CENTRAL LIMIT THEOREM FOR FUNCTIONALS OF A LINEAR PROCESS

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The central limit theorem is proved for stationary related variables η_t, $t\in Z$ of the form $\eta_t = f(X_t)$ under the condition that $\Sigma|r_\eta(t)| < \infty$ and $\Sigma r_\eta(t) > 0$. Here $r_\eta(t) = E\eta_0\eta_t$, $X_t = \Sigma a(t-s)\xi_s$ is a linear process, the variables (ξ_s) are independent and identically distributed, and $f:\mathbf{R}\to\mathbf{R}$ belongs to some class of analytic functions containing, in particular, all polynomials. The proof is based on the method of cumulants.

INTRODUCTION

In the present paper we consider the question of the central limit theorem (c.l.t.) for stationary related variables η_t, $t\in Z$ of the form

$$\eta_t = f(X_t), \tag{1}$$

where

$$X_t = \sum_{s=-\infty}^{\infty} a(t-s)\xi_s \tag{2}$$

is a *linear process* [i.e., the random variables ξ_t, $t\in Z$ are independent and identically distributed, $E\xi_t = 0$, $E\xi_t^2 = 1$, the sequence $a(t)$, $t\in Z$ is nonrandom and satisfies the condition $\Sigma a^2(t) < \infty$]; $f:\mathbf{R}\to\mathbf{R}$ is a given function. In particular, any stationary Gaussian process (X_t) with absolutely continuous spectral density can be represented in the form (1). If the dependence of the process X_t decreases with distance sufficiently slowly (for example, the strong mixing condition does not hold), then the question of the asymptotic behavior of the sums $\sum_{t=0}^{N} f(X_t)$ is sufficiently complicated and the answer depends strongly on the function f considered. For a Gaussian process (X_t) this question was considered in Rosenblatt [11], Taqqu [16, 17], Dobrushin and Major [3], Gorodetskii [6], Giraitis [19], Giraitis and Surgailis [4], Breuer and Major [2], Sun [13], etc. It was established in Dobrushin and Major [3] that if the correlation function r(t) of the Gaussian process (X_t) behaves asymptotically like $|t|^{-\alpha}(0 < \alpha < 1)$, the Hermite rank of the function f (i.e., the index of the first nonzero coefficient c_k in the expansion $f=\sum_0^\infty c_k H_k$ in Hermite polynomials H_k) is equal to $m(\geqslant 1)$ and $\alpha m < 1$, then the processes $N^{1-\alpha m/2}\sum_{s=0}^{[Nt]} f(X_s)$ converge in distribution* to "strongly dependent" self-similar processes $Z_t^{(m)}$, $t > 0$ which are non-Gaussian for $m \geqslant 2$, representable with the help of multiple Ito–Wiener integrals. Surgailis [15] found an analogous result in the case of a (non-Gaussian) linear process (X_t); it is true that here the function f had to satisfy certain stringent analyticity conditions. The question of the c.l.t. for functionals $f(X_t)$ of a Gaussian process (X_t) was considered independently by Giraitis and Surgailis [4] and Breuer and Major [2] (cf. also the earlier paper of Sun [13]). Let $m(\geqslant 1)$ be the Hermite rank of the function f, r(t) be the correlation function of the Gaussian process (X_t). Then the condition $\Sigma|r(t)|^m < \infty$ is sufficient for the convergence $N^{-1/2}\sum_{s=0}^{n} f(X_s) \overset{d}{\Rightarrow} \mathcal{N}(0,\sigma^2)$, where $\sigma^2=\sum_t r_f(t)$ and $r_f(t) = Ef(X_0)f(X_t)$ [2]. An analogous result was found by Giraitis [5]. Moreover, it turned out that the conditions of asymptotic normality of the quantities $f(X_t)$ can

*We shall write "$\overset{d}{\Rightarrow}$" for convergence of finite-dimensional distributions.

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be expressed in terms of one correlation function $r_f(t)$, without using the concept of Hermite rank (although, on the other hand, this concept plays an important role in the course of the proof). Namely, the following theorem is true.

THEOREM 1 [4, 5]. Let (X_t) be a stationary Gaussian process with correlation function $r(t) \to 0 (t \to \infty)$; let the function $f: R \to R$ satisfy the conditions $Ef(X)_t = 0$, $Ef^2(X_t) < \infty$. If, in addition,

$$\sum |r_f(t)| < \infty, \tag{3}$$

$$\sum r_f(t) \equiv \sigma^2 > 0, \tag{4}$$

then

$$N^{-1/2} \sum_{s=1}^{[Nt]} f(X_s) \overset{d}{\Rightarrow} \sigma W(t), \tag{5}$$

where $(W(t))_{t \geqslant 0}$ is a standard Wiener process.

One should note that although conditions (3) and (4) imply $\mathrm{Var}\left(\sum_1^N f(X_t)\right) \sim N$, on the other hand the last relation is not sufficient for (5) (there is a counterexample in [4]).

It seems likely that the analogous theorem is also true for an arbitrary linear process (X_t). The result formulated below is obtained under rather strong restrictions on the function f. The latter are apparently due to the method of proof, based on the expansion of f in Appell polynomials $A_n(x)$, $n = 0, 1, \ldots,$

$$f(x) = \sum_0^\infty c_k A_k(x), \tag{6}$$

defined with the help of the generating function

$$\sum_{n=0}^\infty z_n A_n(x)/n! = \exp(zx)/E\exp(zX_t).$$

THEOREM 2. Let $(X_{(t)})$ be a linear process (1); let the variables ξ_t satisfy the Cramer condition, i.e., $\Delta \equiv E\exp(r_0|\xi_t|) < \infty$ for some $r_0 > 0$. Let us assume that f has the form (6), where

$$\sum_{k=0}^\infty |c_k| k! d^k < \infty \tag{7}$$

for some $d > 3(1 + 2\Delta)/r_0$ and (3) and (4) hold. Then (5) holds.

THEOREM 3. Let all the conditions of Theorem 2 hold except (3) and (4). Let us assume instead of this that $L_1(N) \equiv \sum_{-N}^{N} r_f(t) \to \infty\ (N \to \infty)$ and $r_f(t) = L(t)/|t|$, where $L(\cdot)$ is a slowly varying function (s.v.f.). Then (5) holds with $N^{-1/2}$ replaced by $(NL_1(N))^{-1/2}$.

The role which the expansion (6) plays in the proof of Theorems 2 and 3 is explained by the following facts. Firstly, there exist simple and natural formulas for the cumulants of the Appell polynomials (this is important since the proofs of Theorems 2 and 3 are based on the method of moments). Another important circumstance is the equivalence of the conditions (3) and $\Sigma|r(t)|^m < \infty$ for $m \geqslant 2$, where m is the Appell rank of the function f (cf. Sec. 4). Unfortunately, the Appell polynomials in general do not form an orthogonal system in L_2, and the question of when the function f can be expanded in a series (6) is quite unclear. One can however show that if the function f is analytic on the whole line and the coefficients of its power series decrease sufficiently rapidly, then one has an expansion (6), and (7) holds. In this case the coefficients c_k have a simple probabilistic meaning: $c_k = Ef^{(k)}(X_t)/k!$, where $f^{(k)}$ is the k-th derivative of f.

It is easy to give examples when the conditions of Theorems 1, 2, or 3 hold, but the process (X_t) does not satisfy the strong mixing condition (and is even nonregular). Some weaker mixing conditions introduced in [18] assume, roughly speaking, the convergence of the series $\Sigma|\alpha(t)|$, which can also not hold in the theorems formulated above. We note finally a result of Ibragimov [7], according to which the c.l.t. for the linear process (X_t) (1) itself holds, provided $\mathrm{Var}\left(\sum_0^N X_t\right)\to\infty\ (N\to\infty)$.

We give the content of the remaining sections briefly. Section 1 is devoted to the Appell polynomials and also to some questions of the convergence of series of such polynomials. In Sec. 2 we discuss the diagramatic formalism for the cumulants of the Appell polynomials of a linear process. In Sec. 3 the c.l.t. is proved for polynomials of a linear process, and in Sec. 4 we consider the general case of a function f.

1. Appell Polynomials

Let μ be some probability distribution on the line $\mathbf{R}$. We shall assume that all moments of the measure μ are finite, and the mean $\int x\,d\mu=0$. The *Appell polynomials* $A_n(x)$, $x\in\mathbf{R}$, $n=0, 1,\ldots$ corresponding to the distribution μ are defined with the help of the generating function

$$\sum_{n=0}^{\infty} z^n A_n(x)/n! = e^{zx}\Big/\int e^{zx}\,d\mu. \tag{8}$$

In this case, if the measure μ satisfies the *Cramer condition*

$$\int e^{r|x|}\,d\mu<\infty \tag{9}$$

for some $r > 0$, the series on the left side of (8) converges absolutely in some disk $\{|z|<r_0\}\subset \mathbf{C}$. The Appell polynomials can also be defined by the formula [14]

$$A_n(x)=\sum_{k=0}^{n} x^k \sum_{\{v\}(n-k)} (-1)^r \prod_{i=1}^{r} \chi(|v_i|), \tag{10}$$

where $\chi(k) = \chi(k, \mu)$ is the k-th cumulant of the measure μ, and the sum $\sum_{\{v\}(n)}$ is taken over all partitions $(v_1,\ldots,v_r)$, $r = 1, 2,\ldots$ of the set $\{1,\ldots,n\}$ such that $|v_i| \geqslant 2$ (we set $\sum_{\{v\}(0)}\ldots=1$ $\sum_{\{v\}(1)}\ldots=0$).

An important special case of the Appell polynomials is the Hermite polynomials corresponding to the standard Gaussian distribution $\mu(dx) = e^{-x^2/2}dx/\sqrt{2\pi}$. The Hermite polynomials are the unique orthogonal system of polynomials among the Appell polynomials [12].

We note the differentiation rule:

$$A_n'(x)=nA_{n-1}(x) \tag{11}$$

and the equation

$$\int A_n(x)\,d\mu=0,\quad n=1, 2, \ldots \tag{12}$$

The question of when the system $\{A_n\}_0^\infty$ forms a *basis* in the space $L^2(\mu)$ [i.e., when each function $f\in L^2(\mu)$ can be represented uniquely as a series $f=\sum_0^\infty c_k A_k$, converging in $L^2(\mu)$] is apparently open. In any case, such a requirement imposes stringent conditions on the smoothness of the measure μ. It is known that each basis is a minimal system [9]. The following assertion, given below without proof, was communicated to the author by D. Surgailis.

Proposition 1. Let μ satisfy the Cramer condition. In order that the system $\{A_n\}_0^\infty$ be minimal in the space $L^2(\mu)$, it is necessary and sufficient that there exist a density $p = d\mu/dx \in C^\infty(\mathbf{R})$ such that $Q_n\equiv p^{-1}d^np/dx^n\in L^2(\mu)$, $n = 1, 2,\ldots$. In the latter case the system $\{Q_n\}_0^\infty$ forms a biorthogonal system with the system $\{A_n\}_0^\infty$.

We introduce the class $\mathscr{A}(\mu)$ consisting of all functions $f\in L^2(\mu)$ having the form

$$f(x)=\sum_0^\infty c_k A_k(x), \tag{13}$$

where the series converges in $L^2(\mu)$.

Definition 1. By the Appell rank of a function $f\in\mathcal{A}(\mu)$ we mean the index of the first nonzero coefficient c_k in the expansion (13).

Although the given definition of the Appell rank relates to the coefficients of the series (13) rather than to its sum, in what follows we shall have to do with the narrower space of functions $f\in\mathcal{A}(\mu)$, for which this definition is proper and has a simple probabilistic meaning. We introduce the class $\tilde{\mathcal{A}}_d\,(d>0)$ of for now formal sums (13), where the coefficients c_k satisfy the condition

$$\sum_{k=0}^{\infty}|c_k|\,k!\,d^k<\infty. \tag{14}$$

Proposition 2. Let $r > 0$ be such that the function $\varphi(z)=\int e^{zx}d\mu$ is analytic inside the disk $|z| \leqslant r$ and satisfies there the inequality $|\varphi(z)|\geqslant 1/2$; $f\in\tilde{\mathcal{A}}_d$ and $d > 1/r$. Then the series (13) converges absolutely for any $x\in\mathbf{R}$, and its sum $f\in C^\infty(\mathbf{R})$. If in addition $\int\exp\{2r|x|\}d\mu<\infty$, then $f^{(n)}=d^nf/d^nx\in\mathcal{A}(\mu)$, $n = 0, 1,\ldots$ and the Appell rank of the function f coincides with the smallest number $k = 0, 1,\ldots$ such that

$$\int f^{(k)}(x)\,d\mu\neq 0. \tag{15}$$

Proof. From the analyticity of the function $\varphi(z)=\int e^{zx}\,d\mu$ in the disk $|z| < r$, Eq. (8) and the Cauchy formula, we get that

$$|A_n(x)|\leqslant 2r^{-n}n!\,e^{r|x|}. \tag{16}$$

Thus, in view of (14), the series (13) converges absolutely and uniformly in each interval $x\in[-K, K], K>0$, from which the continuity of the function f follows. Using the differentiation formula (11), we verify analogously that $f\in C^\infty(\mathbf{R})$.

Let $f_M^{(n)}=d^n\left(\sum_{k\leqslant M}c_kA_k(x)\right)/dx^n$, $M\leqslant M'$; $\|\cdot\|$ be the norm in the space $L_2(\mu)$. Then $\|f_M^{(n)}-f_{M'}^{(n)}\|\leqslant \sum_{k=M\vee(M'-n)}^{M'}|c_k|\,k(k-1)\ldots(k-n+1)\,\|A_{k-n}\|\to 0\ (M, M'\to\infty)$ thanks to (11), (16), and the assumptions made, i.e., $f^{(n)}\in\mathcal{A}(\mu)$. From what was said above and (11) and (12) the assertion of Proposition 2 about the rank of the function f follows.

2. Cumulants of Appell Polynomials of a Linear Process

An important virtue of the Appell polynomials is the existence of simple combinatorial rules for the calculation of the (mixed) cumulants, analogous to the familiar diagramatic formalism for the mixed cumulants of the Hermite polynomials with respect to a Gaussian measure [8].

We denote by $\chi(\eta_1,\ldots,\eta_k)$ the (mixed) cumulant of the random variables $\eta_1,\ldots,\eta_k$, i.e.,

$$\chi(\eta_1,\ldots,\eta_k)=d^k\log E\exp\left\{\sum a_j\eta_j\right\}\Big/da_1\ldots da_k|_{a_1=\ldots=a_k=0}.$$

Sometimes we shall also write $\chi(\eta_1,\ldots,\eta_k) = \chi(\eta_i,\ i = 1,\ldots,k)$ and $\chi(\underbrace{\eta,\ldots,\eta}_{k})=\chi_k(\eta)$.

Let the η_i have the form $\eta_i = A_{n(i)}(X_i)$, where X_i, $i = 1, 2,\ldots,k$ are given random variables, $A_{n(i)}$ is the Appell polynomial of degree $n(i)$, corresponding to the (marginal) distribution $\mu_i(dx) = P(X_i\in dx)$. In order to describe the rule according to which one calculates the cumulant $\chi(\eta_1,\ldots,\eta_k)$, we introduce the following terms and notation.

By a *diagram* γ we mean a partition $\gamma = (V_1,\ldots,V_r)$, $r = 1, 2,\ldots$ of an array

$$\begin{pmatrix}(1, 1), \ldots, (1, n(1))\\ (k, 1), \ldots, (k, n(k))\end{pmatrix}=T \tag{17}$$

into (nonempty) sets V_i (*the edges of the diagram*) such that $|V_i| > 1$. We shall call the edge V_i *flat*, if it is contained in one row of the array T. We shall call the diagram *connected*, if any two collections of rows of the array (17) are not split separately by this diagram. We denote the set of all diagrams without flat edges over the array (17) by $\Gamma(T)$, and the subset of all connected $\gamma\in\Gamma(T)$ by $\Gamma_0(T)$. We shall call the diagram $\gamma=(V_1,\ldots,V_r)$ *Gaussian*, if $|V_1|=\ldots=|V_r|=2$.

We define new variables X_{ij}, $(i,j)\in T$, by the equation

$$X_{ij}=X_i.$$

We set $\chi(X^V)=\chi(X_{ij},(i,j)\in V)$, where $V\subset T$.

Proposition 3 [14].

$$\chi(A_{n(i)}(X_i),\ i=1,\ldots,k)=\sum_{\gamma\in\Gamma_0(T)}\chi(X^{V_1})\ldots\chi(X^{V_r}).$$

In what follows, we shall consider the case when the variables X_i have the form $X_i=X_{t_i}$, where

$$X_t=\sum_{s=-\infty}^{\infty}a(t-s)\xi_s,\quad i\in Z, \tag{18}$$

is a *linear process*, i.e., the variables ξ_s, $s\in Z$ are independent and identically distributed, $a(t)$, $t\in Z$ is a nonrandom real sequence. We shall assume in addition that the ξ_s have moments of all orders, $E\xi_s=0$, $E\xi_s^2=1$ and

$$EX^2(t)=\sum_t a^2(t)=1. \tag{19}$$

We note that

$$\chi(X_{t_1},\ldots,X_{t_k})=\chi_k(\xi_0)\sum_s a(t_1-s)\ldots a(t_k-s), \tag{20}$$

in view of the independence of the variables ξ_s and familiar properties of cumulants.

We introduce the space $L^p(Z^n)(p\geqslant 1)$ of real functions $f=f(t_1,\ldots,t_n)$, $(t_1,\ldots,t_n)\in Z^n$ with norm $\|f\|_p=\|f\|_{p,n}=(\Sigma\,|f(t_1,\ldots,t_n)|^p)^{1/p}<\infty$. Let $f_i\in L^2(Z^{n(i)})$, $i=1,\ldots,k$; $\gamma=(V_1,\ldots,V_r)$ be some diagram [over the array T of (17)]. We consider the tensor product

$$f_1\otimes\ldots\otimes f_k=f_1(t_{11},\ldots,t_{1n(1)})\ldots f_k(t_{k1},\ldots,t_{kn(k)}) \tag{21}$$

and we define a new function $f^\gamma:Z^r\to\mathbb{R}$ by means of the replacement of the variables t_{ij}, $(i,j)\in V_s$ in (21) by one new variable $\tilde{t}_s\in Z$, $s=1,\ldots,r$. With the help of Cauchy's inequality, just as in the case of "Gaussian" diagrams (cf. [10, 4]), it is easy to get

Proposition 4. Let $\gamma=(V_1,\ldots,V_r)\in\Gamma(T)$. Then

$$\|f^\gamma\|_{1,r}\leqslant\prod_{i=1}^{r}\|f_i\|_{2,n(i)}. \tag{22}$$

It follows from Propositions 3 and 4, Eq. (20), and the fact that the linear process (18) is stationary that one has

Proposition 5.

$$\chi(A_{n(i)}(X_{t_i}),\ i=1,\ldots,k)=\sum_{\gamma\in\Gamma_0(T)}d_\gamma I_\gamma, \tag{23}$$

where

$$d_\gamma=\chi_{|V_1|}\ldots\chi_{|V_r|},\quad \chi_{|V|}=\chi_{|V|}(\xi_0),$$
$$I_\gamma=\sum_{j_1,\ldots,j_r}f^\gamma(j_1,\ldots,j_r), \tag{24}$$

$f^\gamma\in L^1(Z^r)$ is the function constructed from the tensor product (21) of the functions $f_i\in L^2(Z^{n(i)})$, $i=1,\ldots,k$,

$$f_i(j_1,\ldots,j_{n(i)})=a(t_i-j_1)\ldots a(t_i-i_{n(i)}) \tag{25}$$

according to the rule described above.

In what follows we shall also use the "Fourier-representation" of the cumulants of the variables $A_{n(i)}(X_{t_i})$.

We denote by $\Pi^n = [-\pi, \pi]^n$ the n-dimensional torus with Lebesgue (Haar) measure $d^n x$. We define the Fourier transform $f \to \hat{f}: L^2(Z^n) \to L^2(\Pi^n)$ by the formula (below $\int d^n x = \int\limits_{\Pi^n} d^n x$)

$$\hat{f}(t_1, \ldots, t_n) = \int f(x_1, \ldots, x_n) \exp\left[i \sum t_j x_j\right] d^n x.$$

Let $\gamma=(V_1, \ldots, V_r)$, $f_i \in L^2(Z^{n(i)})$, $i=1, \ldots, k$, f^γ be the same as in Proposition 4, $n = n(1) + \ldots + n(k)$. Then

$$\hat{f}^\gamma(t_1, \ldots, t_r) = \int d^n x \left[\bigotimes_1^k \hat{f}_i\right] \exp[i(t_1 x_{V_1} + \ldots + t_r x_{V_r})],$$

where

$$x_V = \sum_{(i,j) \in V} x_{ij} \,(\mathrm{mod}\, 2\pi), \; |x_V| \leqslant \pi, \; V \subset T. \tag{26}$$

With the help of a change of variables, the latter integral can be transformed to

$$\hat{f}^\gamma(t_1, \ldots, t_r) = \int d^r y g_\gamma(y) \exp[i(t_1 y_1 + \ldots + t_r y_r)], \tag{27}$$

where $g_\gamma: \Pi^r \to C$ is conveniently written symbolically in the form

$$g_\gamma(y) = \int d^n x \left[\bigotimes_1^k \hat{f}_i\right] \delta(y_1 - x_{V_1}) \ldots \delta(y_r - x_{V_r}).$$

δ is the Dirac function. Since $f^\gamma \in L^1(Z^r) \subset L^2(Z^r)$ (cf. Proposition 4), it follows from (27) that $g^\gamma = \hat{f}^\gamma$, and that the function g^γ is continuous. Thus,

$$\sum f^\gamma(t_1, \ldots, t_r) = (2\pi)^r g^\gamma(0). \tag{28}$$

Applying what was said above to the case considered in Proposition 5, we get

COROLLARY 1. Let $\gamma=(V_1, \ldots, V_r) \in \Gamma_0(T)$ and I_γ be defined in (24), (25). Then

$$I_\gamma = (2\pi)^r \int d^n x \prod_{j=1}^k \exp[it_j(x_{j1} + \ldots + x_{jn(j)})] \, \hat{a}(x_{j1}) \ldots \hat{a}(x_{jn(j)}) \, \delta(x_{V_1}) \ldots \delta(x_{V_r}). \tag{29}$$

It also follows from Proposition 4 that one has

COROLLARY 2. Let $g_i \in L^2(\Pi^{n(i)})$, $i=1, \ldots, k$, $\gamma=(V_1, \ldots, V_r) \in \Gamma(T)$, $n=n(1)+ \ldots + n(k)$. Then

$$\int d^n x \bigotimes_1^k |g_i| \, \delta(x_{V_1}) \ldots \delta(x_{V_r}) \leqslant \mathrm{const} \prod_1^n \|g_i\|_2, \tag{30}$$

where $\|g_i\|_2 = \left(\int |g_i|^2 d^{n(i)} x\right)^{1/2}$ is the norm in the space $L^2(\Pi^{n(i)})$.

3. Central Limit Theorem for Polynomials in a Linear Process

In what follows, $A_n(x)$, $n \geqslant 0$ denotes the system of Appell polynomials corresponding to the (stationary) distribution $\mu(dx) = P(X_t \in dx)$; $r(t) = E[X_0 X_t] = a * a(t)$ is the correlation function of the process (18). First we consider the c.l.t. for functions f which are polynomials. We introduce the Dirichlet kernel

$$D_N(x) = \sin(Nx/2)/\sin(x/2). \tag{31}$$

Proposition 6. Let f be a polynomial, $m \geqslant 1$ be its Appell rank. Let us assume that conditions (3), (4) of Theorem 2 hold, and

$$\sum_{t=0}^{\infty} |r(t)|^m < \infty. \tag{32}$$

Then (5) holds [i.e., the c.l.t. holds for the process $f(X_t)$].

Proof. We restrict ourselves to the proof of the convergence of one-dimensional distributions, setting $t = 1$ in (5) for the sake of simplicity. It follows from (3) and (4) that $\mathrm{Var}\left(\sum_1^N f(X_j)\right) \sim N$. We set $S_N^{(n)} = \sum_{j=1}^N A_n(X_j)$. It suffices to verify that

$$\chi(S_N^{(n(1))}, \ldots, S_N^{(n(k))}) = o(N^{k/2}) \tag{33}$$

for any $k \geqslant 3$, $n(1),\ldots,n(k) \geqslant m$.

From Proposition 5 and Corollary 1, we have that the cumulant on the left side of (33) is equal to the sum $\Sigma d_\gamma I_\gamma(N)$ over all $\gamma \in \Gamma_0(T)$, where

$$I_\gamma(N) = C \int d^n x \prod_{j=1}^{k} D_N(x_{j1} + \ldots + x_{jn(j)})\, g_j(x_{j1}, \ldots, x_{jn(j)})\, \delta(x_{V_1}) \ldots \delta(x_{V_p}), \tag{34}$$

where $n = n(1) + \ldots + n(k)$ and

$$g_j = \hat{a} \otimes \ldots \hat{\otimes}\, a\big(n(j)\,\text{pa3}\big). \tag{35}$$

We also set

$$g_{N,j}(x_1, \ldots, x_{n(j)}) = g_j(x_1, \ldots, x_{n(j)})\, D_N(x_1 + \ldots + x_{n(j)}). \tag{36}$$

To prove (33) it suffices to verify the validity of three lemmas.

LEMMA 1.

$$\| g_{N,j} \|_2 \leqslant C N^{1/2}. \tag{37}$$

Lemma 1 is used to prove Lemma 2 below.

LEMMA 2. Let $\gamma \in \Gamma_0(T)$ be a non-Gaussian diagram. Then

$$I_\gamma(N) = o(N^{k/2}). \tag{38}$$

LEMMA 3. Let the diagram $\gamma \in \Gamma_0(T)$ be Gaussian. Then (38) is true again.

We note immediately that Lemma 3 follows from the fact that Proposition 6 and (33) are valid for the case of a Gaussian process X_t (18) (cf. [4, Theorem 6] or [2, Theorem 1]). [We recall that in the Gaussian case the Appell polynomials coincide with the Hermite polynomials $H_n(x)$.] One can say the same thing about Lemma 1, which follows easily from (32) and the equation $\| g_{N,j} \|_2^2 = \operatorname{Var}\left(\sum_{t=1}^{N} H_{n(j)}(X_t)\right)$ under the assumption that the process X_t is Gaussian.

Proof of Lemma 2. Let L_j, $j = 1,\ldots,k$ be the rows of the array T of (17). We consider two cases:

a) $\exists V_i \in \gamma : |V_i \cap L_j| \geqslant 1$ for at least three different rows L_j, $j = 1,\ldots,k$;

b) all other cases.

In case a), without loss of generality we shall assume that $i = 1$ and $V_1 \supset \{(1,1), (2,1), (3,1)\} \equiv \tilde{V}_1$. Let us assume that $\tilde{V}_1 = V_1$. First we estimate the integral in (34) on the hyperplane $x_{V_1} = x_{11} + x_{21} + x_{31} = 0$. Let $u_i = x_{i2} + \ldots + x_{in(i)}$, $i = 1, 2, 3$. Then

$$\iint dx_{11}\, dx_{21}\, | D_N(x_{11} + u_1)\, \hat{a}(x_{11})\, D_N(x_{21} + u_2)\, \hat{a}(x_{21})\, D_N(-x_{11} - x_{21} + u_3)\, \hat{a}(-x_{11} - x_{21}) | \leqslant \prod_{i=1}^{3} \alpha_i(u_i), \tag{39}$$

where $\alpha_i(u) = \| \hat{a}(\cdot)\, D_N(\cdot + u) \|_1$, $i=1$, $= \| \hat{a}(\cdot) D_N(\cdot + u) \|_2$, $i = 2, 3$. Here $\|\cdot\|_p$ is the norm in $L^p(\Pi)$, $p = 1, 2$. We estimate

$$\alpha_1(u) = \int_{-\pi}^{\pi} | \hat{a}(x)\, D_N(x+u) |\, dx = \int_{|\hat{a}| \leqslant \log N} + \int_{|\hat{a}| > \log N} \leqslant \log N \| D_N \|_1 + \| \hat{a}\, 1(|\hat{a}| > \log N) \|_2 \| D_N \|_2 = o(N^{1/2}) \tag{40}$$

thanks to the fact that $\|D_N\|_1 \leqslant C \log N$ and $\|D_N\|_2 \leqslant C N^{1/2}$. Thus,

$$| I_\gamma(N) | \leqslant C \int d^{n-3} x \bigotimes_{j=1}^{k} g'_{N,j}\, \delta(x_{V_2}) \ldots \delta(x_{V_p}), \tag{41}$$

where

$$g'_{N,j}(x_2, \ldots, x_{n(j)}) = \alpha_i(x_2 + \ldots + x_{n(j)})\, |\underbrace{\hat{a} \otimes \ldots \otimes \hat{a}}_{n(j)-1}|$$

for $j = 1, 2, 3$, and $g'_{N,j} = |g_{N,j}|$ for $j > 3$. We note that $g'_{N,j}(\cdot) = \left(\int |g_{N,j}(x, \cdot, \ldots, \cdot)|^2 dx\right)^{1/2}$ for $j = 2, 3$, and thus $\|g'_{N,j}\|_2 \leqslant C N^{1/2}$, $j = 2, 3$, according to Lemma 1, while at the same time $\|g'_{N,1}\|_2=o(N^{1/2})$. Since the diagram $\gamma'=(V_2, \ldots, V_r)\in\Gamma(T\setminus V_1)$ also has no flat edges, we get from (39)-(41) and Corollary 2 that (38) holds. The case $|V_1| > 3$ can be considered analogously.

b) We call the edge $V_t\in\gamma$ *interior*, if $V_t\subset L_1\cup L_2$, and *exterior* if not. According to the proposition, there exists an interior edge (say V_1) such that $|V_1| \geqslant 3$. For simplicity let us assume that the remaining edges $V_2,\ldots,V_r$ are exterior and that $V_1 = \{(1, 1), (1,2), (2, 1)\}$. From the array T of (17) we form a new array T', having k − 1 rows, and we denote the elements of the first row by $(1, 3),\ldots,(1, n(1))$, $(2, 2),\ldots,(2, n(2))$, and the elements of the i-th row by $(i + 1, 1),\ldots,(i + 1, n(i + 1))$, $i = 2,\ldots,k - 1$. With this notation, under the assumptions made above, the partition $(V_2,\ldots,V_r)$ of the set $T\setminus V_1$ induces a (connected) diagram γ' of the array T', having no flat edges. Let $g'_1=\int g_{N,1}\otimes g_{N,2}\,\delta(x_{V_1})\,dx_{11}\,dx_{12}\,dx_{21}$, $g'_i = g_{N,i+1}$, $i = 2,\ldots,k - 1$. Let $u_1 = x_{1,3} + \ldots + x_{1n(1)}$, $u_2 = x_{22} + \ldots + x_{2n(2)}$. Since

$$|g'_1|\leqslant\|D_N(\cdot+u_1)D_N(-u_2)\hat{a}\|_1\|\hat{a}\|_2^2\equiv h(u_1, u_2),$$

just as in case a) one can show that

$$\|g'_1\|_2\leqslant C\left(\int\int du_1du_2\,|h(u_1, u_2)|^2\right)^{1/2}=o(N).$$

It remains to use Corollary 2. ✱

Proposition 7. If m = 1, Proposition 6 remains valid if instead of (32) one has

$$\sum |r(j)|^2<\infty \tag{42}$$

and

$$\mathrm{Var}\left(\sum_{j=1}^{N}X_j\right)\leqslant CN,\ N\geqslant 1. \tag{43}$$

Proof. By (42) and (43) Lemma 1 is true and hence Lemma 2. It remains to prove Lemma 3. Let the diagram $\gamma=(V_1, \ldots, V_r)\in\Gamma_0(T)$ be Gaussian. If $n(1),\ldots,n(k) \geqslant 2$, Lemma 3 is valid by (42). Let us assume that among the numbers $n(1),\ldots,n(k)$, 1 occurs. For simplicity we shall assume that $n(1) = 1$, $n(2),\ldots,n(k) \geqslant 2$, $V_1 = ((1, 1), 2, 1))$ (the general case is considered analogously). We write the right side of (34) in the form

$$I_\gamma(N)=C\int d^{n-2}x\bigotimes_{j=2}^{k}g'_{N,j}\,\delta(X_{V_2})\ldots\delta(X_{V_r}),$$

where $g'_{N,2}(x_2, \ldots, x_{n(2)})=\int g_{N,1}(x)g_{N,2}(x, x_2, \ldots, x_{n(2)})\,dx$, $g'_{N,j}=g_{N,j}$, $j=3, \ldots, k$. Considering Lemma 2 and Corollary 2, it suffices to verify that

$$\|g'_{N,2}\|_2=o(N). \tag{44}$$

We prove (44). Let $l = n(2)$. We have

$$\|g'_{N,2}\|_2^2=\int d^{l-1}x\,|\hat{a}(x_2)\ldots\hat{a}(x_l)|^2\left[\int\int D_N(x_1)\,D_N(x_1+\ldots+x_l)\,\hat{a}(x_1)\,dx_1\right]^2=\int_{D_R}+\int_{D_R^c}\equiv Q_N^{(1)}+Q_N^{(2)},$$

where $R>0$, $D_R=\{(x_2, \ldots, x_l)\in\Pi^{l-1}:|\hat{a}(x_j)|\leqslant R,\ j=2, \ldots, l\}$, $D_R^c=\Pi^{l-1}\setminus D_R$.

Using Cauchy's inequality and (43), we get that

$$Q_N^{(2)}\leqslant\|D_N(\cdot)\hat{a}^2(\cdot)\|_2^2\int_{D_R^c}D_N^2(x_1+\ldots+x_l)\,|\hat{a}(x_2)\ldots\hat{a}(x_l)|^2\,d^lx\leqslant CN\,\mathrm{Var}\left(\sum_1^N X_j\right)\int_{|\hat{a}|\geqslant R}\hat{a}^2(x)\,dx\leqslant\varepsilon N^2$$

for all $N \geqslant 1$ and sufficiently large $R = R(\varepsilon)$, where $\varepsilon > 0$ is arbitrarily small. On the other hand, for each $R > 0$

$$Q_N^{(1)}\leqslant R^{l-1}\int\left[\int|D_N(x)\,D_N(x+u)|\,dx\right]^2du=o(N^2).$$ ✱

4. Central Limit Theorem for Functionals $f(X_t)$ (General Case)

First we give some auxiliary assertions.

Proposition 8. Suppose given a function $f\in\mathscr{A}_d(\mu)\equiv\tilde{\mathscr{A}}_d\cap\mathscr{A}(\mu)$, $m\geq 1$ is its Appell rank, d > 0 satisfies the hypotheses of Theorem 2. Then (a) for m ⩾ 2 conditions (3) and (32) are equivalent; (b) for m = 1, condition (3) implies (42) and (43); (c) for m ⩾ 1 it follows from (3) that

$$\sum_t |r_{f(M)}(t)|\to 0\ (M\to\infty),\ \text{where}\ f(M)=\sum_{k=M}^{\infty} c_k A_k.$$

Proof. According to Proposition 5,

$$r_f(t)=\sum_{k,\,k'\geq m} c_k c_{k'} \sum d_\gamma J_\gamma(t), \tag{45}$$

where the second sum is taken over all diagrams $\gamma=(V_1,\ldots,V_r)\in\Gamma_0(T)$, $r\geq 1$, of the array T = T(k, k') composed of two rows L and L' of length k and k', respectively, and

$$J_\gamma(N)=\sum_{s_1,\ldots,s_r}\prod_{i=1}^{r} a^{n(i)}(t-s_i)\,a^{n'(i)}(s_i), \tag{46}$$

$n(i)=|V_i\cap L|$, $n'(i)=|V_i\cap L'|$. We consider the following cases:

(γ1) among the numbers n(i), n'(i), i = 1,...,r at least two are bigger than 1;

(γ2) n(i) = n'(i) = 1 i (i.e., the diagram γ is Gaussian), k = k' = r = m;

(γ3) conditions (γ1) and (γ2) do not hold and the numbers k, k', and m are bigger than 1;

(γ4) all other cases.

LEMMA 4. One has the relations:

$$|J_\gamma(t)|\leq q(t)\ \text{ in case }\ (\gamma 1); \tag{47}$$

$$J_\gamma(t)=r^m(t)\ \text{ in case }\ (\gamma 2); \tag{48}$$

$$|J_\gamma(t)|\leq \varepsilon|r(t)|^m+q(t)\ \text{ in case }\ (\gamma 3); \tag{49}$$

$$|J_\gamma(t)|\leq p(t)\ \text{ in case }\ (\gamma 4). \tag{50}$$

Here $\Sigma|q(t)|<\infty$, $\Sigma p^2(t)<\infty$, p(t), q(t) do not depend on γ, $\varepsilon > 0$ is arbitrarily small.

The *proof* of (47), (48), (50) involves no difficulties and (49) follows from the inequalities

$$|J_\gamma(t)|\leq|r(t)|^{r-1}\sum|a(t-s)|\,|a(s)|^2\leq\varepsilon|r(t)|^{2(r-1)}+(1/\varepsilon)\sum a^2(t-s)a^2(s),\ r\geq m\geq 2,$$

if the diagram is not Gaussian, and if γ is Gaussian, from (48) and the relation r(t) → 0 (t → ∞). ✱

We get from (45), Lemma 4 (where ε > 0 is sufficiently small), and Lemma 5 below that

$$r_f(t)=\begin{cases} cr^m(t)+\tilde{q}(t),\ m\geq 2, & (51)\\ cr(t)+\tilde{p}(t)+\tilde{q}(t),\ m=1, & (52)\end{cases}$$

where $c\neq 0$, $\Sigma|\tilde{q}(t)|<\infty$, $\Sigma|\tilde{p}(t)|^2<\infty$.

Assertion (a) of the proposition follows from (51), and so does (42) if m = 1. To prove (43), we note that it follows from (3) that Var $\left(\sum_1^N f(X_t)\right)\leq CN$, and from (42) and (a) that Var $\left(\sum_1^N (f(X_t)-c_1X_t)\right)\leq CN$ [we recall that $A_1(x) = x$]. The proof of assertion (c) is analogous. *

LEMMA 5. Let $f\in\tilde{\mathscr{A}}_d$, where d, Δ, r_0 satisfy the hypotheses of Theorem 2. Then $f\in\mathscr{A}_d(\mu)$ $(\mu(dx)=P(X_t\in dx))$ and

$$\sum_{k,\,k'}|c_kc_{k'}|\sum_{\gamma\in\Gamma_0(T(k,\,k'))}|d_\gamma|<\infty. \tag{53}$$

Proof. We use Proposition 2 and the following familiar result [1, pp. 28, 29]. Let $\lambda_k\geq 0$, k = 1,...,n, $\lambda_1 = 0$ be certain numbers satisfying the inequality $\lambda_k\leq ck!\,H^{k-2}(c,H>0)$, and $\nu_n=\Sigma\lambda_{(|V_1|)}\ldots\lambda_{(|V_r|)}$, where the sum is taken over all partitions $(V_1,\ldots,V_r)$, r = 1, 2,... of the set {1,...,n}. Then

$$\nu_n\leqslant H^k(\sqrt{1+2c})^{k-2}ck!. \tag{54}$$

It follows from (20) that $|\chi_k(X_0)|\leqslant|\chi_k(\xi_0)|$, and from the Cramer condition that $E|\xi_0|^k\leqslant\Delta r_0^{-k}k!$. Hence $|\chi_k(\xi_0)|\leqslant r_0^{-k}(\sqrt{1+2\Delta})^{k-2}\Delta k!$ (cf. [1]) and $E|X_0|^k\leqslant(1+2\Delta)^k r_0^{-k}k!$ thanks to (54). From this it follows that $\varphi(z)=\int e^{zx}d\mu$ is analytic in the disk $|z|<r\equiv r_0/(1+2\Delta)$ and that $|\varphi(z)|>1/2$ for $|z|<r/3$.

To prove (53) we again use (54) and the definition of d_γ (cf. Proposition 5), from which we get that $\sum_\gamma|d_\gamma|\leqslant[(1+2\Delta)/r_0]^{k+k'}(k+k')!$. Together with the condition imposed by the lemma on the coefficients c_k and the numbers d, Δ, r_0, this implies (53). *

Proof of Theorem 2. It follows from the preceding lemma that $f\in\mathscr{A}(\mu)$. Let f = f' + f", where $f'=f'_M=\sum_{k=m}^{M}c_kA_k$. We denote by $r'_f(t)$, $r''_f(t)$, S'_N, S''_N, respectively, the correlation functions and partial sums of the processes $f'(X_t)$ and $f''(X_t)$, i.e., $S_N=\sum_1^N f(X_t)=S'_N+S''_N$.

It follows from Proposition 8 (c) that $\operatorname{Var}S''_N\leqslant N\Sigma|r''_f(t)|\leqslant\varepsilon N$ for M sufficiently large, where ε > 0 is arbitrarily small. Thus, $A'_N=(\operatorname{Var}S'_N)^{1/2}=o(N^{1/2})$. It follows from Propositions 6, 7, and 8 that $S'_N/A_N\overset{d}{\Rightarrow}\mathcal{N}(0,1)$. The rest is proved simply. *

Proof of Theorem 3. Arguing just as in the proof of Proposition 7, we get that for m ⩾ 2 one has

$$L(t)/|t|=r_f(t)=cr^m(t)+q(t), \tag{55}$$

where $\Sigma|q(t)|<\infty$, $c\neq 0$, and for m = 1, the inequality

$$\sum r^2(t)<\infty,\quad \operatorname{Var}\left(\sum_1^N X_t\right)\leqslant CA_N, \tag{56}$$

where $A_N=(L_1(N)N)^{1/2}$. From (55), (56) it is easy to derive the estimate $\|g_{N,j}\|\leqslant CA_N$ (cf. Lemma 1), where $g_{N,j}$ is defined in (36). With the help of the latter estimate, just as in the proof of Lemma 2, we get that $I_\gamma(N)=o(A_N^{k/2})$ for all non-Gaussian diagrams $\gamma\in\Gamma_0(T)$ of the array T of (17). For Gaussian diagrams, if m ⩾ 2 the corresponding relation is proved in [4], Vol. 7; if m = 1 it follows from (56) just like Proposition 7. The rest of the proof completely follows the outline of the proof of Theorem 2. *

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ADDITIVE ARITHMETIC FUNCTIONS ON SEMIGROUPS AND THE PRESERVATION OF WEAK CONVERGENCE OF MEASURES

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We use the following notation. $\mathbf{N}$ is the set of positive integers, $\mathbf{N}_0=\mathbf{N}\cup\{0\}$. $\mathbf{R}$ is the set of real numbers, $\mathcal{P}(M)$ is the set of all subsets of the set M.

1. One of the basic problems of probabilistic number theory can be formulated as follows. Let $f:\mathbf{N}\to\mathbf{R}$ be an additive function, i.e., $f(mn) = f(m) + f(n)$ for relatively prime m and n. We consider the sequence of distribution functions

$$F_n^{(1)}(y)=\frac{1}{n}\sum_{\substack{m\leq n\\ f(m)-\alpha_n<y\beta_n}} 1, \qquad (1)$$

where α_n and β_n are sequences of centering and normalizing quantities. The problem is to determine conditions (sufficient and necessary) under which (1) converges weakly to some limit distribution.

A more general formulation is possible: we consider the sequence of distribution functions

$$F_n^{(2)}(y)=\frac{1}{n}\sum_{\substack{m\leq n\\ f_n(m)-\alpha_n<y}} 1,$$

where $\{f_n,\ n\in\mathbf{N}\}$ is a sequence of additive functions. Thus one gets limit theorems for distributions of additive functions. The history of the origin of this problem and the basic results obtained in solving it are reflected in [7, 17].

In addition to this problem we consider its generalization in various directions. Here we indicate two such generalizations.

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