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LIMIT THEOREM FOR POLYNOMIALS OF A LINEAR PROCESS WITH LONG-RANGE DEPENDENCE

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A (noncentral) limit theorem is proved for sums $S_N^{(m)} = \sum_1^N P_m(X_t)$ of polynomials of degree $m \geq 1$ of a linear (or moving average) process $X_t = \sum_{-\infty}^{\infty} a(t-s)\xi_s$ with slowly decreasing coefficients $a(t)$. Conditions assuring the convergence of the distributions $S_N^{(m)}/B_N$ are expressed in terms of the asymptotics of the variances $B_N^2 = \operatorname{Var} S_N^{(m)}$ and $A_N^2 = \operatorname{Var} \sum_1^N X_t$. The limit distribution of the sums $S_N^{(m)}/B_N$ is given by an m-fold stochastic Ito–Wiener integral. The theorem proved develops the results of [1-7], obtained under the hypothesis that the process X_t is Gaussian and (or) of the regularity of the asymptotics of the coefficients $a(t)$.

1. Introduction

Recently, rather a lot of attention has been devoted to limit theorems for random variables with long-range dependence (in what follows, we shall call such variables LRD-variables). Although there does not exist a rigorous definitiion of LRD-variables, usually this term characterizes stationary variables whose normalized sums converge in distribution but either the limit law or the normalization differs from the "classical" ones. In the case of finite variances the Gaussian law and the normalization by $\sqrt{N}$ are "classical." The simplest example of LRD-variables is given by a stationary Gaussian process $(X_t)_{t\in Z}$ with zero mean and covariance

$$r(t) \equiv EX_0X_t \sim \text{const}\,|t|^{-\alpha}, \tag{1.1}$$

where $\alpha \in (0, 1)$. In this case the variance $A_N^2 \equiv \operatorname{Var} S_N$ of the sum $S_N = \sum_1^N X_t$ grows considerably faster than the "classical" N, namely

$$\operatorname{Var} S_N \sim \sigma^2 N^{2-\alpha} \quad (\sigma^2 > 0). \tag{1.2}$$

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The normalized partial sums $(A_N^{-1}S_{[Nt]})_{t\geq 0}$ converge in distribution to a Gaussian process of fractional Brownian motion

$$Z_t = D(1)^{-1/2} \int_{-\infty}^{\infty} \left((e^{itx}-1)/ix\right) W(dx), \tag{1.3}$$

where $W(dx) = \overline{W(-dx)}$ is a complex Gaussian stochastic measure with independent values and variance $E|W(dx)|^2 = |x|^{\alpha-1}dx$; the constant D(1) is defined in (1.6) below.

Another example of LRD-variables is the moving average process

$$X_t = \sum_{s=0}^{\infty} a(s)\xi_{t+s}, \quad t\in\mathbf{Z}, \tag{1.4}$$

where $(\xi_s)_{s\in\mathbf{Z}}$ is a sequence of independent identically distributed random variables with mean 0 and finite variance and the coefficients $a(t)$ decrease like $t^{-(1+\alpha)/2}$ ($\alpha \in (0, 1)$), i.e.,

$$a(t) \sim \text{const}\, t^{-(1+\alpha)/2}. \tag{1.5}$$

(1.5) itself implies (1.1) and (1.2) and also the convergence in distribution of the partial sums of the process X_t (1.4) to the process Z_t (1.3). Of course, (1.2), (1.5) here are unnecessarily restrictive; for the indicated convergence to the process Z_t indeed the one condition (1.2) suffices. We note that (1.2) up to a slowly varying factor is also necessary for the convergence to Z_t as follows from a general result of Lamperti [8].

One can construct more complicated examples of LRD-variables with the help of nonlinear functions of Gaussian or linear processes. One knows rather many papers devoted to limit theorems for nonlinear transformations of Gaussian processes under long-range dependence, in particular Rosenblatt [4, 9], Dobrushin and Major [2], Major [10, 11], Taqqu [6, 7], Gorodetskii [3], et al. We cite one of the most famous results.

Let $K_0(x) = (e^{ix} - 1)/ix$ ($x \in \mathbf{R}$),

$$D(m) = \int_{\mathbf{R}^m} |K_0(x_1 + \ldots + x_m)|^2 |x_1|^{\alpha-1} \ldots |x_m|^{\alpha-1} d^m x, \tag{1.6}$$

$$\gamma = \int_{\mathbf{R}} e^{ix}|x|^{\alpha-1}dx = 2\Gamma(\alpha)\cos(\alpha\pi/2). \tag{1.7}$$

We shall write $A_N \sim B_N$ if $\lim A_N/B_N = 1$. The notations $\overset{d}{=}$, $\overset{d}{\Rightarrow}$ will mean equality and weak convergence of (finite-dimensional) distributions, respectively. Finally, $H_m(x)$, m = 0, 1, ... denotes the Hermite polynomials with leading coefficient 1.

<u>THEOREM 1</u> [2]. Suppose one has a stationary Gaussian process $(X_t)_{t\in\mathbf{Z}}$ with mean 0, variance 1, and covariance function

$$r(t) = L(|t|)|t|^{-\alpha}, \tag{1.8}$$

where $\alpha \in (0, 1)$ and $L:[1, \infty) \to \mathbf{R}$ is a slowly varying function (s.v.f.). Let

$$S_N^{(m)} = \sum_{1}^{N} H_m(X_t). \tag{1.9}$$

Let $\alpha m < 1$. Then

$$S_N^{(m)}/B_N \overset{d}{\Rightarrow} Z_t^{(m)}, \tag{1.10}$$

where

$$B_N^2 \equiv \text{Var}\, S_N^{(m)} \sim m!\left(D(m)/\gamma\right) N^{2-\alpha m} L^m(N) \tag{1.11}$$

and

$$Z_t^{(m)} = (m!\,D(m))^{-1/2}\, t \int_{\mathbf{R}^m} K_0\big(t(x_1+\ldots+x_m)\big)\, W(dx_1)\ldots W(dx_m) \tag{1.12}$$

is a self-similar process which can be represented as a multiple Wiener–Ito integral with respect to a Gaussian measure W(dx) with variance $E|W(dx)|^2 = |x|^{\alpha-1}dx$.

In the same paper [2, Sec. 4], Dobrushin and Major noted that the condition (1.8) of Theorem 1 is too stringent and can be relaxed. As a possible relaxation of (1.8) the authors of [2] considered the condition of locally weak convergence of the renormalized spectral measure of the process X_t to the spectral measure of a fractional Brownian motion. The latter requirement is equivalent to one of the two following conditions:

$$S_{[Nt]}/A_N \overset{d}{\Rightarrow} Z_t^{(1)} \equiv Z_t \tag{1.13}$$

or

$$A_N^2 \sim N^{2-\alpha} L(N), \tag{1.14}$$

where $S_N = \sum_1^N X_t$, $A_N^2 = \operatorname{Var} S_N$, L is an s.v.f.

However, as was shown in [2], the condition (1.13) or (1.14) is insufficient for the convergence (1.10). Most likely this condition does not assure even the relative compactness of finite-dimensional distributions $S_{[Nt]}^{(m)}/B_N$. In this connection there arises the problem of finding supplementary conditions to (1.13) and (1.14) for the convergence in distribution of the sequence of processes $S_{[Nt]}^{(m)}/B_N$. As such a condition we propose the "regular growth" of the variance $B_N^2 = \operatorname{Var} S_N^{(m)}$:

$$B_N^2 \sim C N^{2-\alpha m} L^m(N), \tag{1.15}$$

where α, L are the same as in (1.14), and C = C(m, α) is a constant. The condition (1.15) [together with (1.14)] looks simple and tempting. However, we have only obtained the corresponding limit theorem under the additional assumption of the "regularity" of the constant C.

<u>THEOREM 2.</u> Let $(X_t)_{t\in\mathbf{Z}}$ be a statiionary Gaussian process with mean 0, variance 1, and such that

$$A_N^2 \equiv \operatorname{Var} \sum_1^N X_t = N^{2-\alpha} L(N), \tag{1.16}$$

where α ∈ (0, 1), and L: [1, ∞) → $\mathbf{R}_+$ is an s.v.f. Let us assume in addition that αm < 1 and (1.15) holds, where

$$C = C(m, \alpha) = m!\, D(m)/D^m(1). \tag{1.17}$$

Then the convergence (1.10) holds.

It is easy to verify that Theorem 2 generalizes Theorem 1. Condition (1.17) together with (1.16) and (1.15) means, roughly speaking, that the basic contribution to the variance B_N^2 (and thus to the distribution of the sum $S_{[Nt]}^{(m)}$) is introduced by the frequency of the spectrum of the process (X_t) near the point x = 0. As Rosenblatt [9] showed, cases are possible in which the spectrum of the process (X_t) has singularities away from the point x = 0, (1.15) and (1.16) hold, (1.17) does not hold, and $S_{[Nt]}^{(m)}/B_N$ tends to a self-similar limit different from $Z_t^{(m)}$ (1.12).

The <u>proof</u> of Theorem 2 follows rather simply from the proof of Theorem 1 of [2] (cf. Sec. 2 below). The basic result of the present paper is the corresponding theorem for a linear process (X_t). In what follows, by a linear process we mean a stationary sequence

$$X_t = \sum_{-\infty}^{\infty} a(t-s)\xi_s, \quad t \in \mathbf{Z}. \tag{1.18}$$

where $a(t)$, $t \in Z$ are real numbers satisfying the condition $\sum a^2(t) < \infty$, and $(\xi_s)_{s \in Z}$ is a sequence of independent identically distributed random variables with mean 0, variance 1, and finite moments of any order. It is well known that a stationary Gaussian process $(X_t)_{t \in Z}$ admits a representation (1.18) if and only if its spectral measure is absolutely continuous.

The role of Hermite polynomials for a linear process is played by the Appell polynomials $P_m(x)$, m = 0, 1, ..., defined by

$$P_m(x) = d^m (e^{zx}/Ee^{zX_0})/dz^m |_{z=0}, \tag{1.19}$$

cf. [1, 12, 13]. Obviously, in the case of a Gaussian process (X_t) the polynomials $P_m(x)$ coincide with the Hermite polynomials $H_m(x)$. Let

$$S_N^{(m)} = \sum_1^N P_m(X_t). \tag{1.20}$$

THEOREM 3. Let $(X_t)_{t \in Z}$ be a linear process, $m \geq 1$ and $\alpha \in (0, 1/m)$. Let us assume, in addition, that the variances $A_N^2 = \operatorname{Var} \sum_1^N X_t$ and $B_N^2 = \operatorname{Var} S_N^{(m)}$ satisfy (1.15), (1.16), and (1.17). Then

$$S_{[Nt]}^{(m)}/B_N \overset{d}{\Rightarrow} Z_t^{(m)}, \tag{1.21}$$

where the process $Z_t^{(m)}$ is defined in (1.12).

In the special case of regularly varying coefficients $a(t)$ [cf. (1.5)], Theorem 3 was previously proved by the second author [5] (cf. also Avram and Taqqu [1]). As follows from Theorem 4 below, the asymptotics of variances A_N^2, B_N^2 are determined up to O(N) by just the covariance function of the process (X_t) (1.18), equal to

$$r(t) = \sum_{-\infty}^{\infty} a(t+s)a(s). \tag{1.22}$$

THEOREM 4. Let (X_t) be a linear process. Then

$$B_N^2 \equiv \operatorname{Var} \sum_1^N P_m(X_t) = m! \sum_{t,s=1}^N r^m(t-s) + O(N). \tag{1.23}$$

We note that for a Gaussian process (X_t)

$$B_N^2 \equiv \operatorname{Var} \sum_1^N H_m(X_t) = m! \sum_{t,s=1}^N r^m(t-s). \tag{1.24}$$

The next result follows from Theorems 1, 3, and 4 and (1.24).

COROLLARY 1.1. Let the covariance function r(t) (1.22) of the linear process (X_t) (1.18) satisfy (1.8), where $\alpha \in (0, 1/m)$ $(m \geq 1)$. Then the convergence (1.21) holds.

Theorem 5 below is also based on Theorems 3 and 4. Let

$$\hat{g}(x) = (2\pi)^{-1/2} \sum g(t) e^{-itx}, \quad x \in [-\pi, \pi] \tag{1.25}$$

be the Fourier transform of the sequence $(g(t))_{t\in Z} \in L^2(Z)$.

THEOREM 5. Let us assume that the spectral density $f(x) = |\hat{a}(x)|^2$ of the linear process (X_t) (1.18) has the form

$$f(x)=|x|^{\alpha-1}L(|x|^{-1})\theta(|x|^{-1}), \tag{1.26}$$

where $\alpha \in (0, 1/m)$ $(m \geq 1)$, L: $\mathbf{R}_+\to\mathbf{R}_+$ is an s.v.f. which is bounded on compacta, and the function θ: $\mathbf{R}_+\to\mathbf{R}_+$ is bounded and such that the following limit exists:

$$\lim_{T\to\infty} T^{-1}\int_0^T \theta(t)\,dt\equiv\bar{\theta}>0. \tag{1.27}$$

Then the convergence (1.21) holds.

We note that the function $\theta(1/x)$ in Theorem 5 is not at all necessarily slowly varying and the spectral density $f(x)$ (1.26) can behave quite irregularly near the point $x = 0$ (as, for example, in the case

$$\theta(x)=\sum_{n=0}^{\infty}\mathbf{1}_{[2n,\,2n+1)}(x)).$$

We briefly explain the idea of the proof of the fundamental Theorem 3. We consider the spectral representation

$$X_t=\int_{-\pi}^{\pi} e^{itx}Z(dx) \tag{1.28}$$

of the linear process (X_t) (1.18), where $Z(dx) = \overline{Z(-dx)}$ is an orthogonal stochastic measure corresponding to the spectral measure $f(dx) = f(x)dx = |\hat{a}(x)|^2$. It follows from (1.16) that the renormalized spectral measure

$$F_N(dx)=F(dx/N)\,N^{\alpha}/L(N) \tag{1.29}$$

converges as $N \to \infty$ to a measure on the line with density $D^{-1}(1)|x|^{\alpha-1}$ at the same time that the stochastic measure corresponding to F_N

$$Z_n(dx)=Z(dx/N)\left(N^{\alpha}/L(N)\right)^{1/2} \tag{1.30}$$

converges in distribution to the Gaussian measure $D^{-1/2}(1)W(dx)$ [cf. (1.3)]. Thus, to prove the convergence (1.21) it suffices to show that there exists a polynomial in the random variables $Z_N(\Delta_{-M})$, ..., $Z_N(\Delta_M)$ approximating $S_N^{(m)}/B_N$ in the mean square uniformly in N; here, Δ_{-M}, ..., Δ_M are intervals independent of N. The proof of existence of such an approximation occupies the basic part of the paper and is split into several propositions.

2. Convergence of Spectral Measures

In this section we consider the convergence of the spectral measures $F_N(dx)$ (1.29) and $Z_N(dx)$ (1.30), assuming that the condition (1.16) on the growth of the variance holds, i.e.,

$$A_N^2\equiv\operatorname{Var}\sum_1^N X_t=N^{2-\alpha}L(N), \tag{2.1}$$

where $\alpha \in (0, 1)$, L: $[1, \infty) \to \mathbf{R}_+$ is an s.v.f.

Proposition 2.1. Let us assume that the stationary process $(X_t)_{t\in\mathbf{Z}}$ is Gaussian or linear and (2.1) holds. Then for each bounded Borel set $A \subset \mathbf{R}$

$$F_N(A)\to F_0(A) \tag{2.2}$$

and

$$Z_N(A) \stackrel{d}{\Rightarrow} Z_0(A), \tag{2.3}$$

where $Z_0(dx) \stackrel{d}{\equiv} D^{-1/2}(1)W(dx)$ is a Gaussian stochastic measure with variance

$$F_0(dx) = E|Z_0(dx)|^2 = D^{-1}(1)|x|^{\alpha-1}dx. \tag{2.4}$$

The <u>proof</u> follows the argument of [2, pp. 33-34]. For concreteness we restrict ourselves to the case of a linear process (X_t). According to Theorem 3 of [14], (2.1) is equivalent to

$$A_N^{-1}\sum_1^{[Nt]} X_t \stackrel{d}{\Rightarrow} Z_t^{(1)}, \tag{2.5}$$

where $(Z_t^{(1)})_{t\geq 0}$ is a fractional Brownian motion, i.e., a Gaussian process with mean 0 and covariance

$$r^{(1)}(t, s) \equiv E\left(Z^{(1)}(t)Z^{(1)}(s)\right) = \frac{1}{2}\left(|t|^{2-\alpha} + |s|^{2-\alpha} - |t-s|^{2-\alpha}\right). \tag{2.6}$$

It is well known [6, 7] that the process $Z_t^{(1)} = Z_t$ admits a representation in the form of a stochastic integral (1.3), from which it follows that

$$\varphi_0(t) \equiv r^{(1)}(t+s, s) - r^{(1)}(t, s) = \int_{-\infty}^{\infty}\left(\frac{e^{i(t+s)x}-1}{ix} - \frac{e^{itx}-1}{ix}\right)\left(\frac{e^{-isx}-1}{-ix}\right)F_0(dx) = \int_{-\infty}^{\infty} e^{itx}\mu_0(dx), \tag{2.7}$$

where

$$\mu_0(dx) = |sK_0(sx)|^2 F_0(dx) \tag{2.8}$$

[we recall that $K_0(x) = (e^{ix} - 1)/ix$].

Let $Y_N = A_N^{-1}\sum_1^N X_t$. The convergence in distribution of the vector $(Y_{[Nt]}, Y_{[Ns]})$ [cf. (2.5)] together with the convergence of the vector $(EY_{[Nt]}^2, EY_{[Ns]}^2)$ [this follows from (2.1)] implies the convergence of the covariances

$$E(Y_{[Nt]}Y_{[Ns]}) \to r^{(1)}(t, s). \tag{2.9}$$

Consequently,

$$\varphi_0(t) = \lim_{N\to\infty}\varphi_N(t), \tag{2.10}$$

where

$$\varphi_N(t) = E\left((Y_{[N(t+s)]} - Y_{[Nt]})Y_{[Ns]}\right).$$

Analogously to [2, p. 33], we write $\varphi_N(t)$ in the form

$$\varphi_N(t) = \int_{-\pi N}^{\pi N} e^{ixt'}\mu_N(dx), \tag{2.11}$$

where $t' = [tN]/N$,

$$\mu_N(dx) = \tilde{K}_N^2(x)F_N(dx), \tag{2.12}$$

$$\tilde{K}_N^2(x) = |(e^{ixs''} - 1)(e^{-ixs'} - 1)/\left(N(e^{ix/N} - 1)\right)|^2, \tag{2.13}$$

$s'' = ([(t + s)N] - [tN])/N$, $s' = [sN]/N$. Since $s'' \to s$, $s' \to s$ $(N \to \infty)$ it is easy to see that

$$\tilde{K}_N^2(x) \to |sK_0(sx)|^2 \tag{2.14}$$

uniformly in $x \in [a, b]$ for finite $a < b$.

According to Lemma 2 of [2], from (2.10) we have that $\mu_N \to \mu_0$ weakly as $N \to \infty$. We fix $-\infty < a < b < +\infty$ and choose $t > s \geq 2\pi/\max(|a|, |b|)$. Since the function $K_0(sx)$ is continuous and does not vanish for $|x| < 2\pi/s \leq \max(|a|, |b|)$, by virtue of the weak convergence of measures μ_N to μ_0, and (2.14), we conclude that

$$F_N(a, b) \to F_0(a, b).$$

Since the numbers $a < b$ are arbitrary, and the measure F_0 is continuous, the convergence (2.2) follows from the last relation.

We proceed to the stochastic measure Z(dx) in the spectral representation (1.28) of the process (X_t). It follows from (1.18) and (1.28) that, for each bounded Borel set $A \subset [-\pi, \pi]$,

$$Z(A) = \sum_{-\infty}^{\infty} \Big(\frac{1}{\sqrt{2\pi}} \int_A e^{-isy} \hat{a}(y)\, dy \Big) \xi_s. \tag{2.15}$$

It follows from the definition of Z_N and F_N [cf. (1.29), (1.30)] that

$$Z_N(A) = \big(Z(A/N)/F^{1/2}(A/N)\big) F_N^{1/2}(A) \equiv W_N F_N^{1/2}(A) \tag{2.16}$$

under the condition that $A \subset [-\pi N, \pi N]$ (outside the interval $[-\pi N, \pi N]$ we set measures F_N and Z_N equal to 0). Since $F_N(A) \to F_0(A)$ cf. above), for the convergence (2.3) it suffices to show that It follows from $E|W_N|^2 = 1$ and (2.15), (2.16) that

$$W_N \overset{d}{\Rightarrow} \mathcal{N}(0, 1). \tag{2.17}$$

$$W_N = \sum_{-\infty}^{\infty} q_s \xi_s, \tag{2.18}$$

where

$$q_s = \frac{1}{\sqrt{2\pi}} \int_{A/N} e^{-isy} \hat{a}(y)\, dy \Big/ \Big(\int_{A/N} |\hat{a}(y)|^2 dy \Big)^{1/2}. \tag{2.19}$$

We note that $\sum |q_s|^2 = 1$ and

$$\sqrt{2\pi} \max |q_s| \leq \Big(\int_{A/N} |\hat{a}|^2 \Big)^{1/2} \Big(\int_{A/N} 1 \Big)^{1/2} \Big/ \Big(\int_{A/N} |\hat{a}|^2 \Big)^{1/2} = \lambda^{1/2}(A/N) \to 0 \qquad (N \to \infty). \tag{2.20}$$

where λ is Lebesgue measure.

Let $W_N' = \operatorname{Re} W_N$, $q_s' = \operatorname{Re} q_s$. We consider the k-th cumulant $\chi_k(W_N')$ of the real random variable W_N'; $k = 2, 3, \ldots$. In view of the independence of ξ_s and (2.20),

$$|\chi_k(W_N')| = \Big| \chi_k(\xi_0) \sum (q_s')^k \Big| \leq \mathrm{const}\,(\max |q_s|) \sum |q_s|^2 \to 0,$$

if $k \geq 3$. Analogously one can show that the cumulants of order $k \geq 3$ of the random variable $W_N'' = \operatorname{Im} W_N$ tend to zero, as well as of arbitrary linear combination of W_N' and W_N''. This proves (2.17) and thus also Proposition 2.1.

Remark 2.1. Similarly, one can show that under the hypotheses of Proposition 2.1 the joint distribution of the variables $Z_N(A_1), \ldots, Z_N(A_n)$ converges to the distribution of the variables $Z(A_1), \ldots, Z(A_n)$ for any $n \geq 1$ and any bounded Borel sets $A_1, \ldots, A_n$.

3. Proof of Theorem 2

For simplicity we restrict ourselves to the proof of the convergence of one-dimensional distributions for $t = 1$ (this remark also relates to the proofs of Theorems 3 and 5).

Let $D_N(x)=\sum_{1}^{N} e^{ijx}=e^{ix}(e^{iNx}-1)/(e^{ix}-1)$ and $K_N(x)=D_N(x/N)/N=e^{ix/N}(e^{ix}-1)/(N(e^{ix/N}-1))$.

We note that

$$K_N(x)\to K_0(x) \tag{3.1}$$

uniformly on each compact set of the real line.

We consider the spectral representation

$$X_t=\int_{\Pi} e^{itx}Z(dx),\quad \Pi=[-\pi,\ \pi] \tag{3.2}$$

of the process $(X_t)_{t\in Z}$. Then

$$S_N^{(m)}\equiv\sum_{1}^{N} H_m(X_t)=\int_{\Pi^m} D_N(x_1+\ldots+x_m)\,d^m Z, \tag{3.3}$$

where, on the right side, one has an m-fold Wiener–Ito integral (cf. [11]). We use the change of variables $y_i = x_i N$, $i = 1, \ldots, m$, in the multiple integral [11, Theorem 4.4] and by the definition of the stochastic measure $Z_N(dx)$ (1.30) we get

$$S_N^{(m)}/B_N=\nu_m\int_{\Pi_N^m} K_N(y_1+\ldots+y_m)\,d^m Z_N, \tag{3.4}$$

where $\Pi_N = [-\pi N,\ \pi N]$ and $\nu_m = N^{1-\alpha m/2}L^{m/2}(N)/B_N$.

According to Proposition 2.1, (3.1), and Lemma 3 of [2], for the convergence in distribution of the integral on the right side of (3.4), it suffices that for each $\varepsilon > 0$ there exist a $K > 0$ and an $N_0 = N_0(\varepsilon, K) \geq 1$ such that for all $N \geq N_0$,

$$\delta_{N,K}\equiv\int_{\Pi_N^m\setminus[-K,K]^m} |K_N(x_1+\ldots+x_m)|^2\,d^m F_N\leqslant\varepsilon. \tag{3.5}$$

Since

$$1=\operatorname{Var}(S_N^{(m)}/B_N)=m!\,\nu_m^2\left(\int_{[-K,K]^m} |K_N(x_1+\ldots+x_m)|^2\,d^m F_N+\delta_{N,K}\right),$$

and also $\nu_m^2 \to C^{-1}(m, \alpha)$ $(N \to \infty)$ and for each $K < \infty$

$$\int_{[-K,K]^m} |K_N(x_1+\ldots+x_m)|^2\,d^m F_N\to\int_{[-K,K]^m} |K_0(x_1+\ldots+x_m)|^2\,d^m F_0$$

according to (3.1) and Proposition 2.1, (3.5) follows from

$$\int_{[-K,K]^m} |K_0(x_1+\ldots+x_m)|^2\,d^m F_0\to C(m,\ \alpha)/m! \tag{3.6}$$

as $K \to \infty$. Obviously (3.6) follows directly from (1.16) and the convergence of the integral D(m) (1.6). □

4. Proof of Theorem 3

Suppose we have a finite collection $\eta_1, \ldots, \eta_m$ of random variables having finite moments of any order. By the Appell (or Vick) product of the variables $\eta_1, \ldots, \eta_m$ we mean the random variable (cf. [12, 13])

$$:\eta_1\ldots\eta_m: = \partial^m \left(\exp\left\{\sum_1^m z_j\eta_j\right\} \Big/ E\exp\left\{\sum_1^m z_j\eta_j\right\}\right) \Big/ \partial z_1\ldots\partial z_m \Big|_{z_1=\ldots=z_m=0}. \qquad (4.1)$$

If the collections $\eta_1, \ldots, \eta_n$ and $\eta_{n+1}, \ldots, \eta_m$ are mutually independent, then

$$:\eta_1\ldots\eta_m: = :\eta_1\ldots\eta_n: :\eta_{n+1}\ldots\eta_m:. \qquad (4.2)$$

If $\eta_1 = \ldots = \eta_m = \eta$, (4.1) coincides with the Appell polynomial of η, i.e.,

$$:\eta\ldots\eta: \equiv :\eta^m: = P_m^{(\eta)}(\eta), \qquad (4.3)$$

where

$$P_m^{(\eta)}(x) = d^m(e^{zx}/Ee^{z\eta})/dz^m\big|_{z=0}, \quad m = 0, 1, \ldots \qquad (4.4)$$

are the Appell polynomials connected with the distribution η.

We return to the linear process X_t (1.18). Let $P_m(x)$, $Q_m(x)$ $(m \geq 0)$ be the Appell polynomials connected with the distribution X_0 and ξ_0, respectively.

Proposition 4.1 [1].

$$P_m(X_t) = \sum_{k=1}^m \sum_{(m)_k}^{+} \binom{m}{m_1, \ldots, m_k} \sum_{(s)_k}' a^{m_1}(t-s_1)\ldots a^{m_k}(t-s_k) Q_{m_1}(\xi_{s_1})\ldots Q_{m_k}(\xi_{s_k}), \qquad (4.5)$$

where $\sum\limits_{(m)_k}^{+}$ is taken over all $(m)_k = (m_1, \ldots, m_k) \in \mathbf{Z}_+^k$, $\mathbf{Z}_+ = \{1, 2, \ldots\}$, such that $m_1 + \ldots + m_k = m$, and the sum $\sum\limits_{(s)_k}'$ is taken over all $(s_k) = (s_1, \ldots, s_k) \in \mathbf{Z}^k$ such that $s_i \neq s_j$ for $i \neq j$, $i, j = 1, \ldots, k$. The series (4.5) converges in the mean square.

(4.5) is a special case of the so-called multinomial formula for Appell polynomials [1]. For finite sums of independent random variables it follows directly from (4.2), (4.3), and the multilinearity of the Appell product. Indeed,

$$P_m(X_t) = :X_t^m: = :\sum_{(s)_m} a(t-s_1)\ldots a(t-s_m)\xi_{s_1}\ldots\xi_{s_m}: = \sum_{(s)_m} a(t-s_1)\ldots a(t-s_m):\xi_{s_1}\ldots\xi_{s_m}:. \qquad (4.6)$$

One gets (4.5) from (4.6) with the help of (4.2), (4.3), and a simple transformation of the sum $\sum\limits_{(s)_m}$.

We rewrite the right side of (4.5) as a sum $\sum\limits_{k=1}^m \sum\limits_{(m)_k}^{+} P_{(m)_k}(t)$. Let

$$P_m(X_t) = P(t) + R(t), \qquad (4.7)$$

where

$$P(t) = \sum_{(s)_m}' a(t-s_1)\ldots a(t-s_m)\xi_{s_1}\ldots\xi_{s_m} \qquad (4.8)$$

is the basic term and

$$R(t) = \sum_{k<m} \sum_{(m)_k}^{+} P_{(m)_k}(t) \qquad (4.9)$$

is the error term. One has

Proposition 4.2 (cf. [1, Step 3]).

$$\operatorname{Var}\left(\sum_{1}^{N} R(t)\right)=O(N). \tag{4.10}$$

Proof. Let $(m)_k = (m_1, \ldots, m_k)$, $k < n$ and $m_i \geq 2$ for some $i = 1, \ldots, k$. Let $r^{(k)}(t) = EP_{(m)_k}(0)P_{(m)_k}(t)$. Since

$$\operatorname{Var}\left(\sum_{1}^{N} P_{(m)_k}(t)\right)=\sum_{t,\, s=1}^{N} r^{(k)}(t-s) \leqslant N \sum_{-\infty}^{\infty} |r^{(k)}(t)|,$$

to prove (4.10) it suffices to see that

$$\sum |r^{(k)}(t)| < \infty. \tag{4.11}$$

Considering the definition of $P_{(m)_k}(t)$ and the equality

$$E\left(Q_{m_1}(\xi_{s_1})\ldots Q_{m_k}(\xi_{s_k}) Q_{m_1'}(\xi_{s_1'})\ldots Q_{m_k'}(\xi_{s_k'})\right)=\prod_{j=1}^{k} \delta(s_j, s_j') E\left(Q_{m_j}(\xi_0) Q_{m_j'}(\xi_0)\right), \tag{4.12}$$

which is valid for any $s_1 < \ldots < s_k$, $s_1' < \ldots < s_k'$, and the independence of ξ_k and the relation $EQ_m(\xi_0) = 0 \ \forall\ m \geq 1$, which follow from it, we get

$$r^{(k)}(t)=C \sum \sum_{(s)_k}' \prod_{j=1}^{k} a^{m_j}(t-s_j) a^{m_{i(j)}}(s_j) E\left(Q_{m_j}(\xi_0) Q_{m_{i(j)}}(\xi_0)\right), \tag{4.13}$$

where the first sum is taken over all permutations $i(1), \ldots, i(k)$ of the numbers $1, \ldots, k$ (by the letter C here and below we denote possibly different constants). Since $\Sigma a^2(t) < \infty$, it follows from (4.13) that

$$|r^{(k)}(t)| \leqslant C\left(\sum a^2(t-s) a^2(s)+\sum \sum a^2(t-s_1)|a(t-s_2) a(s_1)| a^2(s_2)\right) \equiv C\left(r'(t)+r''(t)\right).$$

Here $\sum r'(t) < \infty$ and

$$\sum r''(t)=\sum_{s_2} \sum_{\tau} a^2(\tau) \sum_{s_1} |a(s_1+\tau-s_2) a(s_1)| a^2(s_2) \leqslant \left(\sum a^2(t)\right)^3 < \infty$$

according to Cauchy's inequality. This proves (4.11) and thus Proposition 4.2. □

We recall the definition of the functions $D_N(x)=\sum_{1}^{N} e^{itx}$ and $K_N(x) = D_N(x/N)/N$. We also need the equality

$$|D_N(x)|=|\sin(Nx/2)/\sin(x/2)|. \tag{4.14}$$

Proposition 4.3. Let $P_G(t)$ be defined as

$$P_G(t)=\sum_{(s)_m}' b(t-s_1, \ldots, t-s_m) \xi_{s_1}\ldots\xi_{s_m}, \tag{4.15}$$

where

$$b(s_1, \ldots, s_m) = (2\pi)^{-m/2} \int_{\Pi^m} \exp\left\{ i \sum_1^m x_j s_j \right\} \prod_{i=1}^m \hat{a}(x_j) G(x) d^m x, \tag{4.16}$$

and the function G(x), $x = (x_1, \ldots, x_m) \in \Pi^m$ is symmetric in the variables $x_1, \ldots, x_m$ and uniformly bounded in N [G(x) may also depend on N]. Then

$$\operatorname{Var} \sum_1^N P_G(t) = m! \int_{\Pi^m} |D_N(x_1 + \ldots + x_m)|^2 \prod_{j=1}^m |\hat{a}(x_j)|^2 G^2(x) d^m x + O(N). \tag{4.17}$$

Proof. We consider the sum

$$b_N(s_1, \ldots, s_m) \equiv \sum_1^N b(t-s_1, \ldots, t-s_m) = (2\pi)^{-m/2} \int_{\Pi^m} \exp\left\{ -i \sum_1^m x_j s_j \right\} D_N(x_1 + \ldots + x_m) \prod_{j=1}^m \hat{a}(x_j) G(x) d^m x. \tag{4.18}$$

Clearly,

$$\operatorname{Var} \sum_1^N P_G(t) = m! \sum_{(s)_m}' |b_N(s_1, \ldots, s_m)|^2 \equiv m! \sum_{(s)_m} |b_N(s_1, \ldots, s_m)|^2 + m! R_N. \tag{4.19}$$

According to Parseval's identity, the first summands on the right sides of (4.17) and (4.19) coincide. Thus, it remains for us to verify that

$$R_N = \left(\sum_{(s)_m} - \sum_{(s)_m}' \right) |b_N|^2 = O(N).$$

In view of the symmetry of $b_N(s_1, \ldots, s_m)$,

$$R_N \leqslant C \sum_{(s)_m} |b_N|^2 \mathbf{1}_{(s_{m-1}=s_m)} \leqslant C \int_{\Pi^{m-1}} \left(|D_N(x_1 + \ldots + x_{m-2} + u)| \prod_{j=1}^{m-2} |\hat{a}(x_j)| \int_\Pi |\hat{a}(u-v)\hat{a}(v)| dv \right)^2 d^{m-2}x du$$

(in the last inequality we make use of Parseval's identity and the boundedness of the function G). We note that $\int_\Pi |\hat{a}(u-v)\hat{a}(v)| dv \leqslant \int_\Pi |\hat{a}|^2 dv \leqslant C$. On the other hand, using the estimate

$$|D_N(x)| \leqslant \begin{cases} CN & \text{for } |x| < 1/N, \\ C/x & \text{for } |x| \geqslant 1/N \end{cases}$$

[cf. (4.14)], it is easy to verify that

$$\sup_{x \in \Pi} \int_\Pi |D_N(x+u)|^2 du \leqslant CN.$$

Consequently,

$$R_N \leqslant CN \int_{\Pi^{m-2}} \prod_{j=1}^{m-2} |\hat{a}(x_j)|^2 d^{m-2}x = CN.$$

We preface the rest of the proof of Theorem 3 with the

Proof of Theorem 4. We recall that $|\hat{a}(x)|^2$ is the density of the spectral measure F of the process X_t (1.18). We note that if $G(x) \equiv 1$, $P_G(t)$ (4.15) coincides with P(t) (4.8). Using Propositions 4.2 and 4.3, we have that

$$\mathrm{Var} \sum_{1}^{N} P_m(X_t) = m! \int_{\Pi^m} |D_N(x_1 + \ldots + x_m)|^2 d^m F + O(N). \tag{4.20}$$

Using the equality $r(t) = \int_{\Pi} e^{itx} dF$ and the definition of the function $D_N(x)$, we see that the integral on the right side of (4.20) is equal to $\sum_{t,s=1}^{N} r^m(t-s)$. □

Let

$$G'(x) = \mathbf{1}_{[-K/N,\, K/N]^m}(x), \quad G''(x) = \mathbf{1}_{\Pi^m}(x) - G'(x)$$

and $P'(t) = P_{G'}(t)$, $P''(t) = P_{G''}(t)$. It is clear that

$$P(t) = P'(t) + P''(t), \tag{4.21}$$

where $P(t)$ is defined in (4.8). As follows from Proposition 4.4 below, one can neglect the summand $P''(t)$ for our purposes in what follows.

Proposition 4.4. Let (1.15)-(1.17) of Theorem 3 hold. Then for each $\varepsilon > 0$ there exist $K > 0$ and $N_0 \geq 1$ such that for all $N \geq N_0$

$$\mathrm{Var} \sum_{1}^{N} P''(t) \leqslant \varepsilon B_N^2.$$

Proof. According to Proposition 4.3 and the definition of the measure F_N (1.29), we can write

$$\mathrm{Var} \sum_{1}^{N} P'(t) = m! \int_{-K/N}^{K/N} \cdots \int |D_N(x_1 + \ldots + x_m)|^2 d^m F + O(N) =$$

$$= m!\, N^{2-\alpha m} L^m(N) \int_{-K}^{K} \cdots \int |K_N(x_1 + \ldots + x_m)|^2 d^m F_N + O(N).$$

Dividing both sides by B_N^2 and considering (1.15), we get

$$\lim_{N\to\infty} B_N^{-2}\, \mathrm{Var} \sum_{1}^{N} P'(t) = \frac{m!}{C(m, \alpha)} \lim_{N\to\infty} \int_{-K}^{K} \cdots \int |K_N(x_1 + \ldots + x_m)|^2 d^m F_N =$$
$$= \frac{m!}{C(m, \alpha)} \int_{-K}^{K} \cdots \int |K_0(x_1 + \ldots + x_m)|^2 d^m F_0. \tag{4.22}$$

In the last equality, as in the proof of Theorem 2, we have used Proposition 2.1 and the relation

$$\sup_{|x| \leqslant K} |K_N(x) - K_0(x)| \to 0. \tag{4.23}$$

We complete the proof by an argument analogous to that which was used in the proof of Theorem 2. According to (1.17) and (1.6), (2.4),

$$\frac{m!}{C(m, \alpha)} \int_{\mathbf{R}^m} |K_0(x_1 + \ldots + x_m)|^2 d^m F_0 = 1.$$

Consequently, from (4.22) we have that for any $\varepsilon > 0$ there exist $K > 0$ and $N_0 = N_0(\varepsilon, K) \geq 1$ such that, for all $N \geq N_0$,

$$\left| B_N^{-2}\, \mathrm{Var} \sum_{1}^{N} P'(t) - 1 \right| \leqslant \varepsilon. \tag{4.24}$$

On the other hand, since $N/B_N^2 \to 0$, by virtue of Proposition 4.2

$$\lim_{N\to\infty} B_N^{-2} \operatorname{Var} \sum_1^N P(t) = \lim_{N\to\infty} B_N^{-2} \operatorname{Var} \sum_1^N P_m(X_t) = 1. \qquad (4.25)$$

The assertion of Proposition 4.4 follows from (4.24), (4.25), and (4.21). □

Thanks to (4.23) and the continuity of the function K_0, for any $\varepsilon > 0$ and $K > 0$ we can find $N_0 = N_0(\varepsilon, K) \geq 1$ and a step function

$$g_\Delta(x) = g_\Delta(x_1, \ldots, x_m) = \sum_{(\Delta)} g_{\Delta_1\ldots\Delta_m} \mathbf{1}_{\Delta_1\times\ldots\times\Delta_m}(x) \qquad (4.26)$$

such that for all $N \geq N_0$

$$\sup_{x\in[-K,K]^m} |K_N(x_1+\ldots+x_m) - g_\Delta(x_1, \ldots, x_m)| \leqslant \varepsilon. \qquad (4.27)$$

In (4.26) the sum $\sum_{(\Delta)}$ is taken over all intervals $\Delta_1, \ldots, \Delta_m \in \{\Delta^{(-M)}, \ldots, \Delta^{(M)}\}$, where $\Delta^{(-M)}, \ldots, \Delta^{(M)}$ form a partition of the interval $[-K, K]$ satisfying the conditions $\Delta^{(-i)} = -\Delta^{(i)}$, $i = 1, \ldots, M$, and

$$\delta \equiv \max_{i\leqslant M} \lambda(\Delta^{(i)}) \to 0 \quad (\varepsilon \to 0); \qquad (4.28)$$

λ being Lebesgue measure. Let

$$I_\Delta = \sum_{(s)_m}' b_\Delta(s_1, \ldots, s_m)\xi_{s_1}\ldots\xi_{s_m}. \qquad (4.29)$$

where [cf. (4.15), (4.18)]

$$b_\Delta(s_1, \ldots, s_m) = (2\pi)^{-m/2} \int_{-K/N}^{K/N}\!\!\ldots\!\int \exp\left\{-i\sum_1^m x_j s_j\right\} N g_\Delta(xN) \prod_{j=1}^m \hat{a}(x_j)\, d^m x. \qquad (4.30)$$

As in the proof of Proposition 4.3, we have

$$\begin{aligned}\operatorname{Var}\left(\sum_1^N P'(t) - I_\Delta\right) &\leqslant m! \int_{-K/N}^{K/N}\!\!\ldots\!\int |D_N(x_1+\ldots+x_m) - N g_\Delta(x_1 N, \ldots, x_m N)|^2 d^m F = \\ &= C B_N^2 \int_{-K}^{K}\!\!\ldots\!\int |K_N(x_1+\ldots+x_m) - g_\Delta(x_1, \ldots, x_m)|^2 d^m F_N \leqslant C\varepsilon^2 B_N^2 F_N^m[-K, K].\end{aligned} \qquad (4.31)$$

Since $F_N[-K, K] \to F_0[-K, K]$ (cf. Proposition 2.1), we have thus proved

<u>Proposition 4.5.</u> For any $\varepsilon > 0$ and $K > 0$ there exist an $N_0 \geq 1$ and a step function g_Δ (4.26) such that for all $N \geq N_0$

$$\operatorname{Var}\left(\sum_1^N P'(t) - I_\Delta\right) \leqslant \varepsilon B_N^2. \qquad (4.32)$$

Recalling what the spectral measures Z and Z_N are equal to for a linear process [cf. (2.15), (2.16)], one can note that the sum I_Δ (4.29) is almost the polynomial in the random variables $Z_N(\Delta^{(-M)}), \ldots, Z_N(\Delta^{(M)})$ whose convergence is established in Proposition 2.1. In order to get rid of the word "almost" we must take two more steps, namely:

1) to remove from I_Δ the "diagonals" $\Delta_i = \pm\Delta_j$;

2) to replace the sum $\sum'_{(s)_m}$ in (4.29) by the usual sum $\sum_{(s)_m}$.

One takes the first step easily. We denote by $\sum'_{(\Delta)}$ the sum over all $\Delta_1, \ldots, \Delta_m \in \{\Delta^{(-M)}, \ldots, \Delta^{(M)}\}$ such that $\Delta_i \neq \pm\Delta_j$ for $i \neq j$, $i, j = 1, \ldots, m$, and by $g_\Delta'(x)$, $x \in \mathbf{R}^m$ the step function on the right side of (4.26), where in place of the sum $\sum_{(\Delta)}$ one has $\sum'_{(\Delta)}$.

Finally, we shall denote by I_Δ' the corresponding sum of (4.29), (4.30) with g_Δ replaced by g_Δ'. As in (4.31),

$$\operatorname{Var}(I_\Delta - I'_\Delta) \leqslant CB_N^2 \int_{-K}^{K} \cdots \int |g_\Delta - g'_\Delta|^2 d^m F_N \leqslant CB_N^2 F_N^{m-2}[-K, K] \iint_{-K}^{K} \mathbf{1}_{\{||x_1|-|x_2||\leqslant\delta\}} d^2 F_N. \tag{4.33}$$

As $N \to \infty$, the integral on the right side of (4.33) converges to the corresponding integral with respect to the measure F_0 (cf. Proposition 2.1), which one can make arbitrarily small by the choice of a small $\delta > 0$. The following proposition is a consequence of the argument just made:

<u>Proposition 4.6.</u> Proposition 4.5 remains true with I_Δ replaced by I_Δ'.

We rewrite I_Δ' in the form

$$I'_\Delta = N \sum_{(\Delta)}' \sum_{(s)_m}' g_{\Delta_1 \ldots \Delta_m} a_{\Delta_1}(s_1) \ldots a_{\Delta_m}(s_m) \xi_{s_1} \ldots \xi_{s_m}, \tag{4.34}$$

where

$$a_\Delta(s) = (2\pi)^{-1/2} \int_{\Delta/N} e^{-isx} \hat{a}(x)\, dx, \tag{4.35}$$

$\Delta \in \{\Delta^{(-M)}, \ldots, \Delta^{(M)}\}$. We denote by J_Δ the corresponding sum (4.34), where $\sum'_{(s)_m}$ is replaced by $\sum_{(s)_m}$.

<u>Proposition 4.7.</u>

$$\operatorname{Var}(J_\Delta - I'_\Delta) = o(B_N^2). \tag{4.36}$$

We postpone the <u>proof</u> of Proposition 4.7 until the end of this section, and now we finish the proof of Theorem 3.

Since

$$\sum a_\Delta(s) \xi_s = Z(\Delta/N)$$

[cf. (2.15)], one has

$$J_\Delta = N \sum_{(\Delta)}' g_{\Delta_1 \ldots \Delta_m} Z(\Delta_1/N) \ldots Z(\Delta_m/N)$$

and

$$J_\Delta / B_N = (C(m, \alpha))^{-1/2} \sum_{(\Delta)}' g_{\Delta_1 \ldots \Delta_m} Z_N(\Delta_1) \ldots Z_N(\Delta_m).$$

Since on the right side of the last equality there is a polynomial in a finite number of variables $Z_N(\Delta^{(-M)}), \ldots, Z_N(\Delta^{(M)})$, on the basis of Remark 2.1 we conclude that

$$J_\Delta/B_N \overset{d}{\Rightarrow} (C(m, \alpha))^{-1/2} \sum_{(\Delta)}' g_{\Delta_1 \ldots \Delta_m} Z_0(\Delta_1) \ldots Z_0(\Delta_m). \tag{4.37}$$

Of course the sum on the right side of (4.37) is nothing but the multiple Wiener–Ito integral

$$\int_{\mathbf{R}^m} g'_\Delta(x) Z_0(dx_1) \ldots Z_0(dx_m) \equiv \int g'_\Delta d^m Z_0$$

of the step function g_Δ'. It is known that

$$E\left|\int g'_\Delta(x) d^m Z_0 - \int K_0(x_1 + \ldots + x_m) d^m Z_0\right|^2 = m! \int |g'_\Delta(x) - K_0(x_1 + \ldots + x_m)|^2 d^m F_0. \tag{4.38}$$

Using (4.23), (4.27), (4.28), and also the continuity of the measure F_0, we see that the right side of (4.38) can be made arbitrarily small by suitable choice of K and ε. Together with (4.37) and Propositions 4.1-4.7, this completes the proof of Theorem 3. □

<u>Proof of Proposition 4.7.</u> We set $\rho_N = (J_N - I_\Delta')/B_N$. Then

$$\rho_N = \sum_{(\Delta)}' \sum_{(V)} g_{\Delta_1 \ldots \Delta_m} \gamma_{(V), (\Delta)}, \tag{4.39}$$

where the sum $\sum_{(V)}$ is taken over all partitions $(V) = (V_1, \ldots, V_r)$ of the set $\{1, \ldots, m\}$ such that $|V_i| \geq 2$ for some $i = 1, \ldots, r$;

$$\gamma_{(V), (\Delta)} = \sum_{(s)_r}' p_{V_1}(s_1) \ldots p_{V_r}(s_r) \xi_{s_1}^{|V_1|} \ldots \xi_{s_r}^{|V_r|}, \tag{4.40}$$

$$p_V(s) = \prod_{i \in V} q_{\Delta_i}(s) \quad (V \subset \{1, \ldots, m\}) \tag{4.41}$$

and

$$q_\Delta(s) = a_\Delta(s) (N^\alpha / L(N))^{1/2} \tag{4.42}$$

[$a_\Delta(s)$ is defined in (4.35)]. It follows from (2.2) and (2.19) that as $N \to \infty$,

$$\max_{s \in \mathbf{Z}} |q_\Delta(s)| \to 0, \tag{4.43}$$

$$\sum |q_\Delta(s)|^2 = F_N(\Delta) \to F_0(\Delta) \tag{4.44}$$

and

$$\sum q_{\Delta_i}(s) q_{\Delta_j}(s) = F_N(\Delta_i \cap (-\Delta_j)) = 0 \tag{4.45}$$

under the condition that $i \neq j$, $i, j = 1, \ldots, m$ (we recall that the cases $\Delta_i = \pm\Delta_j$ for $i \neq j$ do not occur in the sum $\sum_{(\Delta)}'$). From (4.43)-(4.45) it is easy to derive the following relations for the coefficients $p_V(s)$ (4.41) of the polynomial form (4.40):

$$\sum |p_V(s)| \leq C, \quad |V| \geq 2. \tag{4.46}$$

$$\sum p_V(s) = 0, \quad |V| = 2, \tag{4.47}$$

$$\sum |p_V(s)| \to 0, \quad |V| \geqslant 3. \tag{4.48}$$

To prove (4.36) it suffices to show that as $N \to \infty$,

$$E|\gamma|^2 \to 0 \tag{4.49}$$

for each $\gamma = \gamma_{(V),(\Delta)}$ in (4.39). We have

$$E|\gamma|^2 = \sum_{(s)_r}' \sum_{(s')_r}' p_{V_1}(s_1)\dots p_{V_r}(s_r) \overline{p_{V_1}(s_1')}\dots\overline{p_{V_r}(s_r')}\,\mu_{(V)}, \tag{4.50}$$

where

$$\mu_{(V)} = E(\xi_{s_1}^{|V_1|}\dots\xi_{s_r}^{|V_r|}\xi_{s_1'}^{|V_1|}\dots\xi_{s_r'}^{|V_r|}). \tag{4.51}$$

Let $|V_1| \geq 2$. We divide the double sum on the right side of (4.50) into $r + 1$ sums $\sum_{(i)}$, $i = 0, 1, \dots, r$, as follows. Into the sum $\sum_{(0)}$ we put collections $(s)_r$, $(s')_r$ satisfying the condition $s_1 \neq s_1', \dots, s_1 \neq s_r'$, and into the sum $\sum_{(i)}$ $(i = 1, \dots, r)$ collections $(s)_r$, $(s')_r$ such that $s_1 = s_i'$. Considering that the random variables ξ_s, $s \in \mathbf{Z}$ are independent and have mean 0, and using (4.46) and (4.48), it is easy to conclude that

$$\sum_{(i)} \dots \to 0 \quad (i = 1, \dots, r). \tag{4.52}$$

We consider the remaining summand $\sum_{(0)}$. Let $(\mathbf{Z}^k)'$ be the collection of all collections $(s)_k \in \mathbf{Z}^k$ such that $s_i \neq s_j$ for $i \neq j$, $i, j = 1, \dots, k$. We note that for $s_1 \neq s_1', \dots, s_r'$ the mean $\mu_{(V)}$ (4.51) is independent of s_1. Hence $\sum_{(0)}$ can be represented in the form

$$\sum_{(0)} \dots = \Big(\sum_{-\infty}^{\infty} p_{V_1}(s_1)\Big) \sum_{(s)_{r-1}}' \sum_{(s')_r}' \dots - \sum_{s_1=-\infty}^{\infty} \sum_{k=2}^{2r} \sum^{(k)} \dots \equiv \sum' - \sum'',$$

where $(s)_{r-1} = (s_2, \dots, s_r) \in (\mathbf{Z}^{r-1})'$, and the sum $\sum^{(k)}$ is taken over all collections $(s)_{r-1} \in (\mathbf{Z}^{r-1})'$, $(s')_r \in (\mathbf{Z}^r)'$ satisfying the condition $s_k = s_1$ for $2 \leq k \leq r$ and $s_{k-r}' = s_1$ for $r < k \leq 2r$. But then $\sum'' \to 0$ by virtue of the same reasons as for (4.52), and for $\sum' \to 0$ one must use (4.47) in addition. □

5. Proof of Theorem 5

As usual, let $F(dx) = f(x)dx = |\hat{a}(x)|^2$ and $r(t) = \int_\Pi e^{itx}dF$ denote, respectively, the spectral measure and covariance function of the linear process (X_t).

It follows from Theorems 3 and 4 that to prove Theorem 5 it suffices to verify that the following relations hold:

$$A_N^2 = \sum_{t,s=1}^{N} r(t-s) \sim L_1(N)\,N^{2-\alpha} \tag{5.1}$$

and

$$\tilde{B}_N^2 \equiv m! \sum_{t,s=1}^{N} r^m(t-s) \sim C(m, \alpha) L_1^m(N)\,N^{2-\alpha}. \tag{5.2}$$

where

$$L_1(x) = \bar{\theta} D(1) L(x) \tag{5.3}$$

is the s.v.f. differing by the constant $\bar{\theta}D(1)$ from the s.v.f. $L(\cdot)$ in (1.26).

Passing to the spectral representation of the covariance function just as in the proofs of Theorems 2 and 3, we have

$$A_N^2 / L_1(N) N^{2-\alpha} = \int_{-\pi N}^{\pi N} |K_N(x)|^2 dF_N \tag{5.4}$$

and

$$\tilde{B}_N^2 / L_1^m(N) N^{2-\alpha m} = m! \int_{-\pi N}^{\pi N} \cdots \int |K_N(x_1 + \ldots + x_m)|^2 d^m F_N, \tag{5.5}$$

where

$$F_N(dx) = F(dx/N) N^{\alpha} / L_1(N). \tag{5.6}$$

We consider the convergence of the spectral measures F_N (5.6).

<u>Proposition 5.1.</u> Under the hypotheses of Theorem 5,

$$F_N(A) \to F_0(A), \tag{5.7}$$

where A is an arbitrary bounded Borel set and $F_0(A) = D^{-1}(1) \int_A |x|^{\alpha-1} dx$ [cf. (2.4)].

<u>Proof</u>. According to (1.26) and (5.6),

$$F_N(A) = (\bar{\theta} D(1))^{-1} \int_A |x|^{\alpha-1} (L(N/|x|)/L(N)) \theta(N/|x|) dx$$

(we assume that $A \subset [-\pi N, \pi N]$). According to a well-known property of s.v.f. (cf., e.g., [7], Lemma 4.1) for any $\varepsilon > 0$ and $0 < x_0 < \infty$, one can find a $0 < C < \infty$ such that $L(N/x)/L(N) \leq Cx^{-\varepsilon}$ uniformly with respect to $N \geq 1$ and $x \in (0, x_0)$. Since the function $\theta(\cdot)$ is bounded, one has

$$\lim_{\delta \to 0} \sup_{N \geq 1} F_N(-\delta, \delta) \leq C \lim_{\delta \to 0} \int_{|x| < \delta} |x|^{\alpha + \varepsilon - 1} dx = 0 \tag{5.8}$$

under the condition that $\alpha + \varepsilon < 1$. Thanks to (5.8) and the symmetry of the measures F_N, it suffices to prove the convergence (5.7) for the intervals $A = (a, b)$, $0 < a < b < \infty$.

In view of the fact that $L(N/x)/L(N) \to 1$ uniformly with respect to $x \in A = (a, b)$, (5.7) follows from

$$I_N \equiv \int_a^b x^{\alpha-1} \theta(N/x) dx \to \bar{\theta} \int_a^b x^{\alpha-1} dx.$$

Let $\Xi(x) = \int_0^x \theta(y) dy$. Then $\Xi(x)/x \to \bar{\theta}$ as $x \to \infty$ [cf. (1.27)] and

$$I_N = -N^{-1} \int_a^b x^{\alpha+1} d\Xi(N/x) = -N^{-1} \Xi(N/x) x^{\alpha+1} \Big|_a^b +$$

$$+(\alpha+1)N^{-1}\int_a^b \Xi(N/x)x^\alpha dx \to -\bar{0}(b^\alpha-a^\alpha)+\bar{0}(\alpha+1)\int_a^b x^{\alpha-1}dx = \bar{\theta}\int_a^b x^{\alpha-1}dx. \quad \square$$

To prove (5.1) and (5.2), it suffices to see that on the right sides of (5.4) and (5.5) one can pass to the limit under the integral sign. In view of Lemma 5.1 and the uniform convergence on compacta of the functions $K_N(\cdot)$, for this it suffices that the following condition hold uniformly with respect to $N \geq 1$, where $[K]^c = [-\pi N, \pi N]^m \backslash [-Km, K]^m$:

$$\lim_{K\to\infty} \int_{[K]^c} |K_N(x_1+\ldots+x_m)|^2 d^m F_N = 0. \tag{5.9}$$

Since the measure $F_N(dx)$ is majorized by the measure $\tilde{F}_N(dx) = \tilde{F}(dx/N)N^\alpha/L(N)$, where $\tilde{F}(dx) = \tilde{C}|x|^{\alpha-1}L(1/|x|)dx$ and $\tilde{C} = \sup_{x\geq 0}\theta(x)/(\theta D(1))$, it suffices to verify (5.9) for $\tilde{F}_N$ instead of F_N. Such a verification is made on pp. 35-36 of Dobrushin and Major [2] (the verification of condition (2.8) of Lemma 3 of [2]). $\square$

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