Journal of Applied Mathematics and Physics (ZAMP)
Vol. 37, November 1986
0044-2275/86/006859-08 $ 3.10/0

Stokes flow past a slightly deformed fluid sphere

By H. Ramkissoon, Dept. of Mathematics, University of the West Indies, St. Augustine, Trinidad, W. I.

1. Introduction

The Stokes flow due to the translation of a spherical fluid particle in an unbounded fluid medium has been investigated independently by Rybczynski [1] and Hadamard [2]. However, there are cases when the droplets are not perfectly spherical and it is to this situation that we now address ourselves.

In this note the problem of symmetrical flow past a fluid spheroid whose shape varies slightly from that of a sphere, is examined. Explicit expressions are obtained for both the external and internal flow fields to the first order in the small parameter characterizing the deformation. As a particular case, we consider the flow past an oblate spheroidal fluid particle and determine the drag experienced by it. Special well-known cases are then deduced.

2. Statement and solution of the problem

In the case of axisymmetric incompressible creeping flow, the solution of Stokes equation in spherical polar coordinates is given by [3]

$$\psi(r,\theta) = \sum_{n=2}^{\infty} (A_n r^n + B_n r^{-n+1} + C_n r^{n+2} + D_n r^{-n+3}) I_n(\zeta) \tag{2.1}$$

where ψ is the stream function, $\zeta = \cos\theta$ and $I_n(\zeta)$ is the Gegenbauer function related to the Legendre function $P_n(\zeta)$ by the relation

$$I_n(\zeta) = \frac{P_{n-2}(\zeta) - P_n(\zeta)}{2n-1}, \quad n \geqq 2.$$

These functions have the following special property [3] relevant to our work:

$$I_m I_2 = -\frac{(m-2)(m-3)}{2(2m-1)(2m-3)} I_{m-2} + \frac{m(m-1)}{(2m+1)(2m-3)} I_m - \frac{(m+1)(m+2)}{2(2m-1)(2m+1)} I_{m+2}, \quad m \geqq 2. \tag{2.2}$$

Let the surface S of a spheroid approximating that of the sphere $r = a$ be $r = a[1 + f(\theta)]$. The orthogonality of the Gegenbauer functions permit us, under general circumstances, to assume the expansion $f(\theta) = \sum_{\kappa=1}^{\infty} \alpha_\kappa I_\kappa(\zeta)$. We can therefore take S to be

$$r = a[1 + \alpha_m I_m(\zeta)] \tag{2.3}$$

and neglect terms of $0(\alpha_m^2)$. We now state our main problem.

Consider slow steady flow of an incompressible fluid of viscosity μ_e past a fluid spheroid of viscosity μ_i whose surface S is given by (2.3) and which is assumed to be macroscopically at rest in an otherwise uniform stream of speed U in the direction of the negative z-axis in the absence of body forces. Assuming that the motion is axially symmetric, determine both the internal and external flow fields.

In view of the axial symmetry and incompressibility of the flow we can, in the usual manner, introduce a stream function Ψ defined by

$$u_r = -\frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta}, \quad u_\theta = \frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial r} \tag{2.4}$$

and satisfying (2.1). Here $\underset{\sim}{u}(r, \theta) \equiv (u_r, u_\theta, 0)$ represents the velocity field. There are two distinct fluid motions, namely the internal motion within the spheroid and the external motion of the flow past the spheroid. We shall use superscripts (i) and (e) to distinguish between these separate motions occurring inside and outside of the fluid spheroid respectively. We take the stream function in the exterior of S to be

$$\frac{\Psi^{(e)}}{U a^2} = \left(a_2 \sigma^2 + \frac{b_2}{\sigma} + c_2 \sigma^4 + d_2 \sigma\right) I_2(\zeta) + \sum_{n=3}^{\infty} (A_n \sigma^n + B_n \sigma^{-n+1} + C_n \sigma^{n+2} + D_n \sigma^{-n+3}) I_n(\zeta), \tag{2.5}$$

while in the interior of S we take it as

$$\frac{\Psi^{(i)}}{U a^2} = \left(a_2' \sigma^2 + \frac{b_2'}{\sigma} + c_2' \sigma^4 + d_2' \sigma\right) I_2(\zeta) + \sum_{n=3}^{\infty} (A_n' \sigma^n + B_n' \sigma^{-n+1} + C_n' \sigma^{n+2} + D_n' \sigma^{-n+3}) I_n(\zeta) \tag{2.6}$$

where $\sigma = \frac{r}{a}$. Using the condition

$$\Psi^{(e)} \to \frac{I}{2} U r^2 \sin^2\theta \quad \text{as} \quad r \to \infty \tag{2.7}$$

and the fact that the components of velocity at the origin must be finite, the above representations (2.5) and (2.6) now take the form

$$\frac{\Psi^{(e)}}{U a^2} = \left(\sigma^2 + \frac{b_2}{\sigma} + d_2 \sigma\right) I_2(\zeta) + \sum_{n=3}^{\infty} (B_n \sigma^{-n+1} + D_n \sigma^{-n+3}) I_n(\zeta), \tag{2.8}$$

$$\frac{\Psi^{(i)}}{U a^2} = \left(a_2' \sigma^2 + c_2' \sigma^4\right) I_2(\zeta) + \sum_{n=3}^{\infty} (A_n' \sigma^n + C_n' \sigma^{n+2}) I_n(\zeta). \tag{2.9}$$

The only coefficients which contribute to the flow past a fluid sphere [4] are b_2, d_2, a_2', c_2' and consequently we expect all other coefficients in (2.8) and (2.9) to be of $0(\alpha_m)$. Therefore, except where these coefficients b_2, d_2, a_2', c_2' are encountered, we may take the surface to be $r = a$ instead of the exact form (2.3).

The unknown coefficients appearing in (2.8)–(2.9) must be evaluated from boundary conditions. The kinematic condition of mutual impenetrability at the surface requires that we take

$$\Psi^{(e)} = 0 \quad \text{on} \quad S, \tag{2.10}$$

$$\Psi^{(i)} = 0 \quad \text{on} \quad S. \tag{2.11}$$

We assume that the tangential velocity is continuous across the surface. Hence,

$$\frac{\partial \Psi^{(e)}}{\partial r} = \frac{\partial \Psi^{(i)}}{\partial r} \quad \text{on} \quad S. \tag{2.12}$$

We further assume that the theory of interfacial tension is applicable to our problem. This means that the presence of interfacial tension only produces a discontinuity in the normal stress t_{rr} and does not in anyway affect the tangential stress $t_{r\theta}$. The latter is therefore continuous accross the surface and so $t_{r\theta}^{(e)} = t_{r\theta}^{(i)}$ on S or equivalently

$$\mu_e \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi^{(e)}}{\partial r}\right) = \mu_i \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \Psi^{(i)}}{\partial r}\right) \quad \text{on} \quad S. \tag{2.13}$$

These boundary conditions (2.10)–(2.13) lead respectively to the following:

$$0 = (1 + b_2 + d_2) I_2(\zeta) + (2 - b_2 + d_2) \alpha_m I_m(\zeta) I_2(\zeta) + \sum_{n=3}^{\infty} (B_n + D_n) I_n(\zeta),$$

$$0 = (a_2' + c_2') I_2(\zeta) + (2 a_2' + 4 c_2') \alpha_m I_m(\zeta) I_2(\zeta) + \sum_{n=3}^{\infty} (A_n' + C_n') I_n(\zeta),$$

$$\begin{aligned} 0 = {} & (2 a_2' + 4 c_2' - 2 + b_2 - d_2) I_2(\zeta) \\ & + (2 a_2' + 12 c_2' - 2 b_2 - 2) \alpha_m I_m(\zeta) I_2(\zeta) \\ & + \sum_{n=3}^{\infty} [n A_n' + (n + 2) C_n' + (n - 1) B_n + (n - 3) D_n] I_n(\zeta), \end{aligned} \tag{2.14}$$

$$
\begin{aligned}
0 = {} & [\mu_i(4c_2' - 2a_2') + \mu_e(2 - 4b_2 + 2d_2)] I_2(\zeta) \\
& + [\mu_i(12c_2' - 2a_2') + \mu_e(2 + 8b_2)] \alpha_m I_m(\zeta) I_2(\zeta) \\
& + \sum_{n=3}^{\infty} \mu_i \{n(n-3)A_n' + (n+2)(n-1)C_n'\} \\
& - \mu_e\{(n-1)(n+2)B_n + n(n-3)D_n\}] I_n(\zeta).
\end{aligned}
$$

The leading terms in the above system of Eqs. (2.14) must vanish. Hence,

$$
1 + b_2 + d_2 = 0, \quad a_2' + c_2' = 2a_2' + 4c_2' - 2 + b_2 - d_2 = 0,
$$

$$
\mu_i(4c_2' - 2a_2') + \mu_e(2 - 4b_2 + 2d_2) = 0.
$$

Solving these equations, we get

$$
a_2' = -\frac{\lambda}{2(1+\lambda)}, \quad c_2' = \frac{\lambda}{2(1+\lambda)}, \quad b_2 = \frac{1}{2(1+\lambda)}, \quad d_2 = -\frac{3+2\lambda}{2(1+\lambda)}, \tag{2.15}
$$

where $\lambda = \dfrac{\mu_e}{\mu_i}$. The identical values were obtained for the case of flow past a fluid sphere [4]. Substituting these values into (2.14), we get

$$
\begin{aligned}
& \frac{\lambda}{1+\lambda} \alpha_m I_m(\zeta) I_2(\zeta) + \sum_{n=3}^{\infty} (B_n + D_n) I_n(\zeta) = 0, \\
& \frac{\lambda}{1+\lambda} \alpha_m I_m(\zeta) I_2(\zeta) + \sum_{n=3}^{\infty} (A_n' + C_n') I_n(\zeta) = 0, \\
& \frac{3(\lambda-1)}{1+\lambda} \alpha_m I_m(\zeta) I_2(\zeta) \\
& \quad + \sum_{n=3}^{\infty} [nA_n' + (n+2)C_n' + (n-1)B_n + (n-3)D_n] I_n(\zeta) = 0, \\
& \frac{2\lambda^2 + 13\lambda}{1+\lambda} \alpha_m I_m(\zeta) I_2(\zeta) \\
& \quad + \sum_{n=3}^{\infty} [n(n-3)A_n' + (n+2)(n-1)C_n' - \lambda(n-1)(n+2)B_n \\
& \qquad - \lambda n(n-3)D_n] I_n(\zeta) = 0.
\end{aligned} \tag{2.16}
$$

Solving the above system of Eqs. (2.16) with the aid of the identity (2.2), we see that the coefficients vanish for all n exept when n has the values $m-2, m, m+2$.

These surviving coefficients are

$$\begin{aligned} A'_{m-2} &= \alpha_m E_1 \left[\frac{3 + 2m\lambda - 6\lambda}{2(1+\lambda)} - F_1\right], \\ B_{m-2} &= \alpha_m E_1 \left[\frac{\lambda}{1+\lambda} - F_1\right], \\ C'_{m-2} &= \alpha_m E_1 \left[\frac{8\lambda - 3 - 2m\lambda}{2(1+\lambda)} + F_1\right], \\ D_{m-2} &= \alpha_m E_1 F_1, \end{aligned} \tag{2.17}$$

$$\begin{aligned} A'_m &= -\alpha_m E_2 \left[\frac{3 - 2\lambda + 2\lambda m}{2(1+\lambda)} + F_2\right], \\ B_m &= -\alpha_m E_2 \left[\frac{\lambda}{1+\lambda} + F_2\right], \\ C'_m &= \alpha_m E_2 \left[\frac{3 - 4\lambda + 2\lambda m}{2(1+\lambda)} + F_2\right], \\ D_m &= \alpha_m E_2 F_2, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} A'_{m+2} &= \alpha_m E_3 \left[\frac{2\lambda + 3 + 2\lambda m}{2(1+\lambda)} - F_3\right], \\ B_{m+2} &= \alpha_m E_3 \left[\frac{\lambda}{1+\lambda} - F_3\right], \\ C'_{m+2} &= \alpha_m E_3 \left[\frac{-3 - 2\lambda m}{2(1+\lambda)} + F_3\right], \\ D_{m+2} &= \alpha_m E_3 F_3, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} E_1 &= \frac{(m-2)(m-3)}{2(2m-1)(2m-3)}, \quad E_2 = \frac{m(m-1)}{2(m+1)(2m-3)}, \\ E_3 &= \frac{(m+1)(m+2)}{2(2m-1)(2m+1)}, \\ F_1 &= \frac{2\lambda^2 + 43\lambda + \lambda^2 m^2 - 3\lambda^2 m - 19\lambda m + 6m + 3\lambda m^2 - 15}{2(1+\lambda)^2(2m-5)}, \\ F_2 &= \frac{3 - 17\lambda - \lambda^2 m^2 - m\lambda^2 - 3m^2\lambda + 7\lambda m - 6m}{(1+\lambda)^2(4m-2)}, \\ F_3 &= \frac{6\lambda^2 + 15\lambda + \lambda^2 m^2 + 5m\lambda^2 + 3m^2\lambda + 5\lambda m + 6m + 9}{(1+\lambda)^2(4m+6)}. \end{aligned} \tag{2.20}$$

We have thus determined the stream functions for both the external and internal flow fields. With the aid of (2.8) and (2.9) they are now given by

$$\frac{\Psi^{(e)}}{U a^2} = \left(\sigma^2 + \frac{b_2}{\sigma} + d_2 \sigma\right) I_2(\zeta) + (B_{m-2}\sigma^{3-m} + D_{m-2}\sigma^{5-m}) I_{m-2}(\zeta)$$
$$+ (B_m \sigma^{-m+1} + D_m \sigma^{-m+3}) I_m(\zeta)$$
$$+ (B_{m+2}\sigma^{-m-1} + D_{m+2}\sigma^{1-m}) I_{m+2}(\zeta), \qquad (2.21)$$

$$\frac{\Psi^{(i)}}{U a^2} = (\alpha'_2 \sigma^2 + c'_2 \sigma^4) I_2(\zeta) + (A'_{m-2}\sigma^{m-2} + C'_{m-2}\sigma^m) I_{m-2}(\zeta)$$
$$+ (A'_m \sigma^m + C'_m \sigma^{m+2}) I_m(\zeta) + (A'_{m+2}\sigma^{m+2} + C'_{m+2}\sigma^{m+4}) I_{m+2}(\zeta)$$

where the constants have all been determined. (2.22)

3. Application to a fluid oblate spheroid

As an application of the foregoing, we now consider the particular case of the oblate spheroid

$$\frac{x^2 + y^2}{c^2} + \frac{z^2}{c^2(1-\varepsilon)^2} = 1,$$

where again we are neglecting terms of $0(\varepsilon^2)$. Its polar form is

$$r = a[1 + 2\varepsilon I_2(\zeta)] \quad \text{or} \quad \sigma = 1 + 2\varepsilon I_2(\zeta), \qquad (3.1)$$

where $a = c(1-\varepsilon)$. To apply the results of the last section, we put $m = 2$, $\alpha_m = 2\varepsilon$. It follows from (2.17) that $A'_0 = B_0 = C'_0 = D_0 = 0$, and it can be verified that the external and internal stream functions given by (2.21) and (2.22) respectively, now take the form

$$\Psi^{(e)} = U c^2 \left[\left\{\left(\frac{r}{c}\right)^2 + (d_2 - d_2\varepsilon + 2D_2\varepsilon)\left(\frac{r}{c}\right)\right.\right.$$
$$\left. + (b_2 - 3b_2\varepsilon + 2B_2\varepsilon)\left(\frac{c}{r}\right)\right\} I_2(\zeta)$$
$$\left. + \left\{2B_4\varepsilon\left(\frac{c}{r}\right)^2 + 2D_4\varepsilon\left(\frac{c}{r}\right)\right\} I_4(\zeta)\right], \qquad (3.2)$$

$$\Psi^{(i)} = U c^2 \left[\left\{a'_2\left(\frac{r}{c}\right)^2 + (c'_2 + 2c'_2\varepsilon + 2C'_2\varepsilon)\left(\frac{r}{c}\right)^4\right\} I_2(\zeta)\right.$$
$$\left. + \left\{2A'_4\varepsilon\left(\frac{r}{c}\right)^4 + 2C'_4\varepsilon\left(\frac{r}{c}\right)^6\right\} I_4(\zeta)\Big)\right], \qquad (3.3)$$

where a_2', b_2, c_2', d_2 are given by (2.15) and

$$B_2 = \frac{3}{5(1+\lambda)}, \quad C_2' = -\frac{2\lambda}{5(1+\lambda)}, \quad D_2 = -\frac{2\lambda+3}{5(1+\lambda)},$$
$$A_4' = \frac{22\lambda^2 + 26\lambda}{35(1+\lambda)^2}, \quad B_4 = -\frac{6\lambda^2 + 23\lambda + 21}{35(1+\lambda)^2}, \tag{3.4}$$
$$C_4' = -\frac{8\lambda^2 + 12\lambda}{35(1+\lambda)^2}, \quad D_4 = \frac{20\lambda^2 + 37\lambda + 21}{35(1+\lambda)^2}.$$

The flow fields within the fluid oblate spheroid and outside of it are now completely determined. We now propose to examine a feature of this flow which is of most practical significance – the force experienced by the spheroid.

To evaluate this drag, we appeal to a simple elegant formula derived by Payne and Pell [5]. From this formula, in the case of slow, steady axisymmetric flow past the oblate spheroid, the drag D experienced is given by the expression

$$D = 8\pi\mu_e \lim_{r\to\infty} \frac{\Psi^{(e)} - \Psi_\infty}{r\sin^2\theta}, \tag{3.5}$$

where Ψ_∞ is the stream function corresponding to the fluid motion at infinity. Here $\Psi_\infty = U r^2 I_2(\zeta) = \frac{1}{2} U r^2 \sin^2\theta$. Substituting this and (3.2) into (3.5), gives

$$D = -6\pi\mu_e c U (1 - \tfrac{1}{5}\varepsilon) \frac{1 + \frac{2}{3}\lambda}{1+\lambda}. \tag{3.6}$$

The following special cases can be deduced immediately:

(a) *Rigid Oblate Spheroid* $\left(\lambda = \dfrac{\mu_e}{\mu_i} = 0\right)$. In this case

$$D = -6\pi\mu_e c U (1 - \tfrac{1}{5}\varepsilon),$$

a result obtained by Happel and Brenner [4].

(b) *A Perfect Fluid Sphere* ($\varepsilon = 0$). Here we get the well known result [4]

$$D = -6\pi\mu_e c U \frac{1 + \frac{2}{3}\lambda}{1+\lambda}. \tag{3.7}$$

(c) *A Gaseous Oblate Spheroidal Bubble* ($\mu_e \gg \mu_i$). This gives rise to a new result

$$D = -4\pi\mu_e c U (1 - \tfrac{1}{5}\varepsilon).$$

As a consequence of our results, we see that the force exerted on the fluid oblate spheroid is less than that experienced by a fluid sphere of radius equal to the equatorial radius of the spheroid. We may make another comparison.

A sphere of radius $c(1 - \frac{1}{3}\varepsilon)$ would have the same volume as our spheroid and its resistance from (3.7) would be

$$D = -6\pi\mu_e c U (1 - \tfrac{1}{3}\varepsilon)\frac{1 + \frac{2}{3}\lambda}{1 + \lambda}.$$

On comparison with (3.6), we see that a fluid sphere of equal volume experiences a smaller resistance than the fluid oblate spheroid.

References

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Summary

The Stokes flow past a fluid spheroid whose shape deviates slightly from that of a sphere, is examined. To the first order in the small parameter characterizing the deformation, an exact solution is obtained. As an application, the case of a fluid oblate spheroid is considered and the drag experienced by it is evaluated. Special well-known cases are then deduced.

Zusammenfassung

Die Stokes-Strömung um einen Flüssigkeitstropfen, der nur leicht von einer perfekten Kugel abweicht, wird untersucht. Eine Lösung wird gefunden, die exakt ist bis zur ersten Ordnung im Deformations-Parameter. Als Beispiel wird der Strömungswiderstand eines abgeplatteten Flüssigkeits-Sphäroids berechnet. Die Methode liefert für bekannte Spezialfälle korrekte Lösungen.

(Received: July 7, 1985; revised: May 26, 1986)