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An exact solution for flow in a porous pipe

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1. Introduction

During the past few years the problem of fluid flow in pipes with mass transfer at the walls has received much attention because of its practical applications. There have been numerous theoretical investigations usually dealing with steady, incompressible, laminar flow with either constant injection or suction. The theoretical solutions can be divided into two classes.

The first is the fully developed solution for constant suction or injection which is a similarity solution. The problem reduces to solving an ordinary non-linear differential equation in terms of a suction Reynolds number. Analytical and numerical solutions have been discussed by numerous research workers and their work is typified by references [1] to [7]. The second type of solution deals with the entrance region solution and this requires a completely numerical approach to deal with the changing shape of the non-dimensional velocity profile. Typical papers on this topic are references [8] to [10].

One of the limitations of investigations into this type of problem is that, as far as the author is aware, all the solutions have involved constant suction or injection. Any experiment (for example, [10]) has to be set up so that the fluid extracted from the wall has constant velocity. The present paper produces a fortunate exact solution which could prove to be most useful in both experimental and theoretical investigations of porous channel flow. The solution has the form of a perturbation of the flow in an impermeable pipe but satisfies the Navier-Stokes equations exactly. It turns out that many different types of velocity distributions of suction or injection are possible but some exhibit special features. These have the remarkable property that the Poiseuille velocity profile deforms continuously into the fully developed suction velocity profile and an entry length is not necessary.

The solution can probably be extended to another class of suction distributions and many also have some applications in other types of fluid flow problems.

2. The equations

Consider the axisymmetric motion of an incompressible fluid in an infinitely long cylinder of radius a. Choose a cylindrical polar coordinate system (r, θ, z) where the axis $0z$ lies along the centre of the tube, r is the distance measured radially and θ is the azimuthal angle. Let u and v be the velocity components in the directions of z and r increasing respectively. Then, for axisymmetric flow, the equation of continuity is

$$\frac{\partial}{\partial r}(r v) + r\frac{\partial u}{\partial z} = 0 \tag{1}$$

and the Navier-Stokes equations are

$$u\,\frac{\partial u}{\partial z}+v\,\frac{\partial u}{\partial r}=-\frac{1}{\varrho}\,\frac{\partial p}{\partial z}+\nu\,\nabla^2 u, \tag{2}$$

$$u\,\frac{\partial v}{\partial z}+v\,\frac{\partial v}{\partial r}=-\frac{1}{\varrho}\,\frac{\partial p}{\partial r}+\nu\left(\nabla^2 v-\frac{v}{r^2}\right), \tag{3}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

and where p is the pressure, ϱ the density and ν the kinematic viscosity of the fluid.

The boundary conditions at the pipe wall are

$$u=0 \quad \text{at} \quad r=a \tag{4}$$

while the speed of suction or injection is assumed to be

$$v=\sum_{i=1}^{\infty} V_i\, e^{\alpha_i z/a} \tag{5}$$

where the constants V_i, α_i are to be determined. In addition symmetry implies that

$$v=0 \quad \text{at} \quad r=0. \tag{6}$$

To solve equations (1), (2) and (3) it is convenient to introduce the stream function ψ defined by

$$u=\frac{1}{r}\,\frac{\partial\psi}{\partial r}, \qquad v=-\frac{1}{r}\,\frac{\partial\psi}{\partial z} \tag{7}$$

and the non-dimensional variables

$$\eta=\frac{r^2}{a^2}, \qquad z_1=\frac{z}{a}. \tag{8}$$

Then, after some manipulation, it can be shown that ψ satisfies

$$\frac{\partial\psi}{\partial\eta}\,\frac{\partial}{\partial z_1}\,(\nabla_1^2\,\psi)+\frac{\partial\psi}{\partial z_1}\left\{\frac{1}{\eta}\,\nabla_1^2\,\psi-\frac{\partial}{\partial\eta}\,(\nabla_1^2\,\psi)\right\}=\frac{\nu\,a}{8}\,\nabla_1^2\,(\nabla_1^2\,\psi) \tag{9}$$

where

$$\nabla_1^2=4\,\eta\,\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial z_1^2}.$$

It should be noted that in (9) ψ has the dimensions $\nu\,a$. The boundary conditions (4) and (6) now become

$$\left.\begin{aligned}\frac{\partial\psi}{\partial\eta}&=0 \quad \text{at} \quad \eta=1,\\ \frac{\partial\psi}{\partial z}&=0 \quad \text{at} \quad \eta=0.\end{aligned}\right. \tag{10}$$

3. The solution

Assume a solution of equation (9) in the form

$$\psi=f_0\,(\eta)+\sum_{i=1}^{\infty} e^{\alpha_i z_1} f_i(\eta), \qquad \alpha_i\neq 0 \tag{11}$$

where the α_i are constants to be determined and can be positive or negative. It is expected that $f_0(\eta)$ will correspond to flow through an impermeable pipe. From (7), this form for ψ gives the radial velocity V at the wall as

$$V = -\frac{1}{a^2}\sum_{i=1}^{\infty} \alpha_i \, e^{\alpha_i z_1} f_i(1)$$

which is equivalent to (5). When (11) is substituted into equation (9) it is easily seen that the terms in z_1 will be proportional to $e^{2\alpha_i z_1}$, $e^{(\alpha_i+\alpha_j) z_1}$, $e^{0 z_1}(=1)$ and $e^{\alpha_i z_1}$. The coefficients of each of these terms will now be examined.

Firstly, equating the coefficient of $e^{2\alpha_i z_1}$ to zero yields

$$\alpha_i \left[\left(\eta \frac{df_i}{d\eta} + f_i \right) D^2 f_i - \eta f_i \frac{d}{d\eta}(D^2 f_i) \right] = 0, \tag{12}$$

where

$$D^2 f_i = \eta \frac{d^2 f_i}{d\eta^2} + \frac{\alpha_i^2}{4} f_i. \tag{13}$$

Equation (12) can be readily integrated to give

$$D^2 f_i = K_i \eta f_i \tag{14}$$

where K_i is an arbitrary constant. The coefficient of $e^{(\alpha_i+\alpha_j) z_1}$ is

$$\frac{1}{\eta}\left[\left(\alpha_j \eta \frac{df_i}{d\eta} + \alpha_i f_i \right) D^2 f_j + \left(\alpha_i \eta \frac{df_j}{d\eta} + \alpha_j f_j \right) D^2 f_i - \alpha_i f_i \eta \frac{d}{d\eta}(D^2 f_j) - \alpha_j f_j \eta \frac{d}{d\eta}(D^2 f_i) \right]. \tag{15}$$

However, f_i and f_j must satisfy equation (14) and it is readily shown that the coefficient of $e^{(\alpha_i+\alpha_j) z_1}$ is zero provided

$$K_i = K_j = K \text{ say}, \tag{16}$$

where K is a constant for all the equations (14).

The only term remaining in the expansion for ψ that can contribute to the coefficient of $e^{0 z_1}$ is $f_0(\eta)$. Substitution in (9) gives

$$f_0'''' + 2 f_0''' = 0 \tag{17}$$

where $'$ denotes differentiation with respect to η. The boundary conditions (10) give $f_0'(1) = 0$ and the solution of (17) is

$$f_0 = c_0 (2\eta - \eta^2) \tag{18}$$

where c_0 is an arbitrary constant. [Note the condition that u is finite at $\eta = 0$ has been used.] The solution corresponds to flow in an impermeable pipe.

Finally the coefficient of $e^{\alpha_i z_1}$ is

$$\frac{\alpha_i}{\eta}\left\{ \eta f_0' D^2 f_i + f_i \left[f_0'' - \frac{d}{d\eta}(\eta f_0'') \right] \right\} = \frac{a\,v}{2} D^4 f_i,$$

and, using (14), this can be reduced to

$$K(2 f_i' + K \eta f_i) = \frac{\alpha_i f_i}{2 v}(K f_0' - f_0'''),$$

where f_i and f_0 satisfy (14) and (17) respectively. It is immediately seen that it is necessary for both K and f''' to be equal to zero. The condition on $f_0(\eta)$ means that a solution through an annulus (which involves a term like $\eta \log \eta$) is not possible and only a solution through the pipe (18) can be found. It is also worth noting that the flow through parallel plates can be obtained from the flow through an annulus by taking the correct limit (the method for finding this limit is given in [11]). Consequently there is not a corresponding solution for flow through parallel plates.

Thus it only remains to find the solutions $f_i(\eta)$ of the differential equation

$$\eta f_i'' + \frac{\alpha_i^2}{4} f_i = 0. \tag{19}$$

The transformation $\eta = x^2 \alpha_i^{-2}, f_i = \eta^{1/2} g_i$ changes equation (19) into Bessel's equation and the appropriate solution that is finite at $\eta = 0$ can be shown to be

$$\begin{aligned} f_i &= c_i \eta^{1/2} J_1(\alpha_i \eta^{1/2}) \\ &= c_i \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\alpha_i}{2}\right)^{2n+1} \eta^{n+1}}{n!\,(n+1)!}. \end{aligned} \tag{20}$$

The only other boundary condition that must be satisfied is $u = 0$ at $r = a$; that is

$$\frac{\mathrm{d}f_i}{\mathrm{d}\eta} = 0 \quad \text{at} \quad \eta = 1,$$

or

$$J_0(\alpha_i) = 0.$$

Thus the α_i are the zeros of the Bessel function $J_0(x)$ and have values given by

$$\pm \alpha_i = 2.405,\ 5.520,\ 8.654,\ 11.792,\ 14.931, \ldots, \tag{21}$$

and tending rapidly to the value $\left(\frac{4n+3}{4}\right)\pi$ where n is an integer.

The required solution is

$$\psi = c_0(\eta^2 - 2\eta) + \sum_{i=1}^{\infty} c_i \eta^{1/2} J_1(\alpha_i \eta^{1/2}) \exp(\alpha_i z_1) \tag{22}$$

with velocity components written in terms of r and z given by

$$u = 4 c_0^* (r^2 - a^2) + \sum_{i=1}^{\infty} c_i^* J_0\left(\alpha_i \frac{r}{a}\right) \exp\left(\alpha_i \frac{z}{a}\right)$$

and

$$v = -\sum_{i=1}^{\infty} c_i^* J_1\left(\alpha_i \frac{r}{a}\right) \exp\left(\alpha_i \frac{z}{a}\right) \tag{23}$$

where $c_0^* = c_0/a^2$ and $c_i^* = \frac{\alpha_i c_i}{a^2} \exp\{-\alpha_i a\}$, an arbitrary constant.

Discussion

The solution has some remarkable features which will become apparent. Firstly we observe that in the absence of suction and taking $c_0^* < 0$ we have flow down an impermeable pipe. From (23) it can be seen that $c_i^* > 0$ ($\alpha_i > 0$) corresponds to injection of the fluid and this will increase the velocity along the channel at $r = 0$ which is consistent with the usual deductions made on physical grounds. Similarly $c_i^* < 0$ corresponds to suction and decreases the centre line velocity.

The solution (23) gives fully developed flow through a porous channel with variable suction in the same way that all the previous analytical solutions for constant suction have been fully developed. Consequently all these solutions have required "a development length" before the solution is attained. However (23) has some remarkable features which can be illustrated by choosing a simple velocity of suction V of the form

$$V = 2\lambda J_1(\alpha_i)\cosh\left(\frac{\alpha_i z}{a}\right)$$
$$= \lambda J_1(\alpha_i)\{e^{\alpha_i z/a} + e^{-\alpha_i z/a}\}$$

where λ is a constant. Then the velocity component u is

$$u = 4c_0^*(r^2 - a^2) + \lambda J_0\left(\frac{\alpha_i r}{a}\right)\{e^{-\alpha_i z/a} - e^{\alpha_i z/a}\}$$
$$= 4c_0^*(r^2 - a^2) - 2\lambda J_0\left(\frac{\alpha_i r}{a}\right)\sinh\left(\frac{\alpha_i z}{a}\right).$$

It is immediately obvious that at $z = 0$ the velocity profile of u is that given by the flow in an impermeable pipe and so an entry length is not required. The flow continuously deforms from that in an impermeable channel. Further by adding several suction terms $\lambda_i J_1(\alpha_i)\cosh\left(\alpha_i \frac{z}{a}\right)$ and making an appropriate choice for λ_i we can make the suction commence with zero velocity at $z = 0$ and finish with zero velocity at an appropriate position downstream!

It is worth noting that although $J_0(\alpha_i r)$ is an oscillating function in the pipe for $|\alpha_i| > 2.4$, it does not follow that there is reverse flow because the c_i^* can always be chosen such that c_0^* dominates the flow.

The solution is a useful addition to flows in porous channels and indicates that even stranger solutions are conceivable. For example, a suction of the type $\sum f_i(r)\, e^{\alpha_i z}\cos(\beta_i z + \gamma_i)$ is a natural extension and this leads to complex Bessel functions which are at present being studied by the author.

The author is grateful to Professor Rott for some suggestions and, in particular, to an illuminating way of looking at the solution which will be discussed in Appendix 1.

Appendix 1

The paper has been concerned with obtaining an exact solution of the Navier Stokes equations with variable suction or injection at the pipe walls. An insight into the resulting solution (23) can be obtained by writing it as

$$u = u_0(r) + \frac{\partial\varphi}{\partial z}$$
$$v = \frac{\partial\varphi}{\partial z},$$

where φ is the potential function

$$\varphi = \sum_{c=1}^{\infty} \frac{a\, c_i^*}{\alpha_i} J_0\left(\alpha_i \frac{r}{a}\right) \exp\left(\alpha_i \frac{z}{a}\right).$$

By substituting for u and v in equations (2) and (3) and eliminating p, it can be shown that it is a possible solution provided $u_0(r) = A\, r^2 + B$, where A and B are constants. Thus for a velocity distribution of this type (for example, Poiseuille flow in a pipe), it is possible to superimpose potential flows upon the viscous flow.

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Summary

An exact solution of the Navier-Stokes equations for flow in a porous pipe is presented. This solution allows the suction or injection at the wall to vary with axial distance and will provide new insight into flows through porous pipes.

Resumé

Une solution exacte d'équation de Navier-Stokes est présentée pour l'écoulement d'un liquide visqueux dans un tube perméable. Ce liquide est aspiré ou injecté avec une vélocité variable et la solution donne une nouvelle optique quant aux tubes poreuses.

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