mu}_{k+k'}2^{(k+k'-l-l')/2}$. Thus

$$|E[F_2(X)_jF_2(X)_{j'}]|\leqslant\sum_{k,\,k'=0}^{\infty}|c_k|\,|c_{k'}|\sum_{l=0}^{k}\sum_{l'=0}^{k'}C_k^lC_{k'}^{l'}\bar{\mu}_{k+k'}2^{(k+k'-l-l')/2}\sum_{(V)(k-l)}\sum_{(V)(k'-l')}\sum_{(i),(i')}\left(\sum\nolimits_1+\sum\nolimits_2\right)\times$$
$$\times|h(j-p_1)\ldots h(j-p_l)h^{v_1}(j-q_1)\ldots h^{v_r}(j-q_r)h(j'-p'_1)\ldots h(j'-p'_{l'})h^{v'_1}(j'-q'_1)\ldots h^{v'_{r'}}(j'-q'_{r'}), \tag{46}$$

where Σ_1, Σ_2 denote sums over all collections $(p)_l\in(\mathbf{Z}^d)^l$, $(q)_r\in(\mathbf{Z}^d)^r$, $(p')_{l'}\in(\mathbf{Z}^d)^{l'}$, $(q')_{r'}\in(\mathbf{Z}^d)^{r'}$, $(p)_l\subseteq(p')_{l'}\cup(q')_{r'}$, $(p')_{l'}\subseteq(p)_l\cup(q)_r$, corresponding to cases (1) and (2), respectively. Using (40) we get

$$\sum\nolimits_1\ldots\leqslant k!\ k'!\sum_p h^2(j-q)h^2(j'-q), \tag{47}$$
$$\sum\nolimits_2\ldots\leqslant k!\ k'!\left(\sum_p h^2(j-p)\,|h(j'-p)|\right)\left(\sum_p|h(j-p)|\,h^2(j'-p)\right), \tag{48}$$

where p, q run through the lattice Z^d. We write $G_1(j-j')$, $G_2(j-j')$ for the right sides of (47), (48), respectively, without the factorials. We shall show that

$$\sum_{j,j'\in[K_N]}G_i(j-j')\leqslant CN^d,\quad i=1,\ 2, \tag{49}$$

whence together with (46)-(48), (43), (44), and (9) there follows (32) for i = 2. In turn (49) follows from

$$\sum_p G_i(p)<+\infty,\quad i=1,\ 2. \tag{50}$$

For i = 1 (50) is obvious, since $\sum_p h^2(p)<+\infty$. In the case i = 2 we use the form of the function h (7). Let L: $R_+ \to R_+$ be a function which is slowly varying (s.v.) at infinity. It is known [25] that for any u_0, $\gamma > 0$

$$u^{\gamma}L(Nu)/L(N)\to u^{\gamma}(N\to\infty)\quad\text{uniformly in}\quad u\in(0,\ u_0], \tag{51}$$
$$L(Nu)/u^{\gamma}L(N)\to 1/u^{\gamma}(N\to\infty)\quad\text{uniformly in}\quad u\in[u_0,\ \infty). \tag{52}$$

Thus, for $\gamma < 0$

$$G_2(p)\leqslant C\left(\sum_{q\neq 0,\,p}|p-q|^{\gamma-\beta}|q|^{2(\gamma-\beta)}\right)^2\leqslant C\left(\int_{\mathbf{R}^d}dx\,\varphi_{\beta-\gamma}(x)\,|p-x|^{\gamma-\beta}\right)^2, \tag{53}$$

where $\varphi_\beta(x)=1$, if $|x| \le 1$, $= |x|^{-2\beta}$, if $|x|>1, x\in \mathbf{R}^d$. It is easy to verify that the right side of (53) does not exceed $C/|p|^{2(\beta-\gamma)}$. If γ is sufficiently small, it follows from the condition $\beta > d/2$ that (50) converges for $i = 2$.

We consider (32) for $i = 3$. According to the definition (cf. (28)) and (40), (41) we have

$$|E[F_3(X)_j F_3(X)_{j'}]| \leqslant \sum_{k,k'=0}^{\infty} |c_k||c_{k'}| \sum_{l=0}^{\min(k,k')} C_k^l C_{k'}^l l! \times$$

$$\times \bar{\mu}_{k+k'-2l} \sum_{(V)(k-l)} \sum_{(V)(k'-l)} \left\{ \left(\sum_p |h(j-p)|^3 |h(j'-p)| \right)^2 + \sum_p |h(j-p)h(j'-p)|^3 \right\}.$$

The rest of the proof is just like the case $i = 2$ considered above.

Before the proof of Lemma 3 we give the

Proof of Lemma 4. We denote by δ the left side of (35). Without loss of generality we shall assume that the ratio $\Lambda(|t|)/|t|^\beta$ is bounded at the origin and the function $h(t) = b(t/|t|)\Lambda(|t|)|t|^{-\beta}$ is defined for all $t\in R^d$ and is square integrable in R^d. We have: $\delta \le 2(\delta_1 + \delta_2)$, where

$$\delta_1 = N^{\alpha m-2d}\Lambda^{-2m}(N) \int dx^m \left\{ \sum_{s\in[K_{Nt}]} \left(\prod_{j=1}^{m}{}' h(s-[x_j]) - \prod_{j=1}^{m} h(s-x_j) \right) \right\}^2, \tag{54}$$

$$\delta_2 = N^{\alpha m-2d} \int dx^m \left(\sum_{s\in[K_{Nt}]} \prod_{j=1}^{m} h(s-x_j)/\Lambda(N) - \int_{K_{Nt}} \prod_{j=1}^{m} \bar{h}(s-x_j)\,ds \right)^2. \tag{55}$$

We estimate δ_2. Using the form of the functions h, $\bar{h}$ and the change of variables $y_j = Nx_j$, $\sigma = Ns$, we get

$$\delta_2 = C \int dy^m \left\{ \int_{K_t} d\sigma \left(\prod_{j=1}^{m} b((\sigma'-y_j)/|\sigma'-y_j|)|\sigma'-y_j|^{-\beta} \Lambda(N|\sigma'-y_j|)/\Lambda(N) - \prod_{j=1}^{m} b((\sigma-y_j)/|\sigma-y_j|)|\sigma-y_j|^{-\beta} \right) \right\}^2, \tag{56}$$

where $\sigma' = [N\sigma]/N \to \sigma (N\to\infty)$. Thanks to the continuity of the function $b(\cdot)$, one can write

$$\delta_2 \leqslant \varepsilon(N) J_\beta + \delta_2',$$

where $\varepsilon(N)\to 0\,(N\to\infty)$, $J_\beta = \int dy^m \left(\int_{K_t} d\sigma \prod_{j=1}^{m} |y_j-\sigma|^{-\beta} \right)^2$ and

$$\delta_2' = C \int dy^m \left\{ \int_{K_t} d\sigma \left(\prod_{j=1}^{m} |y_j-\sigma'|^{-\beta} \Lambda(N|y_j-\sigma'|)/\Lambda(N) - \prod_{j=1}^{m} |y_j-\sigma|^{-\beta} \right) \right\}^2. \tag{57}$$

It is known that

$$J_\beta = C \int dx^m |\hat{1}_{K_t}(x_1+\ldots+x_m)|^2 \prod_{j=1}^{m} |x_j|^{2(\beta-d)} < +\infty \tag{58}$$

for $d/2 < \beta < d$, $\alpha = 2\beta - d < d/m$ (cf. [7, p. 26] and also [22, Proposition 3.7]). To estimate δ_2' we need

LEMMA 6. Let $y, \sigma\in\mathbf{R}^d$, $\sigma'=[\sigma N]/N$, $\gamma>0$. Then

$$\tilde{\delta} \equiv \left| \Lambda(N|\sigma'-y|)|\sigma'-y|^{-\beta}/\Lambda(N) - |\sigma-y|^{-\beta} \right| \leqslant \varepsilon(N)|\sigma-y|^{-\beta\pm\gamma}, \tag{59}$$

where $\varepsilon(N) = \varepsilon(N, y, \sigma)$ is a function of y, σ, bounded uniformly in N, tending to zero for fixed y, σ, as $N\to\infty$, and $|x|^{-\beta\pm\gamma} = \max(|x|^{-\beta+\gamma}, |x|^{-\beta-\gamma})$.

Proof. We note that $|\sigma'-\sigma|\leqslant\sqrt{d}/N$. Let $|\sigma-y|\leqslant 2\sqrt{d}/N$, so $N|\sigma'-y|\leqslant N(|\sigma-y|+|\sigma'-\sigma|)\leqslant 3\sqrt{d}$ and

$$\tilde{\delta}\leqslant N^{\beta}\Lambda(N|\sigma'-y|)/(N|\sigma'-y|)^{\beta}\Lambda(N)+|\sigma-y|^{-\beta}\leqslant C|\sigma-y|^{-\beta-\gamma}$$

by virtue of (52) and the stipulation about the boundedness of $\Lambda(|t|)|t|^{-\beta}$ at the origin. Now if $2\sqrt{d}/N \leq |\sigma - y| \leq 1$, then $2|\sigma-y|\geqslant|\sigma'-y|\geqslant|\sigma-y|-|\sigma'-\sigma|\geqslant|\sigma-y|/2$ and

$$\tilde{\delta}\leqslant C|\sigma-y|^{-\beta}\left|\Lambda(N|\sigma'-y|)/\Lambda(N)-1\right|+|\sigma-y|^{-\beta}\left|(|\sigma-y|/|\sigma'-y|)^{\beta}-1\right|\leqslant\varepsilon(N)|\sigma-y|^{-\beta-\gamma},$$

where $\varepsilon(N)$ satisfies the conditions of the lemma. The case $|\sigma - y| \geq 1$ is considered analogously.

From Lemma 6 and (58) for γ sufficiently small it follows that $\delta_2^1 \to 0$ $(N \to \infty)$ and thus $\delta_2 \to 0$ $(N \to \infty)$. In particular, we get from this the estimate

$$\int dx^m\Big(\sum_{s\in[K_{Nt}]}\prod_{j=1}^{m}\Lambda(|s-x_j|)|s-x_j|^{-\beta}\Big)^2\leqslant CJ_{\beta}N^{2d-\alpha m}\Lambda^{2m}(N), \tag{60}$$

which is used below.

We proceed to estimate δ_1. We note immediately that in (54) the product $\prod_{j=1}^{m} h(s-[x_j])$ (37) can be replaced by the ordinary product $\prod_{j=1}^{m}{}' h(s-[x_j])$, since

$$\int dx^m\Big\{\sum_{s\in[K_{Nt}]}\Big(\prod_{j=1}^{m}{}' h(s-[x_j])-\prod_{j=1}^{m}h(s-[x_j])\Big\}^2= \tag{61}$$

$$=\sum{}'\Big(\sum_{s\in[K_{Nt}]}h(s-p_1)\ldots h(s-p_m)\Big)^2\leqslant C\sum_{s_1,s_2\in[K_{Nt}]}\sum_{p\in Z^d}h^2(s_1-p)h^2(s_2-p)\leqslant CN^d,$$

where the sum Σ' is taken over all $(p)_m\in(Z^d)^m\backslash(Z^d)^m_0$, and $N^d N^{\alpha m-2d}\Lambda^{2m}(N)=o(1)$ for $\alpha m < d$. Thus, after the change of variables $x_j = Ny_j$, $s = N\sigma$ we get

$$\delta_1\leqslant o(1)+CN^{-2d}\Lambda^{-2m}(N)\int dy^m\Big\{\sum_{N\sigma\in[K_{Nt}]}\prod_{j=1}^{m}b\big((\sigma-y_j')/|\sigma-y_j'|\big)\times$$

$$\times\Lambda(N|\sigma-y_j'|)|\sigma-y_j'|^{-\beta}-\prod_{j=1}^{m}b\big((\sigma-y_j)/|\sigma-y_j|\big)\Lambda(N|\sigma-y_j|)\,|\sigma-y_j|^{-\beta}\Big\}^2\leqslant o(1)+\varepsilon(N)J'+\delta_1', \tag{62}$$

where $y_j' = [Ny_j]/N$,

$$J'=\Lambda^{-2m}(N)N^{-2d}\int dy^m\Big(\sum_{N\sigma\in[K_{Nt}]}\prod_{j=1}^{m}\Lambda(N|\sigma-y_j|)|\sigma-y_j|^{-\beta}\Big)^2\leqslant CJ_{\beta}$$

(cf. (60)); $\varepsilon(N) \to 0$ $(N \to \infty)$ thanks to the continuity of the function $b(\cdot)$ and

$$\delta_1'=C\Lambda^{-2m}(N)N^{-2d}\int dy^m\Big\{\sum_{N\sigma\in[K_{Nt}]}\Big(\prod_{j=1}^{m}\Lambda(N|\sigma-y_j'|)/\Lambda(N)\ |\sigma-y_j'|^{\beta}-\prod_{j=1}^{m}\Lambda(N|\sigma-y_j|)/\Lambda(N)|\sigma-y_j|^{\beta}\Big)\Big\}^2. \tag{63}$$

Using Lemma 6 and (60), one can prove the relation $\delta_1^1 = o(1)$ analogously to the proof of the relation $\delta_2^1 = o(1)$ above. Lemma 4 is proved.

Proof of Lemma 3. Since

$$E\Big[\Big(\sum_{j\in[K_N]}\sum_{(p)_m}{}'\prod_{k=1}^{m}h(j-p_k)\,\xi(p_k)\Big)^2\Big]=m!\sum_{(p)m}\Big(\sum_{j\in[K_N]}\prod_{k=1}^{m}{}' h(j-p_k)\Big)^2,$$

(33) follows from (35). To prove (34) we consider the relation

$$a_t(o) = \sum_p h(p)h(t-p)/(\Lambda^2(|t|)|t|^{-\alpha}) =$$
$$= \int_{\mathbf{R}^d} dy\, b(y'/|y'|)\, b((o-y')|o-y'|)\,\Lambda(|t|\,|y'|)\,\Lambda(|t|\,|o-y'|)\times$$
$$\times |y'|^{-\beta}|o-y'|^{-\beta}/\Lambda^2(|t|),$$

where $y'=[|t|y]/|t|$, $o=t/|t|$. Arguing just as in the proof of the preceding lemma, it is easy to verify that $a_t(o) \to a(o)$ ($|t| \to \infty$) uniformly in $o \in S_{d-1}$, i.e., (14) holds. In the same way one can prove

$$\sum_p |h(p)h(t-p)| \leqslant \min(1,\ \Lambda^2(|t|)|t|^{-\alpha}),\quad t \in \mathbf{Z}^d. \tag{64}$$

We denote by δ_N the left side of (34). Then

$$\delta_N = \sum_{l=m+1}^{\infty} \sum_{(p)_l} \Big(\sum_{j \in [K_N]} {\prod_{k=1}^{l}}' h(j-p_k)\Big)^2 e_l^2/l! \equiv \delta'_N + \delta''_N,$$

where in δ'_N the sum over $l \geq m+1$, $l < d/\alpha$, appears and in δ''_N that over $l \geq d/\alpha$. By (33), $\delta'_N = o(\Lambda^{2m}(N)\, N^{2d-\alpha m})$. We note that for $l \geq [d/\alpha]$

$$i_l \equiv \sum_{(p)_l} \Big(\sum_{j \in [K_N]} {\prod_{k=1}^{l}}' h(j-p_k)\Big)^2 \leqslant CN^{d+\varepsilon}, \tag{65}$$

where $\varepsilon > 0$ is arbitrarily small and $C = C(\varepsilon)$ is independent of l. In fact, according to (64),

$$i_l \leqslant \sum_{j_1, j_2 \in [K_N]} \Big(\sum_p |h(j_1-p)h(j_2-p)|\Big)^l \leqslant C(\varepsilon) \sum_{j_1, j_2 \in [K_N]} \min(1, |j_1-j_2|^{\varepsilon-\alpha l^*}),$$

where $l^* = [d/\alpha]$. Thus (65) is true and also $\delta''_N \leqslant CN^{d+\varepsilon} \sum_{l=m+1}^{\infty} e_l^2/l!$. It is easy to see that the last series converges. For $\varepsilon > 0$ sufficiently small, $N^{d+\varepsilon}$ has the order indicated on the right side of (34).

4. In this section we shall prove Lemma 1 of [20] on the convergence with respect to distribution of the discrete multiple integrals to the "continuous" Ito—Wiener integrals, and with its help we complete the proof of Theorem 2.

We consider a sequence $(\Delta)_N$, N = 1, 2, ..., of partitions of the space R^d into d-dimensional cubes Δ of identical dimensions; diam $\Delta \to 0$ ($N \to \infty$). Let there be given for any $N \geq 1$ a family of real random variables $\xi_N = (\xi_N(\Delta))_{\Delta \in (\Delta)_N}$ with mean 0, variance equal to the Lebesgue measure of Δ, and finite moments of any order. The collection of such families of random variables we denote by $\mathfrak{X}_N$. We consider the Hilbert space $(L^2)^n = L^2((R^d)^n, dx^n)$ of functions f: $(R^d)^n \to$ C and the subspace $(L^2)^n_N \subset (L^2)^n$, formed by functions f, assuming constant value $f^{\Delta_1 \ldots \Delta_n}$ on sets $\Delta_1 \times \ldots \times \Delta_n \in (\Delta)^n_N$, and vanishing on "diagonals" $f^{\Delta_1 \ldots \Delta_n} = 0$, if $\Delta_i = \Delta_j$ for $i \neq j$, i, j = 1, ..., n.

By *the discrete multiple integral* of the function $f \in (L^2)^n_N$ with respect to the family $\xi_N \in \mathfrak{X}_N$ we shall mean the sum

$$I_N(f;\ N)=\sum_{\Delta_1,\dots,\Delta_n} f^{\Delta_1\dots\Delta_n}\xi_N(\Delta_1)\dots\xi_N(\Delta_n) \tag{66}$$

(the series on the right side of (66) converges in mean square). For any $f\in(L^2)^n_N$, $g\in(L^2)^m_N$, n, m = 1, 2, ..., one has

$$E[I_n(f;\ N)\overline{I_m(g;\ N)}]=\delta_{mn}n!\,(\mathrm{sym}\,f,\ g)_n, \tag{67}$$

$$E[I_n(f;\ N)]=0, \tag{68}$$

where δ_{mn} is the Kronecker symbol, $(\cdot,\ \cdot)_n$ is the scalar product in $(L^2)^n$, and sym f is the symmetrization of the function $f=f(x_1,\dots,x_n)$, $x_1,\dots,x_n\in\mathbf{R}^d$.

We also introduce the set $\mathscr{Z}$ of all homogeneous real (stationary) random measures (r.m.) $\xi=\xi(dx)$ in R^d with zero mean and variance $E[(\xi(dx))^2]=dx$, assuming independent values on disjoint subsets (such random measures are also called "generalized white noise" [9]). The distribution of any r.m. $\xi\in\mathscr{Z}$ is infinitely divisible, and its characteristic functional can be represented by the Levy–Khinchin formula [9]. Important special cases of such r.m. are *Gaussian white noise* and (centered) *Poisson* r.m.

We shall call a function $f:(R^d)^n\to C$ *simple*, if $f\in(L^2)^n_N$ for some $N\geq 1$ and $f^{\Delta_1\dots\Delta_n}=0$ everywhere except for a finite number of "rectangles" $\Delta_1\times\dots\times\Delta_n\in(\Delta^n_N)$. The *multiple Ito–Wiener integral* $I_n(f)\times\int f(x_1,\dots,x_n)\xi(dx_1)\dots\xi(dx_n)$ of the function $f\in(L^2)^n$ with respect to the random measure $\xi\in\mathscr{Z}$ is defined as the limit in $L^2(\Omega)$ of the integral sums $I_n(f_N;\ N)$ of the form (66), where $\xi_N(\Delta)=\xi(\Delta)$ and $(f_N)_{N\geq 1}$ is a sequence of simple functions converging to f in $(L^2)^n$. The integral $I_n(f)$ also has the orthogonality properties (67), (68) (cf. for more details [7, 9, 11, 16]).

LEMMA 7. Let us assume that there is given a r.m. $\xi\in\mathscr{Z}$ and a sequence $\xi_N\in\mathscr{Z}_N$, $N\geqslant 1$, such that for any $N\geqslant 1$, $\Delta\in(\Delta)_N$ the distribution of the sum $\tilde{\xi}_M(\Delta)=\sum_{\Delta'\in(\Delta)_M:\Delta'\subset\Delta}\xi_M(\Delta')$ converges as $M\to\infty$ to the distribution of the random variable $\xi(\Delta)$. Suppose given a sequence of functions $f_N\in(L^2)^n_N$, $N\geqslant 1$, which converges as $N\to\infty$ to function f in the norm of space $(L^2)^n$, n = 1, 2, ... Then the distribution of the random variable $I_n(f;\ N)$ converges as $N\to\infty$ to the distribution of the integral $I_n(f)$.

Proof. Let us assume for simplicity that the functions f_N are real. By virtue of the hypotheses of the theorem, for any $\varepsilon>0$ there exists a $N\geq 1$ and a simple function $g\in(L^2)^n_N$ such that $||f-g||_n<\varepsilon$ and $||f_M-g||_n<\varepsilon$ for all $M\geq n$, where $||\cdot||_n$ is the norm in $(L^2)^n$. Since

$$|E[\exp\{iaI_n(f_M;\ M)\}]-E[\exp\{iaI_n(g;\ M)\}]|\leqslant$$
$$\leqslant|a|E^{1/2}[|I_n(f_M-g;\ M)|^2]\leqslant|a|\sqrt{n!}\,\varepsilon,\quad M\geqslant N,\quad a\in\mathbf{R}$$

(the analogous inequality is true upon change of the discrete integrals $I_n(\cdot;\ M)$ to $I_n(\cdot)$), then

$$|E[\exp\{iaI_n(f_M;\ M)\}]-E[\exp\{iaI(f)\}]|\leqslant 2|a|(n!)\,\varepsilon^{1/2}+$$
$$+|E[\exp\{iaI_n(g;\ M)\}]-E[\exp\{iaI_n(g)\}]|. \tag{69}$$

By definition, $I_n(g)=P(\xi(\Delta):\Delta\in(\Delta)_N)$, where $P(\cdot)$ is a polynomial depending on a finite number of real variables x_Δ, $\Delta\in(\Delta)_N$. Let us assume that the sequence $(\Delta)_N$, $N\geq 1$, is monotone, i.e., that $(\Delta)_N\subseteq(\Delta)_{N+1}$ $N\geqslant 1$. Then the discrete integral $I_n(g;\ M)$ is equal to the same polynomial in the random variables $\tilde{\xi}_M(\Delta)$, $\Delta\in(\Delta)_N$. From the hypotheses of the lemma and the independence of the values of the r.m. ξ on disjoint sets it follows that the distribution $P(\tilde{\xi}_M(\Delta): \Delta \in (\Delta)_N)$ converges as $M \to \infty$ to the distribution of the random variable $P(\xi(\Delta): \Delta \in (\Delta)_N)$. In view of the arbitrariness of $\varepsilon > 0$ in (69), from what was said above there follows the assertion of Lemma 7. In the case of nonmonotone partitions $(\Delta)_N$ these arguments follow after slight modifications.

Analogously one can get conditions for the convergence in distribution of "continuous" multiple integrals corresponding to different random measures [21]. In the case when the "variational measures" $E[(\xi_N(dx))^2] = \mu_N(dx)$ depend on N, the condition $f_N \to f (N \to \infty)$ in $(L^2)^n$ should be replaced by the condition of "uniform approximability of the sequence $(f_N)_{N=1,\ldots,\infty}$ by simple function" [21]. The convergence in distribution of the multiple integrals corresponding to different Gaussian measures was used in [8, 16].

Proof of Lemma 5. We denote by [k/N, (k + 1)/N) the d-dimensional cube $[k^{(1)}/N, (k^{(1)}+1)/N) \times \ldots \times [k^{(d)}/N, (k^{(d)}+1)/N) \subset \mathbf{R}^d$, $k = (k^{(1)}, \ldots, k^{(d)}) \in \mathbf{Z}^d$. We consider the sequence of partitions $(\Delta)_N$, N = 1, 2, ... of the space R^d into cubes $\Delta = [k/N, (k+1)/N) \in (\Delta)_N$, $k \in \mathbf{Z}^d$ and we set

$$\xi_N(\Delta) = \xi_k / N^{d/2}, \quad \Delta = [k/N, (k+1)/N),$$

$$f(x_1, \ldots, x_m) = \int_{K_t} \prod_{i=1}^{m} \bar{h}(s - x_j)\, ds, \quad x_1, \ldots, x_m \in \mathbf{R}^d, \tag{70}$$

$$f_N(x_1, \ldots, x_m) = \Lambda^m(N) N^{-d+(\alpha+d)m/2} \sum_{s \in [K_{Nt}]} {\prod_{j=1}^{m}}' h(s - k_j), \tag{71}$$

if $(x_1, \ldots, x_m) \in [k_1/N, (k_1+1)/N) \times \ldots \times [k_m/N, (k_m+1)/N) \in (\Delta)_N^m$, $k_1, \ldots, k_m \in \mathbf{Z}^d$. It follows from the central limit theorem that the distribution of the sum $\tilde{\xi}_M(\Delta)$ converges as $M \to \infty$ to the distribution $\xi(\Delta)$ for any $\Delta \in (\Delta)_N$, $N \geqslant 1$, where $\xi(dx)$ is real Gaussian white noise in R^d with variance dx. According to Lemma 4, $f_N \to f$ $(N \to \infty)$ in $(L^2)^m$. The last assertion is also true for arbitrary linear combinations of functions (70) and (71), corresponding to different points $t \in \mathbf{R}_+^d$. It remains to use Lemma 7.

LITERATURE CITED

1. R. Bentkus and Ya. Rudzkis, "Exponential estimates of the distribution of random variables," Liet. Mat. Rinkinys, 20, No. 1, 15-30 (1980).
2. B. V. Gnedenko and A. N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables, Addison-Wesley (1968).
3. V. V. Gorodetskii, "Convergence to semistable Gaussian processes," Teor. Veroyatn. Primen., 22, No. 3, 513-522 (1977).
4. V. V. Gorodetskii, "Estimate of the rate of convergence in limit theorems for random processes," Author's Abstract of Candidate's Dissertation, Leningrad (1977).
5. Yu. A. Davydov, "Invariance principle for random processes," Teor. Veroyatn. Primen., 15, No. 3, 498-509 (1979).
6. R. L. Dobrushin, "Self-similarity and the renormalization group of generalized random fields," in: Multicomponent Random Systems [in Russian], Nauka, Moscow (1978), pp. 179-213.
7. R. L. Dobrushin, "Gaussian and their subordinated self-similar random generalized fields," Ann. Probab., 7, 1-28 (1979).
8. R. L. Dobrushin and P. Major, "Noncentral limit theorems for nonlinear functionals of Gaussian fields," Z. Wahr. Verw. Geb., 50, 27-52 (1979).
9. T. Hida, Stationary Stochastic Processes, Princeton Univ. Press (1970).
10. D. Iagolnitzer and B. Souillard, "Random fields and limit theorems," Ecole Polytechnique, Preprint (1979).
11. K. Ito, "Multiple Wiener integral," J. Math. Soc. Jpn., 3, 157-164 (1951).
12. I. A. Ibragimov and Yu. V. Linnik, Independent and Stationarily Connected Quantities [in Russian], Nauka, Moscow (1965).

13. H. Kesten and F. Spitzer, "A limit theorem related to a new class of self-similar processes," Z. Wahr. Verw. Geb., 50, 5-25 (1979).
14. A. N. Kolmogorov, "Wiener spiral and some other interesting curves in Hilbert space," Dokl. Akad. Nauk SSSR, 26, No. 2, 115-118 (1940).
15. J. Lamperti, "Semistable stochastic processes," Trans. Am. Math. Soc., 104, 62-78 (1962).
16. P. Major, "Multiple Wiener—Ito Integrals," Lect. Notes Math., Vol. 849, Springer-Verlag (1981).
17. M. Rosenblatt, "Independence and dependence," in: Proc. Fourth Berkeley Symp. Math. Statist. Probab., Berkeley (1961), pp. 431-443.
18. M. Rosenblatt, "Limit theorems for Fourier transforms of functionals of Gaussian sequences," Z. Wahr. verw. Geb., 55, 123-132 (1981).
19. Ya. G. Sinai, Theory of Phase Transitions [in Russian], Nauka, Moscow (1980).
20. D. Surgailis, "Convergence of sums of nonlinear functions of moving averages to self-similar processes," Dokl. Akad. Nauk SSSR, 257, No. 1, 51-54 (1981).
21. D. Surgailis, "Linear random fields and those subordinate to them," Doctoral Dissertation, Vilnius (1981).
22. D. Surgailis, "On infinitely divisible self-similar random fields," Z. Wahr. Verw. Geb., 80, 453-477 (1981).
23. D. Surgailis, "On L^2 and non-L^2-multiple stochastic integrals," in: Proc. Third IFIP Working Conf. Stoch. Diff. Syst., Lecture Notes Control Information Sciences, Vol. 36, Springer (1981), pp. 212-226.
24. M. S. Taqqu, "Self-similar processes and related ultraviolet and infrared catastrophes," Cornell Univ., Preprint (1979).
25. M. S. Taqqu, "Convergence of integrated processes of arbitrary Hermite rank," Z. Wahr. Verw. Geb., 50, 53-83 (1979).
26. W. Feller, Introduction to Probability Theory and Its Applications, Wiley (1968).